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POSITIVE SOLUTION FOR HÉNON TYPE EQUATIONS WITH CRITICAL SOBOLEV GROWTH

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Abstract. We investigate the Hénon type equation involving the critical Sobolev exponent with Dirichret boundary condition

$$-\Delta u = \lambda \Psi u + |x|^{\alpha} u^{2^*-1}$$

in Ω included in a unit ball, under several conditions. Here, Ψ is a non-trivial given function with $0 \le \Psi \le 1$ which may vanish on $\partial\Omega$. Let λ_1 be the first eigenvalue of the Dirichret eigenvalue problem $-\Delta\phi = \lambda\Psi\phi$ in Ω. We show that if the dimension $N \ge 4$ and $0 < \lambda < \lambda_1$, there exists a positive solution for small $\alpha > 0$. Our methods include the mountain pass theorem and the Talenti function.

1. Introduction

We consider the Hénon type equation with critical Sobolev growth

$$\begin{gathered} -\Delta u = \lambda \Psi u + |x|^{\alpha} u^{2^*-1} \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

We set $N \ge 3$. We use $2^* = 2N/(N-2)$ to denote the critical Sobolev exponent. Let $\Omega \subset \mathbb{R}^N$ be a piecewise C^1-class bounded domain satisfying $\Omega \subset B(0,1)$. Here, $B(p,r) = \{x \in \mathbb{R}^N : |x-p| < r\}$. Let $x_0 = (1,0,\dots,0) \in \mathbb{R}^N$. We assume that $x_0 \in \partial\Omega$ and Ω satisfies the interior ball condition at x_0, i.e., there exists an open ball $B \subset \Omega$ with $x_0 \in \partial B$. We consider the case $\lambda < \lambda_1$, where λ_1 is the first eigenvalue of the Dirichret eigenvalue problem: $-\Delta\phi = \lambda\Psi\phi$ in Ω. We set $\alpha > 0$ and $\Psi \in L^\infty(\Omega) \setminus \{0\}$ with $0 \le \Psi \le 1$ in Ω.

Next we state our main theorem.

Theorem 1.1. *Let $N \ge 4$ and $0 < \lambda < \lambda_1$. Suppose that there exist $a > 0$, $\beta \ge 0$ and an open ball $B \subset \Omega$ with $x_0 \in \partial B$ such that $\Psi_0 \le \Psi \le 1$ in Ω, where*

$$\Psi_0(x) = \begin{cases} a|x-x_0|^{\beta} & x \in B, \\ 0 & x \notin B. \end{cases}$$

Then, the main problem (1.1) has a solution $u \in H_0^1(\Omega)$ for sufficiently small $\alpha > 0$.

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We give two examples. The first one is simple: Let $N \geq 4$, $0 < \lambda < \lambda_1$ and $\Omega = B(0,1)$. Assume that Ψ is a continuous function defined on $\overline{\Omega}$ with $0 \leq \Psi \leq 1$. Suppose that there exists $\overline{x} \in \partial\Omega$ such that $\Psi(\overline{x}) > 0$. Then, (1.1) has a solution for small $\alpha > 0$. To confirm this example, we set $\beta = 0$ and some small $a > 0$ and some small $B \subset \Omega$ with $\overline{x} \in \partial\Omega$. The second example is for the case where Ψ vanishes on $\partial\Omega$. We state it as the following corollary.

Corollary 1.2. *Let $N \geq 4$, $0 < \lambda < \lambda_1$, $\Omega = B(0,1)$ and $\beta_0 > 0$. Assume that $\Psi(x) = (1 - |x|)^{\beta_0}$. Then,* (1.1) *has a solution for small $\alpha > 0$.*

This corollary follows from elementary geometries. We prove it in Section 6.

In [8], the following Hénon equation for the case $N = 1$ is proposed

$$\begin{gathered} -\Delta u = |x|^\alpha |u|^{p-1} \quad \text{in } B(0,1), \\ u = 0 \quad \text{on } \partial B(0,1). \end{gathered} \tag{1.2}$$

In the subcritical case $p < 2^*$, the existence of solution is proved by standard compactness argument. In [11], it is proved that if $1 < p < 2^*(\alpha) = 2(N+\alpha)/(N-2)$, (1.2) has a positive radial solution. Hénon equation is widely studied in recent times. Many authors study whether there exists a positive non-radial solution of (1.2) for the case $1 < p < 2^*(\alpha)$. We refer [2, 14, 15]. Many authors also study the subcritical case $p < 2^*$ and investigate the behavior of solutions where $p \to 2^*$. We refer [7, 12]. General bounded domain cases of (1.2) are studied in [5, 6, 9] and so on.

If $\alpha = 0$ and $\Psi = 1$ in Ω, (1.1) becomes the original Brézis–Nirenberg problem. In [4], it is proved that under these conditions there exists a solution if $N \geq 4$ and $0 < \lambda < \lambda_1'$, or if $N = 3$, $\lambda_1'/4 < \lambda < \lambda_1'$ and Ω is a ball. Here λ_1' is the first eigenvalue of the Dirichret eigenvalue problem: $-\Delta\phi = \lambda\phi$ in Ω. Over three decades many authors have studied existence and nonexistence of Brézis–Nirenberg type problems.

Our problem (1.1) is regarded as a combination of Hénon equations and Brézis-Nirenberg problems. In [10] and [13], the following problem directly related to (1.1) is studied

$$\begin{gathered} -\Delta u = \lambda u + |x|^\alpha |u|^{2^*-2} u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where $\alpha > 0$ and $\lambda > \lambda_1'$. They show that (1.3) has a sign-changing solution for sufficiently small $\alpha > 0$ when $N \geq 7$ with smooth $\partial\Omega$ and $N \geq 5$ with $\Omega = B(0,1)$ in [10] and [13], respectively. In this paper, we seek for a positive solution for the case that $0 < \lambda < \lambda_1$, $N \geq 4$, Ω is more generalized and Ψ is not necessarily a constant.

Our method is based on the mountain pass theorem and Talenti functions presented in [4]. Since the coefficient $|x|^\alpha$ is not achieved its maximum in Ω, we use the function

$$u_{\epsilon,l}(x) = \frac{\xi_l(x)}{(\epsilon + |x - x_l|^2)^{(N-2)/2}}.$$

Here, $\epsilon > 0$, $x_l = (1 - l, 0, \dots, 0) \in \mathbb{R}^N$ and $\xi_l \in C_c^\infty(\Omega)$ is a cut-off function supported on $B(x_l, l)$. We regard $l = l(\epsilon)$ as a function that satisfies $l \to 0$ as $\epsilon \to 0$. To prove Theorem 1.1, we set $l = l(\epsilon) = \epsilon^\gamma$ for $0 < \gamma < 1/2$ for the case $N \geq 5$ and $l = l(\epsilon) = |\log\epsilon|^{-k}$ for $k > 0$ for the case $N = 4$. For details, see Section 3. If we take $\epsilon \to 0$, the support is getting smaller and x_l is getting closer to x_0. This type of functions has been already introduced in [10] and [13] with $l = l(\epsilon) = \epsilon^\gamma$ for fixed γ. In our case we choose the parameters γ and k appropriately since Ψ may vanish on $\partial\Omega$.

