METRIZATION THEOREMS

THESIS

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PREFACE

The motivation for writing this thesis with the emphasis on metric spaces and metrization of topological spaces arose from the pleasant properties that allow us to relate the abstract realm of topological spaces to our very familiar concrete environment in which exists the concept of distance, size, and order between objects.

With the general reader in mind, this paper is written and developed to be self-contained. Therefore, some of the material included provides a background for the development of ideas and proofs. This work contains a sequence of theorems dealing with metrization of topological spaces, which serve to prove Urysohn's Metrization Theorem.

In 1924, shortly before his drowning at the age of twenty-six, the Russian mathematician Paul Urysohn first stated and proved this metrization theorem, concurrently with Alexandroff. The work of Urysohn andAlexandroff opened a new area for research in topology that gained the interest of many mathematicians worldwide.

I wish to thank my adviser Dr. Singh for his patience and unconditional support and encouragement during my writing of this thesis as well as Dr. Gu and Dr. McCabe for their suggestions and help in preparing my paper. I am grateful to my committee members and extend my thanks to all the members of the Department of Mathematics at Southwest Texas State University for inspiring me and imparting their knowledge during my studies of mathematics.

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ABSTRACT

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The goal of this study is to prove The Urysohn Metrization Theorem. This paper represents an introduction to topological spaces with the focus on metric spaces. We provide a background in set theory and function theory first, then proceed introducing the distance function and looking at some examples of metric spaces, especially the Euclidean n-space. The overview of topological spaces in general leads us to product spaces. Our study of connectedness and separability of topological spaces paves the way to the separation by continuous functions. In conclusion, the proofs of Urysohn's Lemma and The Tietze Extension Theorem enable us to prove The Urysohn Metrization Theorem.

Chapter 1

BACKGROUND

This chapter begins with a table containing the explanations of symbols used throughout this paper. It furthermore provides a collection of definitions and propositions selected from topics of set theory and some basics of function theory.

1.1 Table of Symbols

SYMBOL:	DEFINITION:
$\mathbb{Z}$	set of integers
$\mathbb{R}$	set of real numbers
$\mathbb{R}^2$	Cartesian plane
$\mathbb{R}^n$	Euclidean space of n-dimensions
$\mathbb{H}$	Hilbert space
θ	origin of n-dimensional Euclidean space
(x, y) , $(x_1,x_2, \ldots , x_n)$	ordered pair, ordered n-tuple

SYMBOL:	DEFINITION:
A^C	complement of set A
A'	set of all limit points of a set A
$\overline{A}$	closure of A
$\in$	an element of, belongs to
$\notin$	not an element of
$\sum$	summation
$\prod$	product of sets
(a,b)	open interval, segment, subset of the real number line
$[a,b]$	closed interval, subset of the real number line
$\{a,b,c,\ldots n\}$	set or collection of elements
$\{a\}$	singleton set
$\{a_n\}$	sequence
$\emptyset$	empty set
$\neq$	not equal to
$<, \leq, >, \geq$	inequality symbols
$\exists$	there exists
$:$	given that
$\ni$	such that
$\subset$	subset of a set
$\bigcup, \cup$	union of sets
$\bigcap, \cap$	intersection of sets

SYMBOL:	DEFINITION:
$f : X \to Y$	function from a set X to a set Y
f^{-1}	inverse of a function f
$\forall$	for every
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if
$\therefore$	conclusion, therefore
$f \circ g$	function composition
I where $I \subset \mathbb{Z}^+$	indexing set

1.2 Set Theory

Definition 1 *A **set** is a collection of elements. A is a **subset** of B,* $A \subset B$*, if every element of A is in B.* $A = B$ *if and only if* $A \subset B$ *and* $B \subset A$.

Definition 2 *Then the **union** of sets A and B is a set consisting of all elements x which belong to at least one of the sets A and B:* $A \cup B = \{x : x \in A \text{ or } x \in B\}$.