We set $I\colon H_0^1(\Omega) \to \mathbb{R}$ as

$$I(u) = \frac{1}{2}\int_\Omega |Du|^2 dx - \frac{\lambda}{2}\int_\Omega \Psi u^2 dx - \frac{1}{2^*}\int_\Omega |x|^\alpha (u_+)^{2^*} dx.$$

Here we write $f_+ = \max(f, 0)$ for a function f. Note that $u \in H_0^1(\Omega) \setminus \{0\}$ is a solution of (1.1) if u is a critical point of I. This is because if u is a critical point of I; then we have

$$(-\Delta - \lambda\Psi)u = |x|^\alpha (u_+)^{2^*-1} \geq 0.$$

Since $\lambda < \lambda_1$, we see that $u > 0$ in Ω by the strong maximum principle.

This paper consists of four sections. In Section 2, we prove the mountain pass geometry of I and the convergence of a $(\mathrm{PS})_c$ sequence for some small $c > 0$. In Section 3, we show estimates of integrals of $u_{\epsilon,l}$. In Section 4, we prove Theorem 1.1. In Section 5, we show a technical convergence lemma. In Section 6, we prove Corollary 1.2.

Throughout the present paper, all functions are real-valued. We use $L^r(\Omega)$ for $r \geq 1$ to denote the Lebesgue space equipped with the norm

$$\|v\|_{L^r(\Omega)} = \begin{cases} \left(\int_\Omega |v|^r dx\right)^{1/r} & 1 \leq r < \infty, \\ \operatorname{ess\,sup}_{x\in\Omega} |v(x)| & r = \infty. \end{cases}$$

The inner product of $L^2(\Omega)$ is denoted by

$$(v, w)_{L^2(\Omega)} = \int_\Omega vw dx.$$

The Sobolev space $H_0^1(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|v\|_{H_0^1(\Omega)} = \sqrt{(v, v)_{H_0^1(\Omega)}}, \quad \text{where} \quad (v, w)_{H_0^1(\Omega)} = (Dv, Dw)_{L^2(\Omega)} = \int_\Omega Dv \cdot Dw dx.$$

We write $\langle f, v\rangle$ for the canonical pairing of $f \in H^{-1}(\Omega)$ and $v \in H_0^1(\Omega)$. We remark two notations. If $f = -\Delta w$ for some $w \in H_0^1(\Omega)$, then

$$\langle f, v\rangle = \int_\Omega Dw \cdot Dv dx = (w, v)_{H_0^1(\Omega)}.$$

If we regard $w \in L^2(\Omega)$ as an element of $H^{-1}(\Omega)$, then

$$\langle w, v\rangle = \int_\Omega wv dx = (w, v)_{L^2(\Omega)}.$$

We use S to denote the best Sobolev constant defined by

$$S = \inf_{u\in H_0^1(\Omega), u\not\equiv 0} \frac{\|Du\|_{L^2(\Omega)}^2}{\|u\|_{L^{2^*}(\Omega)}^2}.$$

It is known that S does not depend on $\Omega \subset \mathbb{R}^N$. Without definitions we use the characters $C, C', C'', C_1, C_2 > 0$ to denote positive constants which is not important and may vary by line to line.

2. $(\mathrm{PS})_c$ Condition and Mountain Pass Theorem

In this section we assume that $N \geq 3$ and $\lambda < \lambda_1$. We recall the $(\mathrm{PS})_c$ condition and the mountain pass theorem without (PS) condition.

Definition 2.1. (i) Let $c \in \mathbb{R}$. We say that a sequence $\{u_k\}_{k=0}^{\infty}$ in $H_0^1(\Omega)$ is a Palais-Smale sequence of I at the mountain pass level c if the following conditions hold:
(1) $I(u_k) \to c$ $(k \to \infty)$,
(2) $I'(u_k) \to 0$ in $H^{-1}(\Omega)$ $(k \to \infty)$.

(ii) Let $c \in \mathbb{R}$. We say that I satisfies the $(\mathrm{PS})_c$ condition if any Palais-Smale sequence of I at the mountain pass level c has a convergent subsequence in $H_0^1(\Omega)$.

Proposition 2.2 (The mountain pass theorem without (PS) condition [1])**.** *Suppose that there exist $r, l > 0$ such that $I(u) > l$ for all $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1(\Omega)} = r$. Assume that there exists $v \in H_0^1(\Omega)$ such that $I(v) \leq 0$ and $\|u\|_{H_0^1(\Omega)} > r$. Let*

$$c = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} I(u), \tag{2.1}$$

where Γ is the set of paths in $H_0^1(\Omega)$ connecting 0 and any end point $v \in H_0^1(\Omega)$ with $I(v) \leq 0$ and $\|v\|_{H_0^1(\Omega)} > r$. Then, there exists a Palais-Smale sequence of I at the mountain pass level c.

Lemma 2.3. *For $u \in H_0^1(\Omega)$, we have*

$$\|Du\|_{L^2(\Omega)}^2 - \lambda \int_\Omega \Psi u^2 dx \geq \Big(1 - \frac{\max(\lambda, 0)}{\lambda_1}\Big) \|Du\|_{L^2(\Omega)}^2, \tag{2.2}$$

$$1 - \frac{\max(\lambda, 0)}{\lambda_1} > 0. \tag{2.3}$$

Proof. If $\lambda \leq 0$, we have

$$\|Du\|_{L^2(\Omega)}^2 - \lambda \int_\Omega \Psi u^2 dx \geq \|Du\|_{L^2(\Omega)}^2.$$

If $0 \leq \lambda < \lambda_1$, we have

$$\|Du\|_{L^2(\Omega)}^2 - \lambda \int_\Omega \Psi u^2 dx \geq \Big(1 - \frac{\lambda}{\lambda_1}\Big) \|Du\|_{L^2(\Omega)}^2.$$

Here we used the Poincaré type inequality. Combining these cases, we have (2.2). The inequality (2.3) follows, since $\lambda < \lambda_1$. $\square$

We check the mountain pass geometry of I. We admit that I is a C^1-class functional on $H_0^1(\Omega)$ with $I(0) = 0$.

Lemma 2.4. *There exist $r > 0$ and $l > 0$ such that $I(u) > l$ for all $u \in H_0^1(\Omega)$ with $\|u\|_{H_0^1(\Omega)} = r$.*

Proof. By the Sobolev inequality, there exists $C > 0$ such that

$$\|u\|_{L^{2^*}(\Omega)}^{2^*} \leq C \|Du\|_{L^2(\Omega)}^{2^*}$$

for any $u \in H_0^1(\Omega)$. Thus we have

$$I(u) \geq \frac{1}{2}\Big(1 - \frac{\max(\lambda, 0)}{\lambda_1}\Big) \|Du\|_{L^2(\Omega)}^2 - \frac{1}{2^*} \|u\|_{L^{2^*}(\Omega)}^{2^*}$$

$$\geq C_1\|Du\|_{L^2(\Omega)}^2 - C_2\|Du\|_{L^2(\Omega)}^{2^*}.$$

Since $2 < 2^*$, the proof is complete. $\square$.

Lemma 2.5. *For any $r > 0$, there exists $u \in H_0^1(\Omega)$ such that $I(u) \leq 0$ and $\|u\|_{H_0^1(\Omega)} > r$.*

Proof. Let $v \in H_0^1(\Omega) \setminus \{0\}$ and $t > 0$. We have

$$I(tv) = \frac{t^2}{2}\Big(\|Dv\|_{L^2(\Omega)}^2 - \lambda\int_\Omega \Psi v^2 dx\Big) - \frac{t^{2^*}}{2^*}\int_\Omega |x|^\alpha (v_+)^{2^*} dx.$$

It follows that $\lim_{t\to\infty} I(tv) = -\infty$ since $2 < 2^*$. Set $u = tv$ for large $t > 0$ to complete the proof. $\square$

Next, we study which mountain pass level c satisfies the $(\mathrm{PS})_c$ condition on I.

Lemma 2.6. *Let $\{u_k\}_{k=0}^\infty$ be a Palais-Smale sequence of I at the mountain pass level $c \in \mathbb{R}$. Then, $\{u_k\}$ is bounded in $H_0^1(\Omega)$.*

Proof. Let $\epsilon > 0$. Then, by the condition (2) of Definition 2.1 (i), we have

$$|\langle I'(u_k), u_k\rangle| \leq \epsilon\|Du_k\|_{L^2(\Omega)}$$

for large k. Set $\epsilon = 2^*$ and combine the condition (1) of Definition 2.1 (i) to have

$$I(u_k) - \frac{1}{2^*}\langle I'(u_k), u_k\rangle \leq C + \|Du_k\|_{L^2(\Omega)}.$$

It also follows that

$$\begin{aligned} I(u_k) - \frac{1}{2^*}\langle I'(u_k), u_k\rangle &= \Big(\frac{1}{2} - \frac{1}{2^*}\Big)\Big(\|Du_k\|_{L^2(\Omega)}^2 - \lambda\int_\Omega \Psi u_k^2 dx\Big) \\ &\geq \Big(\frac{1}{2} - \frac{1}{2^*}\Big)\Big(1 - \frac{\max(\lambda,0)}{\lambda_1}\Big)\|Du_k\|_{L^2(\Omega)}^2. \end{aligned}$$

Combining these inequalities, we have

$$C'\|Du_k\|_{L^2(\Omega)}^2 \leq C + \|Du_k\|_{L^2(\Omega)}.$$

We see that $\|Du_k\|_{L^2(\Omega)}$ is bounded, which completes the proof. $\square$

Lemma 2.7. *Let*

$$0 < c < \frac{1}{N}S^{N/2}. \tag{2.4}$$

Then, I satisfies $(PS)_c$ condition.

Proof. Let $\{u_k\}_{k=0}^\infty$ be a Palais-Smale sequence of I at the mountain pass level c satisfying (2.4). By Lemma 2.6, $\{u_k\}$ is a bounded sequence of $H_0^1(\Omega)$. Thus there exists $u \in H_0^1(\Omega)$ such that, taking a subsequence,

$$\begin{gathered} u_k \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \\ u_k \to u \quad \text{in } L^r(\Omega) \quad (r < 2^*), \\ u_k \to u \quad \text{a.e. in } \Omega \end{gathered} \tag{2.5}$$

as $k \to \infty$. Let $\psi \in H_0^1(\Omega)$. By Lemma 5.1, we have

$$\begin{aligned} \langle I'(u_k), \psi\rangle &= \int_\Omega Du_k \cdot D\psi dx - \lambda\int_\Omega \Psi u_k\psi dx - \int_\Omega |x|^\alpha (u_k)_+^{2^*-1}\psi dx \\ &\xrightarrow{k\to\infty} \int_\Omega Du \cdot D\psi dx - \lambda\int_\Omega \Psi u\psi dx - \int_\Omega |x|^\alpha u_+^{2^*-1}\psi dx \end{aligned}$$

$$= \langle I'(u), \psi \rangle.$$

Since $\lim_{k\to\infty}\langle I'(u_k), \psi\rangle = 0$, it follows that

$$\langle I'(u), \psi \rangle = 0. \tag{2.6}$$

We show that

$$u_k \to u \quad \text{in } H_0^1(\Omega). \tag{2.7}$$

Note that $u_+ = u$ since either $u \equiv 0$ or $u > 0$ in Ω. Set $\psi = u$ in (2.6) to have

$$\int_\Omega |Du|^2 dx - \lambda \int_\Omega \Psi u^2 dx - \int_\Omega |x|^\alpha u^{2^*} dx = 0. \tag{2.8}$$

Then, we see that

$$I(u) = \Big(\frac{1}{2} - \frac{1}{2^*}\Big) \int_\Omega |x|^\alpha u^{2^*} dx \geq 0. \tag{2.9}$$

Let $w_k = u_k - u$. We have

$$\begin{gathered} w_k \rightharpoonup 0 \quad \text{weakly in } H_0^1(\Omega), \\ w_k \to 0 \quad \text{in } L^r(\Omega) \quad (r < 2^*), \\ w_k \to 0 \quad \text{a.e. in } \Omega \end{gathered} \tag{2.10}$$

as $k \to 0$. It follows that

$$\int_\Omega |Du_k|^2 dx = \int_\Omega |Dw_k|^2 dx + \int_\Omega |Du|^2 dx + o(1).$$

Let $\widetilde{w_k} = (u_k)_+ - u$. By Brézis–Lieb Lemma [3], we have

$$\int_\Omega |x|^\alpha (u_k)_+^{2^*} dx = \int_\Omega |x|^\alpha u^{2^*} dx + \int_\Omega |x|^\alpha |\widetilde{w_k}|^{2^*} dx + o(1).$$

Thus,

$$I(u_k) - I(u) = \frac{1}{2} \int_\Omega |Dw_k|^2 dx - \frac{1}{2^*} \int_\Omega |x|^\alpha |\widetilde{w_k}|^{2^*} dx + o(1).$$

Then

$$I(u) + \frac{1}{2} \int_\Omega |Dw_k|^2 dx - \frac{1}{2^*} \int_\Omega |x|^\alpha |\widetilde{w_k}|^{2^*} dx = c + o(1). \tag{2.11}$$

Since $\langle I'(u_k), u_k\rangle \to 0$ as $k \to \infty$, we have

$$\lim_{k\to\infty} \Big(\int_\Omega |Du_k|^2 dx - \lambda \int_\Omega \Psi u_k^2 dx - \int_\Omega |x|^\alpha (u_k)_+^{2^*} dx \Big) = 0.$$

Combining this equation with (2.8) we obtain

$$\lim_{k\to\infty} \Big(\int_\Omega |Dw_k|^2 dx - \int_\Omega |x|^\alpha |\widetilde{w_k}|^{2^*} dx \Big) = 0.$$

Taking a subsequence, we have

$$\lim_{k\to\infty} \int_\Omega |Dw_k|^2 dx = \lim_{k\to\infty} \int_\Omega |x|^\alpha |\widetilde{w_k}|^{2^*} dx.$$

We write $l \geq 0$ as this limit. By the Sobolev inequality, we have

$$\|Dw_k\|_{L^2(\Omega)}^2 \geq S \|w_k\|_{L^{2^*}(\Omega)}^2 \geq S \|\widetilde{w_k}\|_{L^{2^*}(\Omega)}^2 \geq S \Big(\int_\Omega |x|^\alpha |\widetilde{w_k}|^{2^*} dx \Big)^{2/2^*},$$

which implies $l \geq S l^{2/2^*}$. Here we note that $w_k \geq \widetilde{w_k}$ in Ω since either $u \equiv 0$ or $u > 0$ in Ω. We show $l = 0$. Assume to the contrary that $l > 0$. Then, we have $l \geq S^{N/2}$. By (2.11), we have

$$I(u) + \Big(\frac{1}{2} - \frac{1}{2^*}\Big) l = c.$$

By (2.9), it follows that $S^{N/2}/N \leq c$, which contradicts (2.4). Thus we conclude that $l = 0$, which implies (2.7) as desired. □