Definition 3 *Suppose A,B are sets. The **intersection** of A and B is a set consisting of all elements x which belong to both A and B:* $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

Definition 4 *Suppose A,B are sets. A and B are **disjoint** if* $A \cap B = \emptyset$.

Definition 5 *Suppose A and B are sets. The **set difference*** $B \backslash A$ *is the set of all points of B which do not belong to A:* $B \backslash A = \{x : x \in B \text{ and } x \notin A\}$.

Definition 6 *The **complement** of* $A \subset X$ *is the set* $A^C = \{x : x \in X \text{ and } x \notin A\}$.

Definition 7 *A number u is an **upper bound** for a set A of real numbers provided that* $a \leq u$ *for all* $a \in A$*. If there is a smallest upper bound* u_0 *for A, an upper bound* u_0 *less than all other upper bounds for A, then* u_0 *is called the **least upper bound** of A, denoted by lubA. A number l is a **lower bound** for a set A of real numbers provided that* $l \leq a$ *for all* $a \in A$*. If there is a largest lower bound* l_0 *for A, that is, a lower bound greater than all other lower bounds for A, then* l_0 *is called the **greatest lower bound** of A, denoted by glbA.*

Proposition 8 *De Morgan's Laws: Let X be a set and let* $\{A_i\}_{i \in I}$ *be a family of subsets of X. Then,* $\left(\bigcap_{i \in I} A_i\right)^C = \bigcup_{i \in I} A_i^C$ *and* $\left(\bigcup_{i \in I} A_i\right)^C = \bigcap_{i \in I} A_i^C$.

1.3 Functions

Definition 9 *Let X and Y be two sets. A function f from X to Y is a rule which assigns to each member $x \in X$ a unique member $y = f(x) \in Y$, and f is denoted by $f : X \to Y$.*

Definition 10 *The **Cartesian plane** $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.*

Definition 11 *The **Cartesian product** $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$. Infinite products are defined in Chapter 4.*

Definition 12 *Suppose X and Y are sets, $A \subset X$ and $B \subset Y$, and $f : X \to Y$. Then,*

1. $f(A) = \{y \in Y : y = f(x) \text{ for some } x \in A\}$ is the **image** of A under f.
2. $f^{-1}(B) = \{x \in X : f(x) \in B\}$ is the **inverse image** of B under f.
3. f is **one-to-one**, if for $x_1, x_2 \in X$ and $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.
4. f is **onto** if $f(X) = Y$.

Definition 13 *Suppose A_i is a set, $i \in I$ where I is an indexing set. Then,*

1. $\bigcup_{i \in I} A_i = \{x : x \in A_i, \quad \text{for at least one } i \in I\}$.
2. $\bigcap_{i \in I} A_i = \{x : x \in A_i, \ \forall\, i \in I\}$.

Proposition 14 *Suppose $f : X \to Y$, A_1, $A_2 \subset X$, and $B_1, B_2 \subset Y$. Then,*

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
2. $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.
3. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
4. $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

Proposition 15 *Let $f : X \to Y$ be a function, and let $\{A_i\}_{i\in I}$ be a family of subsets of X, and $\{B_j\}_{j\in J}$ be a family of subsets of Y. Then,*

1. $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$.
2. $f\left(\bigcap_{i\in I} A_i\right) \subset \bigcap_{i\in I} f(A_i)$.
3. $f^{-1}\left(\bigcap_{j\in J} B_j\right) = \bigcap_{j\in J} f^{-1}(B_j)$
4. $f^{-1}\left(\bigcup_{j\in J} B_j\right) = \bigcup_{j\in J} f^{-1}(B_j)$.

Proposition 16 *Let $f : X \to Y$ be a function. Let $A \subset X$ and $B \subset Y$. Then,*

1. $A \subset f^{-1}(f(A))$.
2. $f(f^{-1}(B)) \subset B$.
3. *if f is one-to-one, then* $f^{-1}(f(A)) = A$.
4. *if f is onto, then* $f(f^{-1}(B)) = B$.