Proposition 2.8. *Assume that there exists a Palais-Smale sequence of I at the mountain pass level c satisfying* (2.4)*. Then,* (1.1) *has a solution.*

Proof. Let $\{u_k\}_{k=0}^{\infty}$ be a Palais-Smale sequence of I at the mountain pass level c satisfying (2.4). By Lemma 2.7, $\{u_k\}$ has a convergent subsequence. We use $u \in H_0^1(\Omega)$ to denote the limit of it. Then, (2.6) holds for any $\psi \in H_0^1(\Omega)$, i.e., u is a critical point of I. In addition, it follows that

$$I(u) = \lim_{k \to \infty} I(u_k) = c > 0,$$

which implies $u \not\equiv 0$. Hence, u is a solution of (1.1). □

3. Evluations of Integrals

We set $U \colon \mathbb{R}^N \to \mathbb{R}$ as

$$U(x) = \frac{1}{(1 + |x|^2)^{(N-2)/2}}.$$

We set $x_l = (1 - l, 0, \dots, 0) \in \mathbb{R}^N$ for $0 < l < 1$. For $0 < l < 1$, we set cut-off functions $\xi_l \in C_c^\infty(\Omega)$ which satisfies the following conditions:

(1) $0 \leq \xi_l \leq 1$.
(2)
$$\xi_l(x) = \begin{cases} 1 & x \in B(x_l, l/2), \\ 0 & x \notin B(x_l, l). \end{cases}$$
(3) $|D\xi_l| \leq C/l$ for some constant $C > 0$.
(4) $D\xi_l(x) \cdot (x - x_l) \leq 0$.

We set $u_{\epsilon,l}, v_{\epsilon,l} \in H_0^1(\Omega)$ for $\epsilon > 0$ and $0 < l < 1$ as follows:

$$u_{\epsilon,l}(x) = \frac{\xi_l(x)}{(\epsilon + |x - x_l|^2)^{(N-2)/2}},$$
$$v_{\epsilon,l}(x) = \frac{u_{\epsilon,l}(x)}{\| |x|^{\alpha/2^*} u_{\epsilon,l} \|_{L^{2^*}(\Omega)}}.$$

Hereinafter, we regard $l = l(\epsilon)$ as a function of $\epsilon > 0$ which satisfies $l \to 0$ as $\epsilon \to 0$ and $\epsilon \leq l$.

Lemma 3.1. *Suppose that $N \geq 3$. There exist positive constants $C_1, C_2, C > 0$ such that the following inequalities hold for small $\epsilon > 0$:*

$$\begin{aligned} \|DU\|_{L^2(\mathbb{R}^N)}^2 \epsilon^{-(N-2)/2} - C_1 l^{-(N-2)} &\leq \|Du_{\epsilon,l}\|_{L^2(\Omega)}^2 \\ &\leq \|DU\|_{L^2(\mathbb{R}^N)}^2 \epsilon^{-(N-2)/2} + C_2 l^{-(N-2)}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} (1-2l)^{2\alpha/2^*} (\|U\|_{L^{2^*}(\mathbb{R}^N)}^{2^*} \epsilon^{-N/2} - C l^{-N})^{2/2^*} &\leq \| |x|^{\alpha/2^*} u_{\epsilon,l} \|_{L^{2^*}(\Omega)}^2 \\ &\leq \|U\|_{L^{2^*}(\mathbb{R}^N)}^2 \epsilon^{-(N-2)/2}. \end{aligned} \tag{3.2}$$

Proof. First, we investigate

$$I = \int_\Omega |Du_{\epsilon,l}|^2 dx.$$

We have

$$Du_\epsilon(x) = \frac{D\xi_l(x)}{(\epsilon + |x - x_l|^2)^{(N-2)/2}} - \frac{(N-2)\xi_l(x)(x - x_l)}{(\epsilon + |x - x_l|^2)^{N/2}}.$$

We divide I into three terms, i.e., $I = I_1 + I_2 + I_3$;

$$\begin{gathered} I_1 = \int_\Omega \frac{|D\xi_l(x)|^2}{(\epsilon + |x - x_l|^2)^{N-2}} dx, \\ I_2 = \int_\Omega \frac{-2(N-2)\xi_l(x)(D\xi_l(x) \cdot (x - x_l))}{(\epsilon + |x - x_l|^2)^{N-1}} dx, \\ I_3 = \int_\Omega \frac{(N-2)^2 \xi_l(x)^2 |x - x_l|^2}{(\epsilon + |x - x_l|^2)^N} dx. \end{gathered}$$

We start by getting an upper bound. We have

$$I_3 \le \int_{\mathbb{R}^N} \frac{(N-2)^2|x|^2}{(\epsilon + |x|^2)^N} dx = \|DU\|^2_{L^2(\mathbb{R}^N)} \epsilon^{-(N-2)/2}.$$

The integrals I_2 and I_1 are estimated as follows:

$$\begin{aligned} I_2 &\le \frac{C}{l} \int_{B(x_l,l)\setminus B(x_l,l/2)} \frac{|x - x_l|}{(\epsilon + |x - x_l|^2)^{N-1}} dx \\ &= \frac{C}{l} \int_{B(0,l)\setminus B(0,l/2)} \frac{|x|}{(\epsilon + |x|^2)^{N-1}} dx \\ &\le \frac{C}{l} \int_{B(0,l)\setminus B(0,l/2)} \frac{dx}{|x|^{2N-3}} = \frac{C}{l} \int_{l/2}^{l} r^{-N+2} dr \le Cl^{-N+2}, \end{aligned}$$

$$\begin{aligned} I_1 &\le \frac{C}{l^2} \int_{B(0,l)\setminus B(0,l/2)} \frac{1}{(\epsilon + |x|^2)^{N-2}} dx \\ &\le \frac{C}{l^2} \int_{B(0,l)\setminus B(0,l/2)} \frac{dx}{|x|^{2N-4}} \\ &= \frac{C}{l^2} \int_{l/2}^{l} r^{-N+3} dr \le Cl^{-N+2}. \end{aligned}$$

Note that the last integrals of above two inequalities are calculated differentially by the dimension $N \ge 3$. However, the resulting evaluations are the same $I_2, I_1 \le Cl^{-N+2}$. Thus we have the upper bound of (3.1). Next we consider the lower bound. We have $I_1, I_2 \ge 0$. We estimate I_3 as follows:

$$\begin{aligned} I_3 &> \int_{B(x_l,l/2)} \frac{(N-2)^2|x - x_l|^2}{(\epsilon + |x - x_l|^2)^N} dx \\ &= \|DU\|^2_{L^2(\mathbb{R}^N)} \epsilon^{-(N-2)/2} - \int_{\mathbb{R}^N \setminus B(x_l,l/2)} \frac{(N-2)^2|x - x_l|^2}{(\epsilon + |x - x_l|^2)^N} dx. \end{aligned}$$

Here, we obtain

$$\int_{\mathbb{R}^N \setminus B(x_l,l/2)} \frac{(N-2)^2|x - x_l|^2}{(\epsilon + |x - x_l|^2)^N} dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N\setminus B(0,l/2)} \frac{(N-2)^2|x|^2}{(\epsilon+|x|^2)^N}dx\\
&< C\int_{\mathbb{R}^N\setminus B(0,l/2)} \frac{1}{|x|^{2N-2}}dx\\
&= C\int_{l/2}^{\infty} r^{-N+1}dr = Cl^{-N+2},
\end{aligned}
$$

which implies the lower bound of (3.1).

Second, we study

$$
I = \int_\Omega |x|^\alpha u_{\epsilon,l}^{2^*}dx.
$$

We have

$$
I = \int_{B(x_l,l)} \frac{|x|^\alpha \xi_l(x)^{2^*}}{(\epsilon+|x-x_l|^2)^N}dx = \int_{B(0,l)} \frac{|x+x_l|^\alpha \xi_l(x+x_l)^{2^*}}{(\epsilon+|x|^2)^N}dx.
$$

Thus it follows that $(1-2l)^\alpha \widetilde{I} \le I \le \widetilde{I}$. Here we set

$$
\widetilde{I} = \int_{B(0,l)} \frac{\xi_l(x+x_l)^{2^*}}{(\epsilon+|x|^2)^N}dx.
$$

We obtain

$$
\begin{aligned}
\widetilde{I} &= \Big(\int_{B(0,l)} \frac{\xi_l(x+x_l)^{2^*}}{(\epsilon+|x|^2)^N}dx - \int_{B(0,l)} \frac{1}{(\epsilon+|x|^2)^N}dx\Big)\\
&\quad + \Big(\int_{B(0,l)} \frac{1}{(\epsilon+|x|^2)^N}dx - \int_{\mathbb{R}^N} \frac{1}{(\epsilon+|x|^2)^N}dx\Big) + \int_{\mathbb{R}^N} \frac{1}{(\epsilon+|x|^2)^N}dx\\
&= \int_{B(0,l)\setminus B(0,l/2)} \frac{\xi_l(x+x_l)^{2^*}-1}{(\epsilon+|x|^2)^N}dx\\
&\quad - \int_{\mathbb{R}^N\setminus B(0,l)} \frac{1}{(\epsilon+|x|^2)^N}dx + \|U\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}\epsilon^{-N/2}.
\end{aligned}
$$

Thus we have

$$
\widetilde{I} \le \|U\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}\epsilon^{-N/2}.
$$

For the lower bound, it follows that

$$
\begin{aligned}
&\Big|\int_{B(0,l)\setminus B(0,l/2)} \frac{\xi_l(x+x_l)^{2^*}-1}{(\epsilon+|x|^2)^N}dx - \int_{\mathbb{R}^N\setminus B(0,l)} \frac{1}{(\epsilon+|x|^2)^N}dx\Big|\\
&\le \int_{\mathbb{R}^N\setminus B(0,l/2)} \frac{dx}{(\epsilon+|x|^2)^N}\\
&\le \int_{\mathbb{R}^N\setminus B(0,l/2)} \frac{dx}{|x|^{2N}} = Cl^{-N}.
\end{aligned}
$$

Thus we have

$$
\widetilde{I} \ge \|U\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}\epsilon^{-N/2} - Cl^{-N}.
$$

Finally we conclude that

$$
(1-2l)^\alpha\big(\|U\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}\epsilon^{-N/2} - Cl^{-N}\big) \le I \le \|U\|_{L^{2^*}(\mathbb{R}^N)}^{2^*}\epsilon^{-N/2},
$$

which implies (3.2). $\square$

Lemma 3.2. *Let $c > 0$ be a positive constant. Assume that $\lim_{\epsilon\to 0}(\sqrt{\epsilon}/l) = 0$. Then, we have*

$$\int_{B(0,cl)} \frac{dx}{(\epsilon + |x|^2)^{N-2}} = \begin{cases} O(\epsilon^{-(N-4)/2}) & N \geq 5, \\ O(|\log(\sqrt{\epsilon}/l)|) & N = 4, \\ O(l) & N = 3, \end{cases} \tag{3.3}$$

as $\epsilon \to 0$.

Note that this is not a direct conclusion argued in [4, p. 445]. We have to take it into account that $l \to 0$ as $\epsilon \to 0$. If $N = 3$, the integral converges to 0, which does not in [4].

Proof. Let I denote the integral on the left side of (3.3). First, we investigate the case $N \geq 5$. We have

$$I = \epsilon^{-(N-4)/2} \int_{B(0,cl/\sqrt{\epsilon})} \frac{dx}{(1 + |x|^2)^{N-2}}.$$

Since $\lim_{\epsilon\to 0}(cl/\sqrt{\epsilon}) = \infty$, we obtain $I = O(\epsilon^{-(N-4)/2})$ as $\epsilon \to 0$.

Second, we investigate the case $N = 4$. We have

$$I = C \int_0^{cl} \frac{r^3}{(\epsilon + r^2)^2} dr.$$

To investigate how l affects the conclusion, we evaluate the integral on the right side by direct calculation. We start by getting the lower bound. It follows that

$$\begin{aligned} \int_0^{cl} \frac{r^3}{(\epsilon + r^2)^2} dr &> \int_0^{cl} \frac{r^3}{(\sqrt{\epsilon} + r)^4} dr \\ &= \int_0^{cl} \frac{((\sqrt{\epsilon} + r) - \sqrt{\epsilon})^3}{(\sqrt{\epsilon} + r)^4} dr \\ &= \sum_{i=0}^{3} (-1)^{3-i} \binom{3}{i} I_i, \end{aligned}$$

where

$$I_i = \epsilon^{(3-i)/2} \int_0^{cl} (r + \sqrt{\epsilon})^{i-4} dr$$

for $i = 0, 1, 2, 3$. For $i = 0, 1, 2$, we have

$$\begin{aligned} I_i &= \epsilon^{(3-i)/2} \Big[\frac{1}{i-3} (r + \sqrt{\epsilon})^{i-3} \Big]_0^{cl} \\ &= \frac{\epsilon^{(3-i)/2}}{i-3} \Big(\Big(cl + \sqrt{\epsilon} \Big)^{i-3} - \epsilon^{(i-3)/2} \Big) \\ &= \frac{1}{i-3} \Big(\Big(\frac{\sqrt{\epsilon}}{cl + \sqrt{\epsilon}} \Big)^{3-i} - 1 \Big) = O(1) \end{aligned}$$

as $\epsilon \to 0$. By contrast, it follows that

$$\begin{aligned} I_3 &= \int_0^{cl} \frac{dr}{r + \sqrt{\epsilon}} = \Big[\log(r + \sqrt{\epsilon}) \Big]_0^{cl} \\ &= \log(cl + \sqrt{\epsilon}) - \log \sqrt{\epsilon} \\ &= \log \Big(c + \frac{\sqrt{\epsilon}}{l} \Big) - \log \Big(\frac{\sqrt{\epsilon}}{l} \Big) = O\Big(\Big| \log \Big(\frac{\sqrt{\epsilon}}{l} \Big) \Big| \Big). \end{aligned}$$

Next, we have the upper bound as follows:

$$
\begin{aligned}
\int_0^{cl} \frac{r^3}{(\epsilon+r^2)^2} dr < \int_0^{cl} \frac{(\epsilon+r^2)^{3/2}}{(\epsilon+r^2)^2} dr &= \int_0^{cl} \frac{1}{\sqrt{\epsilon+r^2}} dr \\
&= \Big[\log\Big(r+\sqrt{r^2+\epsilon}\Big)\Big]_0^{cl} \\
&= \log\Big(cl+\sqrt{c^2l^2+\epsilon}\Big) - \log\sqrt{\epsilon} \\
&= \log\Big(c+\sqrt{c^2+\frac{\epsilon}{l^2}}\Big) - \log\Big(\frac{\sqrt{\epsilon}}{l}\Big) \\
&= O\Big(\Big|\log\Big(\frac{\sqrt{\epsilon}}{l}\Big)\Big|\Big).
\end{aligned}
$$

Hence we have $I = O(|\log(\sqrt{\epsilon}/l)|)$.