Proposition 17 *If f is one-to-one and onto, then f^{-1} is one-to-one and onto.*

Definition 18 *Let $f : X \to Y$ and $g : Y \to Z$. The **composite function** $g \circ f : X \to Z$ is defined by $(g \circ f)(x) = g(f(x))$ for $x \in X$.*

Definition 19 *Let $a = (a_1, a_2) \in \mathbb{R}^2$ and let r be a positive number. The **open ball** $B(a, r)$ with center a and radius r is the set $B(a, r) = \{x = (x_1, x_2) \in \mathbb{R}^2 : d(a, x) < r\}$ where the distance between a and x is defined by $d(a, x) = \sqrt{(a_1 - x_1)^2 + (a_2 - x_2)^2}$.*

Definition 20 *A set $O \subset \mathbb{R}^2$ is **open** if it is the union of some family of open balls.*

Definition 21 *A set $A \subset \mathbb{R}^2$ is **closed** if its complement $\mathbb{R}^2 \backslash A$ is open.*

Chapter 2

METRIC SPACES

In this section we define the distance function and the metric space, then focus on the Euclidean space showing that the Euclidean Metric is indeed a metric. We also include the definition for the discrete metric, among other definitions, and a proof that shows that the conditions of the definition of the metric are safisfied for the discrete metric.

Definition 22 *Let X be a set and $d : X \times X \to \mathbb{R}$ be a function such that for all $x, y, z \in X$, $d(x, y) \geq 0$, and the conditions*

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$

hold, then d is a **metric** or **distance function** on $X \times X$ and $d(x, y)$ is the distance from x to y.

Definition 23 *The set (X, d) denotes a **metric space**.*

Definition 24 *Euclidean n-space* $(\mathbb{R}^n, d)$ *or the* ***usual topology*** *for* $\mathbb{R}^n$ *is the set*

$\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n\}$ with the metric d defined for each n by

$$d(x,y) = \sqrt{\left(\sum_{i=1}^{n} (x_i - y_i)^2\right)}, \quad x = (x_1, x_2, x_3, \ldots, x_n),\ y = (y_1, y_2, y_3, \ldots, y_n) \in \mathbb{R}^n.$$

Proposition 25 *The Euclidean Metric on* $\mathbb{R}^n$ *is indeed a metric.*

PROOF:

$$a = (a_1, a_2, \ldots, a_n), \quad b = (b_1, b_2, \ldots, b_n), \quad d(a,b) = \sqrt{\left(\sum_{i=1}^{n} (a_i - b_i)^2\right)}.$$

1. $d(a,b) = 0 \quad \Rightarrow \quad (a_i - b_i)^2 = 0 \quad \forall i \in \mathbb{Z}^+ \quad \Rightarrow a_i = b_i \quad \forall i \in \mathbb{Z}^+$

   $a_i = b_i \quad \forall i \in \mathbb{Z}^+ \quad \Rightarrow \quad d(a,b) = 0$

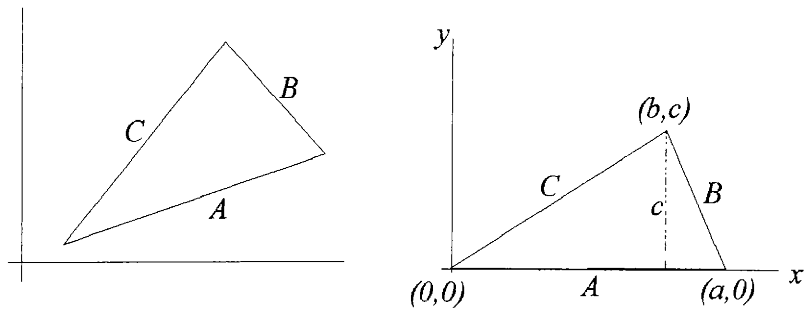
   $\therefore \quad d(a,b) = 0 \iff a_i = b_i \quad \forall i \in \mathbb{Z}^+.$