Finally, we investigate the case $N = 3$. First, we have

$$
I < \int_{B(0,cl)} \frac{dx}{|x|^2} = Cl.
$$

Next, since $\lim_{\epsilon\to 0}(\sqrt{\epsilon}/l) = 0$, it follows that

$$
\begin{aligned}
I &\geq \int_{B(0,cl)\setminus B(0,c\sqrt{\epsilon})} \frac{dx}{\epsilon+|x|^2} \geq \int_{B(0,cl)\setminus B(0,c\sqrt{\epsilon})} \frac{dx}{C|x|^2} \\
&= C' \int_{c\sqrt{\epsilon}}^{cl} dr \geq C'' l.
\end{aligned}
$$

Thus we have $I = O(l)$. We complete the proof. $\square$

Lemma 3.3. *Let $0 < \gamma < 1/2$. Set $l = l(\epsilon) = \epsilon^{\gamma}$. Then*

$$
\int_\Omega \Psi_0 u_{\epsilon,l}^2 dx = \begin{cases} O(\epsilon^{\beta\gamma-(N-4)/2}) & N \geq 5, \\ O(\epsilon^{\beta\gamma}|\log\epsilon|) & N = 4, \\ O(\epsilon^{(\beta+1)\gamma}) & N = 3, \end{cases} \tag{3.4}
$$

as $\epsilon \to 0$.

Proof. We investigate

$$
I = \frac{1}{a}\int_\Omega \Psi_0 u_{\epsilon,l}^2 dx = \int_{B(x_l,l)} \frac{|x-x_0|^\beta \xi_l(x)^2}{\big(\epsilon+|x-x_l|^2\big)^{N-2}} dx.
$$

We have

$$
I \leq (2l)^\beta \int_{B(0,l)} \frac{1}{(\epsilon+|x|^2)^{N-2}} dx,
$$

and

$$
\begin{aligned}
I &\geq \int_{B(x_l,l/2)} \frac{|x-x_0|^\beta}{(\epsilon+|x-x_l|^2)^{N-2}} dx = \int_{B(0,l/2)} \frac{|x-x_0+x_l|^\beta}{(\epsilon+|x|^2)^{N-2}} dx \\
&\geq \Big(\frac{l}{2}\Big)^\beta \int_{B(0,l/2)} \frac{1}{(\epsilon+|x|^2)^{N-2}} dx.
\end{aligned}
$$

By Lemma 3.2, we obtain

$$\int_\Omega \Psi_0 u_{\epsilon,l}^2 dx = \begin{cases} O(l^\beta \epsilon^{-(N-4)/2}) & N \geq 5, \\ O(l^\beta |\log(\sqrt{\epsilon}/l)|) & N = 4, \\ O(l^{\beta+1}) & N = 3, \end{cases} \tag{3.5}$$

as $\epsilon \to 0$. Letting $l = \epsilon^\gamma$, we have (3.4). □

Corollary 3.4. *Let $k > 0$. Set $l = l(\epsilon) = |\log \epsilon|^{-k}$. Then*

$$\int_\Omega \Psi_0 u_{\epsilon,l}^2 dx = \begin{cases} O(|\log \epsilon|^{-\beta k} \epsilon^{-(N-4)/2}) & N \geq 5, \\ O(|\log \epsilon|^{1-\beta k}) & N = 4, \\ O(|\log \epsilon|^{-(\beta+1)k}) & N = 3, \end{cases} \tag{3.6}$$

as $\epsilon \to 0$.

Proof. Set $l = |\log \epsilon|^{-k}$ in (3.5). The conclusion immediately follows for the case $N \geq 5$ and $N = 3$. For the case $N = 4$, we see

$$l^\beta |\log(\sqrt{\epsilon}/l)| = |\log \epsilon|^{-\beta k} |\log(\sqrt{\epsilon} |\log \epsilon|^k)|.$$

For small $\epsilon > 0$, it follows that $\sqrt{\epsilon} \leq \sqrt{\epsilon}|\log \epsilon|^k \leq \sqrt[4]{\epsilon}$. Then, we have

$$|\log \epsilon|^{-\beta k} |\log(\sqrt{\epsilon}|\log \epsilon|^k)| = O(|\log \epsilon|^{1-\beta k}),$$

which completes the proof. □

4. Proof of Theorem 1.1

By Proposition 2.2 and Proposition 2.8, it suffice to prove (2.4) for $c > 0$ defined by (2.1).

By elementary calculations, we have

$$\begin{aligned} c &\leq \sup_{t>0} I(tv_{\epsilon,l}) = \sup_{t>0} \Big(\frac{t^2}{2} \Big(\|Dv_{\epsilon,l}\|^2_{L^2(\Omega)} - \lambda \int_\Omega \Psi v_{\epsilon,l}^2 dx \Big) - \frac{t^{2^*}}{2^*} \Big) \\ &= \frac{1}{N} \Big(\|Dv_{\epsilon,l}\|^2_{L^2(\Omega)} - \lambda \int_\Omega \Psi v_{\epsilon,l}^2 dx \Big)^{N/2} \xrightarrow{\epsilon \to 0} \frac{1}{N} S^{N/2}. \end{aligned}$$

We define

$$A(\epsilon) = \|Dv_{\epsilon,l}\|^2_{L^2(\Omega)} - \lambda \int_\Omega \Psi v_{\epsilon,l}^2 dx - S.$$

We show that there exists $\epsilon > 0$ such that $A(\epsilon) < 0$ to completes the proof. We write

$$I = \int_\Omega \Psi v_{\epsilon,l}^2 dx, \quad I_0 = \int_\Omega \Psi_0 v_{\epsilon,l}^2 dx.$$

Assume that $\lim_{\epsilon \to 0}(\sqrt{\epsilon}/l) = 0$. By Lemma 3.1, it follows that

$$\begin{aligned} A(\epsilon) &= \frac{\|Du_{\epsilon,l}\|^2_{L^2(\Omega)} - \lambda I}{\||x|^{\alpha/2^*} u_{\epsilon,l}\|^2_{L^{2^*}(\Omega)}} - S \\ &\leq \frac{\|DU\|^2_{L^2(\mathbb{R}^N)} \epsilon^{-(N-2)/2} + C' l^{-(N-2)} - \lambda I_0}{(1-2l)^{2\alpha/2^*} (\|U\|^{2^*}_{L^{2^*}(\mathbb{R}^N)} \epsilon^{-N/2} - Cl^{-N})^{2/2^*}} - S \\ &= \frac{S + C' l^{-(N-2)} \epsilon^{(N-2)/2} - C'' I_0 \epsilon^{(N-2)/2}}{(1-2l)^{2\alpha/2^*} (1 - Cl^{-N} \epsilon^{N/2})^{2/2^*}} - S. \end{aligned}$$

We set

$$B(\epsilon) = S + C'l^{-(N-2)}\epsilon^{(N-2)/2} - C''I_0\epsilon^{(N-2)/2} - S(1-2l)^{2\alpha/2^*}\big(1 - Cl^{-N}\epsilon^{N/2}\big)^{2/2^*}.$$

The condition $A(\epsilon) < 0$ is equivalent to $B(\epsilon) < 0$. We have

$$\begin{aligned} B(\epsilon) &\le S - S(1-2l)^{2\alpha/2^*}\big(1 - Cl^{-N}\epsilon^{N/2}\big) \\ &\quad + C'l^{-(N-2)}\epsilon^{(N-2)/2} - C''I_0\epsilon^{(N-2)/2} \\ &\le \big(S - S(1-2l)^{2\alpha/2^*}\big) \\ &\quad + \Big(Cl^{-N}\epsilon^{N/2} + C'l^{-(N-2)}\epsilon^{(N-2)/2} - C''I_0\epsilon^{(N-2)/2}\Big). \end{aligned}$$

Note that

$$\lim_{\epsilon\to 0} \frac{l^{-N}\epsilon^{N/2}}{l^{-(N-2)}\epsilon^{(N-2)/2}} = 0.$$

Hereinafter, we divide the proof into two cases; (i) $N \ge 5$ and (ii) $N = 4$.

(i) Let $N \ge 5$, $0 < \gamma < 1/2$ and $l = l(\epsilon) = \epsilon^\gamma$. By Lemma 3.3, we have

$$I_0\epsilon^{(N-2)/2} = O(\epsilon^{\beta\gamma+1})$$

as $\epsilon \to 0$. We show that there exists $0 < \gamma < 1/2$ such that

$$(N-2)\Big(\frac{1}{2} - \gamma\Big) > \beta\gamma + 1. \tag{4.1}$$

This inequality is equivalent to $\gamma < (N-4)/2(\beta + N - 2)$. Thus the condition we are now considering is equivalent to

$$\frac{N-4}{2(\beta+N-2)} > 0,$$

which is always true since $\beta > 0$ and $N \ge 5$. Fix such $0 < \gamma < 1/2$ that satisfies (4.1). Thus we obtain

$$\lim_{\epsilon\to 0} \frac{\epsilon^{(N-2)(1/2-\gamma)}}{\epsilon^{\beta\gamma+1}} = 0.$$

Therefore we admit the existence of $\epsilon > 0$ such that

$$Cl^{-N}\epsilon^{N/2} + C'l^{-(N-2)}\epsilon^{(N-2)/2} - C''I_0\epsilon^{(N-2)/2} < 0.$$

Fix such $\epsilon > 0$ and take $\alpha > 0$ so small that $B(\epsilon) < 0$ to obtain the conclusion.

(ii) Let $N = 4$. By Corollary 3.4, We have

$$I_0\epsilon^{(N-2)/2} = O(\epsilon|\log\epsilon|^{1-\beta k}).$$

We see that there exists $k > 0$ such that $1 - \beta k > 2k$, which is equivalent to $k < 1/(2+\beta)$. Fix such $k > 0$ to obtain

$$\lim_{\epsilon\to 0} \frac{\epsilon|\log\epsilon|^{2k}}{\epsilon|\log\epsilon|^{1-\beta k}} = 0.$$

The rest of the argument is the same as (i). We complete the proof.

5. Appendix: Convergence of integrals with critical growth

Lemma 5.1. *Let $v,\psi \in H_0^1(\Omega)$. Let $\{v_k\}_{k=0}^\infty$ be a bounded sequence in $H_0^1(\Omega)$. Assume that $v_k \to v$ a.e. in Ω. Then, we have*

$$\int_\Omega |x|^\alpha (v_k)_+^{2^*-1}\psi dx \to \int_\Omega |x|^\alpha v_+^{2^*-1}\psi dx \tag{5.1}$$

as $k \to \infty$.

Proof. Let $\epsilon > 0$. We set

$$W_{\epsilon,k} = \Big(||x|^\alpha (v_k)_+^{2^*-1}\psi - |x|^\alpha v_+^{2^*-1}\psi| - \epsilon |x|^\alpha (v_k)_+^{2^*}\Big)_+.$$

By the Young inequality, there exists $C > 0$ such that

$$\big||s|_+^{2^*-1}t\big| \le \epsilon s_+^{2^*} + C|t|^{2^*}$$

for $s,t \in \mathbb{R}$. Thus we have

$$|W_{\epsilon,k}| \le \epsilon |x|^\alpha v_+^{2^*} + 2C|x|^\alpha |\psi|^{2^*} \le \epsilon v_+^{2^*} + 2C|\psi|^{2^*}.$$

The right side of above inequality is integrable. Since $v_k \to v$ a.e. in Ω, it follows that $W_{\epsilon,k} \to 0$ a.e. in Ω. Thus we have

$$\lim_{k\to\infty} \int_\Omega W_{\epsilon,k} dx = 0.$$

By the definition of $W_{\epsilon,k}$, we have

$$\begin{aligned}
\int_\Omega ||x|^\alpha (v_k)_+^{2^*-1}\psi - |x|^\alpha v_+^{2^*-1}\psi| dx &\le \int_\Omega W_{\epsilon,k} dx + \epsilon \int_\Omega |x|^\alpha (v_k)_+^{2^*} dx \\
&\le \int_\Omega W_{\epsilon,k} dx + \epsilon \int_\Omega (v_k)_+^{2^*} dx.
\end{aligned}$$

Since $\{v_k\}$ is a bounded sequence of $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, we have $\int_\Omega (v_k)_+^{2^*} dx \le C$. Therefore,

$$\limsup_{k\to\infty} \int_\Omega ||x|^\alpha (v_k)_+^{2^*-1}\psi - |x|^\alpha v_+^{2^*-1}\psi| dx \le C\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we obtain (5.1). $\square$

6. Appendix: Proof of Corollary 1.2

We use notation of elementary geometries. Let X, Y, Z be points of the Euclidean space $\mathbb{R}^N$. We write $\overline{XY}$ as the length of the segment XY, $\angle XYZ$ as the angle of XYZ and $\triangle XYZ$ as the triangle of XYZ.

Corollary 1.2 is a direct conclusion of Theorem 1.1 and the following lemma.

Lemma 6.1. *Let $x_0 \in \partial\Omega$ and $B \subset \Omega$ be an open ball whose radius is $0 < r_0 < 1/2$ and where ∂B come in contact with $\partial\Omega$ at x_0. Let $\beta = 2\beta_0$. Then, there exists $a > 0$ such that $\Psi(P) \ge \Psi_0(P)$ for any $P \in B$.*

Proof. Let T to denote the point x_0. Let O and O' be the center of Ω and B, respectively. Let $P \in B$. If P is on the segment OT, just taking $\beta \ge \beta_0$ and $0 < a < 1$ will do. Hereinafter we assume P is not on the segment OT. We argue on the plane containing O, O', T and P (Figure 1). Let Q and R be the intersection point of the the half line OP with ∂B and $\partial\Omega$, respectively. Let $l = \overline{QT}$ and $k = \overline{QR}$. Let $\theta = \angle TO'Q$. Then, we see that $\overline{PT} > \overline{QT}$ since $\angle PQT$ is an obtuse angle. We can take a point S on the segment PT so that $\overline{ST} = \overline{QT}$. Let $x = \overline{PS}$ and $y = \overline{PQ}$. Let $\rho = \angle PQS$, $\sigma = \angle PSQ$ and $\tau = \angle QPS$ (Figure 2).