2. $(a_i - b_i)^2 = (b_i - a_i)^2 \quad \Rightarrow \quad \sqrt{\left(\sum_{i=1}^{n} (a_i - b_i)^2\right)} = \sqrt{\left(\sum_{i=1}^{n} (b_i - a_i)^2\right)}$

   $\therefore \quad d(a,b) = d(b,a)$

3. To show that $d(x,z) \leq d(x,y) + d(y,z)$ for $x, y, z \in \mathbb{R}^n$, it suffices to study the three sides of the triangle in $\mathbb{R}^2$. Consider a triangle in the plane. Place the triangle with one of its vertices at the origin and, if needed, rotate the triangle such that one of its sides coincides with the positive side of the x-axis.

$$\begin{aligned} A - b &\leq B \\ b &\leq C \end{aligned} \quad \Rightarrow A - b + b \leq B + C \quad \Rightarrow A \leq B + C$$



Triangle Inequality

Choosing a set of arbitrary three points $x, y, z \in \mathbb{R}^n$, we can construct a triangle with each of x, y, z as one of the vertices. Three points determine a plane, thus the triangle resides in a plane $\mathbb{R}^k$ whose properties correspond to $\mathbb{R}^2$. Then, our triangle with the vertices x, y, z existing in $\mathbb{R}^k$ is equivalent to the triangle explored in the above example for $\mathbb{R}^2$. By induction then, the triangle inequality holds for $\mathbb{R}^n$.

$\therefore$ Since $d(x,z) \leq d(x,y) + d(y,z)$ for $x, y, z \in \mathbb{R}^n$ holds, the Euclidean Metric is indeed a metric.

The triangle inequality can also be proved using the Minkowski Inequality which can be derived from the Cauchy-Schwarz inequality. One version of the proof is found on page 57 [1] and a different version is utilizing La Grange multipliers [11].

Definition 26 *Let X be a set and define $d : X \times X \to \mathbb{R}$ by* $d(x,y) = \begin{cases} 0 & \textit{if } x = y \\ 1 & \textit{if } x \neq y \end{cases}$. *Then d is called the "**discrete metric.**"*

Proposition 27 *The discrete metric d on $X \times X$ as defined above is indeed a metric.*

PROOF: Suppose x,y,$z \in X$.

Need to show: 1) $d(x,y)=0 \quad \Leftrightarrow x=y$

2) $d(x,y)=d(y,x)$

3) $d(x,z)\le d(x,y)+d(y,z)$.

1) Suppose $d(x,y)=0$ and assume $x\neq y$.

$x\neq y \quad \Rightarrow \quad d(x,y)=1.$ But, by hypothesis, $d(x,y)=0$.

$d(x,y)=0 \quad \Rightarrow \quad x=y$ By way of contraposition.

$\therefore \quad d(x,y)=0 \Leftrightarrow x=y.$

2) (*i*) Suppose $x=y$.

$x=y \quad \Rightarrow \quad d(x,y)=0$ and since $(x=y)\equiv(y=x)$,

$y=x \quad \Rightarrow \quad d(y,x)=0$

$\Rightarrow \quad d(x,y)=d(y,x)=0$

(*ii*) Suppose $x\neq y$.

$x\neq y \quad \Rightarrow \quad d(x,y)=1$ and since $(x\neq y)\equiv(y\neq x)$,

$y\neq x \quad \Rightarrow \quad d(y,x)=1$

$\Rightarrow \quad d(x,y)=d(y,x)=1$

$\therefore \quad d(x,y)=d(y,x)$ holds.

3) Cases to examine:

(*i*) $x=y=z$

(*ii*) $x\neq y$, $y\neq z$, and $x\neq z$.

(*iii*) $x=y$, but $x\neq z$.

The case where $x=z$, but $y\neq z$ is similar and we omit the proof.

(*i*) $x=y=z \quad \Rightarrow d(x,y)=d(y,z)=d(x,z)=0$

$\Rightarrow d(x,z)=d(x,y)+d(y,z)$

(ii) Suppose $\quad x \neq y,\ y \neq z,$ and $x \neq z.$

$$
\begin{aligned}
x \neq z \quad & \Rightarrow d(x,z) = 1 \\
x \neq y, \text{ and } y \neq z \quad & \Rightarrow d(x,y) = 1 \text{ and } d(y,z) = 1 \\
& \Rightarrow d(x,z) < d(x,y) + d(y,z) = 2
\end{aligned}
$$

(iii) Suppose $\quad x = y$ and $x \neq z.$

$$
\begin{aligned}
x \neq z \quad & \Rightarrow d(x,z) = 1 \\
x = y \text{ and } y \neq z \quad & \Rightarrow d(x,y) = 0 \text{ and } d(y,z) = 1 \\
& \Rightarrow d(x,z) = d(x,y) + d(y,z)
\end{aligned}
$$

$\therefore \quad d(x,z) \leq d(x,y) + d(y,z)$

$\therefore$ The function $d : X \times X \to \mathbb{R}$ satisfies the definition for a metric. Therefore, the discrete metric is indeed a metric.

Definition 28 *Let (X,d) be a metric space and $A \subset X$. A point $x \in X$ is a* ***limit point*** *of A if every open set containing x contains a point of A distinct from x. The* ***derived set*** *A' is the set of limit points of A.*

Definition 29 *Let (X,d) be a metric space, $A \subset X$ and $A \neq \emptyset$.. If $\{d(x,y) : x,y \in A\}$ has an upper bound, then A is called a* ***bounded set*** *and $lub\{d(x,y) : x,y \in A\} = D(A)$ is called the* ***diameter*** *of A. The diameter of the empty set is zero. If the set X is bounded, then (X,d) is a* ***bounded metric space****.*

Definition 30 *Let (X,d) be a metric space, $A \subset X$ be non-empty, and $x \in X$. The* ***distance*** *$d(x,A)$ from x to A is defined by $d(x,A) = glb\{d(x,y) : y \in A\}$.*

Definition 31 *Let (X,d) be a metric space, $a \in X$, and $r \in \mathbb{R}^+$. The* ***open ball*** *$B_d(a,r)$ is the set $B_d(a,r) = \{x \in X : d(a,x) < r\}$, with center a and radius r.*

The ***closed ball*** *is the set* $B_d[a,r] = \{x \in X : d(a,x) \leq r\}$*, with center* a *and radius* r*. When dealing with a single metric, we use the notation* $B(a,r)$ *and* $B[a,r]$*.*

Definition 32 *Let* (X,d) *and* (Y,d') *be metric spaces and* $f : X \to Y$ *a function. Then* f *is* ***continuous at a point*** $a \in X$ *if for each* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that if* $x \in X$ *and* $d(x,a) < \delta$*, then* $d'(f(x), f(a)) < \varepsilon$*. A function is* ***continuous*** *if it is continuous at each point of its domain.*

Chapter 3

TOPOLOGICAL SPACES, GENERAL

In this chapter we consider topological spaces from a general perspective. Here, examples of topological spaces with different kinds of topologies are given, definitions are included, and several proofs are worked out.

Definition 33 *Let X be a set and let $\mathcal{T}$ be a family of subsets of X satisfying the following conditions:*

1. The set X and the empty set $\emptyset$ belong to $\mathcal{T}$.

2. The union of any family of members of $\mathcal{T}$ is a member of $\mathcal{T}$.

3. The intersection of any finite family of members of $\mathcal{T}$ is a member of $\mathcal{T}$.

*Then $\mathcal{T}$ is a **topology** for X and the members of $\mathcal{T}$ are **open sets**. The ordered pair $(X, \mathcal{T})$ is called a **topological space**. Often we refer to the topological space $(X, \mathcal{T})$ simply as "the space X."*

Example 34 *A topology induced on a set X by a metric is a* ***metric topology****. A set with the metric topology is a metric space.*