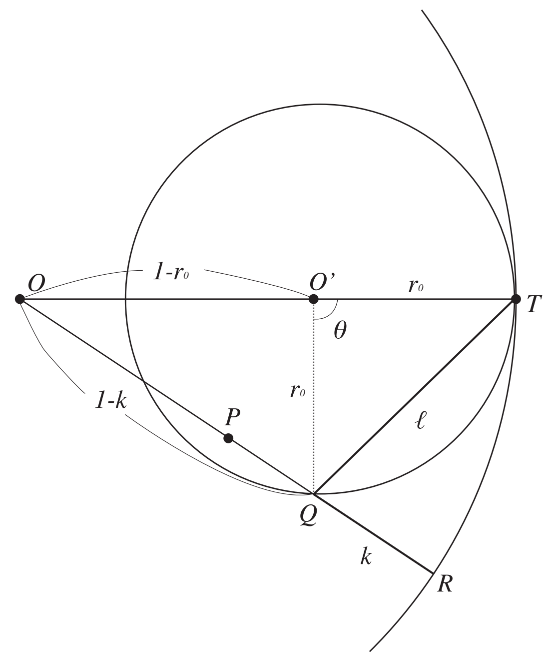

Figure 1. The plane containing O, O', T and P.

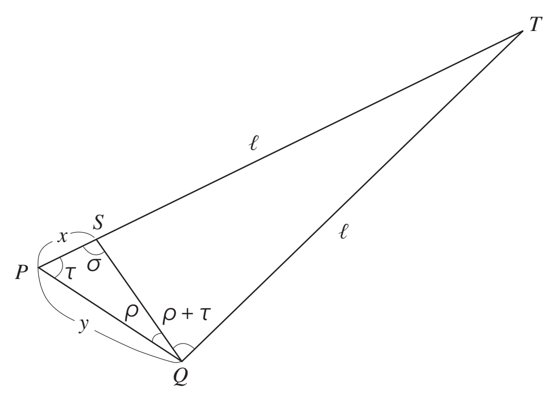

Figure 2. Focusing on $\triangle PQT$.

First, we prove that if we set $\beta = 2\beta_0$, there exists $a > 0$ such that $k^{\beta_0} > al^{\beta}$ independently on Q. Considering $\triangle O'TQ$ and $\triangle OO'Q$, we have $l = 2r_0 \sin(\theta/2)$ and

$$(1 - r_0)^2 + r_0^2 + 2r_0(1 - r_0)\cos\theta = (1 - k)^2,$$

respectively. By the formula $\cos\theta = 1 - 2\sin^2(\theta/2)$, we have

$$l^2 = \frac{r_0}{1-r_0}(1-(1-k)^2).$$

Therefore

$$\begin{aligned} k^{2\beta_0} - a^2 l^{2\beta} &= k^{2\beta_0} - a^2\Big(\frac{r_0}{1-r_0}\Big)^{\beta} k^{\beta}(2-k)^{\beta} \\ &> k^{2\beta_0} - a^2 2^{\beta}\Big(\frac{r_0}{1-r_0}\Big)^{\beta} k^{\beta}. \end{aligned}$$

We set $\beta = 2\beta_0$ and take $a > 0$ so small that

$$1 - a^2 2^{2\beta_0}\Big(\frac{r_0}{1-r_0}\Big)^{2\beta_0} > 0.$$

Then, we have $k^{\beta_0} > al^{\beta}$ independently on Q as desired.

Next, we prove that $x < y$. Since $\angle SQT = \angle QST = \rho + \tau$, by $\triangle PQT$, we have $2\rho + 2\tau < \pi$. Combining this with $\rho + \sigma + \tau = \pi$, we have $\sigma > \pi/2 > \rho$. Thus we have $x < y$.

Finally, we prove that there exists $a > 0$ such that $(y+k)^{\beta_0} > a(x+l)^{2\beta_0}$ independently on P. Since $k^{\beta_0} > al^{2\beta_0}$ and $x < y$, it follows that

$$\begin{aligned} (y+k)^{\beta_0} - a(x+l)^{2\beta_0} &= (y+k)^{\beta_0} - \big(a^{1/2\beta_0}x + a^{1/2\beta_0}l\big)^{2\beta_0} \\ &> (y+k)^{\beta_0} - \big(a^{1/2\beta_0}y + \sqrt{k}\big)^{2\beta_0}. \end{aligned}$$

Observing $0 < k < 1$ and $0 < y < 2r_0$, we have

$$\begin{aligned} (y+k) - \big(a^{1/2\beta_0}y + \sqrt{k}\big)^2 &= y\big(1 - 2a^{1/2\beta_0}\sqrt{k} - a^{1/\beta_0}y\big) \\ &> y(1 - 2a^{1/2\beta_0} - 2r_0 a^{1/\beta_0}). \end{aligned}$$

If we need, we can again take $a > 0$ so small that the right side above is positive. Therefore we have $(y+k)^{\beta_0} > a(x+l)^{2\beta_0}$ independently on P, which completes the proof. □

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References

[1] Antonio Ambrosetti and Paul H. Rabinowitz; *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381. MR 0370183 (51 #6412)

[2] Marino Badiale and Enrico Serra; *Multiplicity results for the supercritical Hénon equation*, Adv. Nonlinear Stud. **4** (2004), no. 4, 453–467. MR 2100908

[3] Haïm Brézis and Elliott Lieb; *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), no. 3, 486–490. MR 699419 (84e:28003)

[4] Haïm Brézis and Louis Nirenberg; *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477. MR 709644 (84h:35059)

[5] Jaeyoung Byeon and Zhi-Qiang Wang; *On the Hénon equation: asymptotic profile of ground states. II*, J. Differential Equations **216** (2005), no. 1, 78–108. MR 2158918

[6] Jaeyoung Byeon and Zhi-Qiang Wang; *On the Hénon equation: asymptotic profile of ground states. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), no. 6, 803–828. MR 2271694

[7] Daomin Cao and Shuangjie Peng; *The asymptotic behaviour of the ground state solutions for Hénon equation*, J. Math. Anal. Appl. **278** (2003), no. 1, 1–17. MR 1963460

[8] Michel Hénon; *Numerical experiments on the stability of spherical stellar systems*, Astronomy and astrophysics **24** (1973), 229.

[9] Norimichi Hirano, *Existence of positive solutions for the Hénon equation involving critical Sobolev terms*, J. Differential Equations **247** (2009), no. 5, 1311–1333. MR 2541412

[10] Wei Long and Jianfu Yang; *Existence for critical Hénon-type equations*, Differential Integral Equations **25** (2012), no. 5-6, 567–578. MR 2951742

[11] Wei Ming Ni; *A nonlinear Dirichlet problem on the unit ball and its applications*, Indiana Univ. Math. J. **31** (1982), no. 6, 801–807. MR 674869

[12] Shuang-jie Peng; *Multiple boundary concentrating solutions to Dirichlet problem of Hénon equation*, Acta Math. Appl. Sin. Engl. Ser. **22** (2006), no. 1, 137–162. MR 2191725

[13] Simone Secchi; *The Brezis-Nirenberg problem for the Hénon equation: ground state solutions*, Adv. Nonlinear Stud. **12** (2012), no. 2, 383–394. MR 2951722

[14] Enrico Serra; *Non radial positive solutions for the Hénon equation with critical growth*, Calc. Var. Partial Differential Equations **23** (2005), no. 3, 301–326. MR 2142066

[15] Didier Smets, Michel Willem, and Jiabao Su; *Non-radial ground states for the Hénon equation*, Commun. Contemp. Math. **4** (2002), no. 3, 467–480. MR 1918755

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