Example 35 *For a set X, the topology generated by the discrete metric (Def. 26) is the* ***discrete topology****. In the discrete topology, every subset of X is open. Thus, the discrete topology consists of all open subsets of X, and it is the largest possible collection of open subsets of X. A set with the discrete topology is a discrete space.*

Example 36 *The topology $\mathcal{T}_T = \{X, \emptyset\}$ is called the* ***trivial*** *or* ***indiscrete topology****.*

Example 37 *The topology $\mathcal{T}'$ consisting of $\emptyset$ and all $O \subset X$ for which $X \backslash O$ is finite is called the* ***finite complement topology****.*

Definition 38 *Let $(X, \mathcal{T})$ be a topological space. A* ***basis*** *$\mathcal{B}$ for $\mathcal{T}$ is a subcollection of elements of $\mathcal{T}$ such that each element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$. The elements of $\mathcal{B}$ are called* ***basic open sets****, and $\mathcal{T}$ is the topology* ***generated*** *by $\mathcal{B}$. Usually we refer to $\mathcal{B}$ as the basis for X.*

Proposition 39 *Let X be a set on which the discrete metric is defined. The topology $\mathcal{T}_X$ induced by the discrete metric on the set X is the discrete topology.*

PROOF:

Let $x \in X$. Then $\forall y \in X \ni y \neq x$, $d(x,y) = 1$ Def. 26

$\Rightarrow$ $B(x, \frac{1}{2}) = \{x\}$ and $B(y, \frac{1}{2}) = \{y\}$ are open balls.

Let $U \subset X$, and $u_i \in U$ for $i \in I$. Then, $U = \bigcup_{i \in I} \{u_i\} = \bigcup_{i \in I} B\left(u_i, \frac{1}{2}\right)$.

$\{u_i\}$ is an open ball $\forall i \in I$ $\Rightarrow \{u_i\}$ is an open set $\forall i \in I$

$\Rightarrow U \subset X$ is open (arbitrary union of open sets is open)

$\Rightarrow X$ is a union of open sets

$\Rightarrow \mathcal{T}_X$ is the discrete topology

$\therefore$ The topology induced on a set X by the discrete metric is the discrete topology.

Definition 40 *A subset C of a topological space X is **closed** provided its complement $X \backslash C$ is open.*

Definition 41 *Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. A point $x \in X$ is a **limit point** of A if every open set containing x contains a point of A distinct from x. The set of limit points A' is called the **derived** set of A.*

Definition 42 *Let A be a subset of a topological space X. A point $x \in A$ is an **interior point** of A if there is an open set O containing x and contained in A. The **interior** of A is the set of all interior points of A. The **closure** of A is the set $\overline{A} = A \cup A'$. A point $x \in X$ is a **boundary point** of A if x belongs to both $\overline{A}$ and $\overline{X \backslash A}$. The set of boundary points of A is called the **boundary** of A.*

Definition 43 *Let A be a subset of a topological space X. A collection $\mathcal{O} = \{U_\alpha : \alpha \in \Lambda\}$ of subsets of X is an **open cover** of A if $\bigcup \{U_\alpha\}_{\alpha \in \Lambda}$ contains A. A **subcover** derived from an open cover $\mathcal{O}$ is a subcollection $\mathcal{O}$' of $\mathcal{O}$ whose union contains A. An open cover of a space X is a family of open subsets of X whose union is X.*

Definition 44 *A topological space X is **compact** if every open cover of X has a finite subcover. X is **countably compact** if every countable open cover of X has* *a finite subcover. X is a* ***Lindelöf space*** *if every open cover of X has a countable subcover.*

Lemma 45 *If there exists a basis $\mathcal{B}$ for a topological space X such that every open cover of X by elements of $\mathcal{B}$ has a finite subcover, then X is compact*

PROOF:

Let X be a space and $\mathcal{B}$ be a basis for X such that every open cover by elements of $\mathcal{B}$ has a finite subcover.

Let $\mathcal{O}$ be an open cover of X composed of elements of $\mathcal{B}$, $x \in X$.

Then there is an open set $\mathcal{O}_x \in \mathcal{O}$ that contains x.

$\Rightarrow \exists\, B_x \in \mathcal{B} \ni x \in B_x \subset \mathcal{O}_x$ (basis)

$\Rightarrow$ the collection $\{B_x : x \in X\}$ is an open cover of X which by hypothesis has a finite subcollection $\{B_{x_i}\}_{i=1}^n$ for $i \in \mathbb{Z}^+$. Then, $\{B_{x_i}\}_{i=1}^n$ is a finite subcover that covers X. The corresponding collection $\{\mathcal{O}_{x_i}\}_{i=1}^n$ is a finite subcover of $\mathcal{O}$ that covers X.

$\therefore$ X is a compact space.

Definition 46 *A subset A of a space X is* ***dense*** *in X if for each point $p \in X$, p is a limit point of A. Thus, A is dense in X if $\overline{A} = X$. If X has a countable dense subset, then X is a* ***separable*** *space.*

Example 47 *$\mathbb{R}$ is a separable space since the set of rational numbers $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$.*

Theorem 48 *A family $\mathcal{B}$ of subsets of a set X is a basis for some topology for X if and only if both of the following conditions hold:*

1. $\bigcup_{B\in\mathcal{B}} B = X$.

2. $\forall B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 \cap B_2, \exists B_x \in \mathcal{B} \ni x \in B_x \subset (B_1 \cap B_2)$.

Definition 49 *Let $(X, \mathcal{T})$ be a space, $a \in X$. A* ***local basis*** *at a is a subcollection $\mathcal{B}_a$ of $\mathcal{T}$ such that*

1. *a belongs to each member of $\mathcal{B}_a$*

2. *for each open set Q containing a, there is a member r of $\mathcal{B}_a$ such that $r \in Q$.*

Definition 50 *A space X is* ***first countable*** *provided that there is a countable local basis at each point of X. The space X is* ***second countable*** *provided the topology of X has a countable basis.*

Definition 51 *Let $\mathcal{B}$ and $\mathcal{B}$' be bases for topologies $\mathcal{T}$ and $\mathcal{T}$' for a set X. Then $\mathcal{B}$ and $\mathcal{B}$' are* ***equivalent bases*** *provided that the topologies $\mathcal{T}$ and $\mathcal{T}$' are identical.*

Definition 52 *Let $(X, \mathcal{T})$ be a space. A subcollection $\mathcal{S}$ of $\mathcal{T}$ is a* ***subbasis*** *for $\mathcal{T}$ if the family $\mathcal{B}$ of all finite intersections of members of $\mathcal{S}$ is a basis for $\mathcal{T}$.*

Definition 53 *A function $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is* ***continuous*** *means that for each open set V in Y, $f^{-1}(V)$ is an open set in X.*

Proposition 54 *Suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $\mathcal{T}_X$ is the discrete topology. Then any function $f : X \to Y$ is continuous.*

PROOF:

Let f be a function from X to Y and $O \in \mathcal{T}_Y$ be an open set. Note that $O \subset f(X)$.

f is a function $\quad \Rightarrow \quad f^{-1}(O)$ exists and $f^{-1}(O) \in \mathcal{T}_X$

$\mathcal{T}_X$ is discrete $\quad \Rightarrow \quad f^{-1}(O) \subset X$ is open

$f^{-1}(O) \in \mathcal{T}_X \quad \forall O \in \mathcal{T}_Y \quad \Rightarrow \quad f : X \to Y$ is continuous.

$\therefore$ If $\mathcal{T}_X$ is the discrete topology, then any function $f : X \to Y$ is continuous.

Definition 55 *Let $f : X \to Y$ be a function on the indicated spaces. Then f is an **open function** if for each open set $O \subset X$, $f(O)$ is open in Y. The function f is a **closed function** if for each closed set $C \subset X$, $f(C)$ is closed in Y.*

Proposition 56 *Let $f : X \to Y$ be a continuous function between the spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. Then $f^{-1}(F)$ is a closed subset of X for every closed subset F of Y.*

PROOF:

Let $F \subset Y$ be closed. Then $Y \backslash F$ is open and $f^{-1}(Y \backslash F) \subset X$ is open since f is continuous.

To complete the proof, it suffices to show: $f^{-1}(Y \backslash F) = X \backslash (f^{-1}(F))$.

(a) Let $x \in f^{-1}(Y \backslash F)$.

$\Rightarrow x \notin f^{-1}(F) \quad \Rightarrow x \in [X \backslash f^{-1}(F)] \quad \Rightarrow f^{-1}(Y \backslash F) \subset X \backslash f^{-1}(F)$.

(b) Let $z \in f^{-1}(F)$.

$\Rightarrow z \notin f^{-1}(Y \backslash F) \quad \Rightarrow z \in [X \backslash (f^{-1}(Y \backslash F))]$

$\Rightarrow f^{-1}(F) \subset X \backslash (f^{-1}(Y \backslash F)) \quad \Rightarrow (X \backslash f^{-1}(F)) \subset (f^{-1}(Y \backslash F))$.

By the results of (a) and (b): $f^{-1}(Y \backslash F) = X \backslash (f^{-1}(F))$.

$\Rightarrow X \backslash (f^{-1}(F))$ is open, and

$f^{-1}(Y \backslash F)$ is open $\quad \Rightarrow X \backslash f^{-1}(Y \backslash F)$ is closed, and

$X\backslash f^{-1}(Y\backslash F) = f^{-1}(F) \quad \Rightarrow f^{-1}(F) \subset X$ is a closed set.

$\therefore \quad f^{-1}(F)$ is a closed subset of X for every closed subset F of Y.

Definition 57 *Topological spaces X and Y are **topologically equivalent** or **homeomorphic** if there is a one-to-one and onto function $f : X \to Y$ such that each of f and f^{-1} is continuous. The function f is called a **homeomorphism**.*

Definition 58 *If space X is homeomorphic to a subspace A of Y, then X is said to be **embedded** in Y and the homeomorphism $f : X \to A \subset Y$ is an **embedding** of X in Y.*

Definition 59 *A topological space $(X, \mathcal{T})$ is **metrizable** if and only if there exists a metric d for X such that the metric topology generated by d is identical to the original topology $\mathcal{T}$.*

Definition 60 *A topological space X is a **Hausdorff space** if for each pair a, b of distinct points of X there exist disjoint open sets U and V such that $a \in U$ and $b \in V$.*

Theorem 61 *Let (X, d) be a metric space and $x, y \in X$ such that $x \neq y$. Then, there exist two open sets $U, V \subset X$ in $\mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Thus, every metric space is Hausdorff.*

PROOF:

$x \neq y \qquad \Longrightarrow d(x, y) > 0.$ Define $\varepsilon < \frac{1}{2} d(x, y)$.

$\exists B(x, \varepsilon) \subset X$, and $\exists B(y, \varepsilon) \subset X$, and $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$.

Suppose the intersection is not empty. Assume $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$. Then,

$d(x,z) < \varepsilon$ and $d(z,y) < \varepsilon$

$d(x,y) \leq d(x,z) + d(z,y) < \varepsilon + \varepsilon < 2\left(\frac{1}{2}d(x,y)\right) = d(x,y)$

But, $d(x,y) \not< d(x,y)$; contradiction, means the assumption was false.

Call $U = B(x,\varepsilon)$ and $V = B(y,\varepsilon)$.

U is an open set containing x,

V is an open set containing y, and

$U \cap V = \emptyset$.

$\therefore$ Any metric space $(X, \mathcal{T})$ is Hausdorff.

Proposition 62 *The indiscrete topology on a set with more than one element is not metrizable.*

PROOF:

For simplicity, we prove the statement holds for a set of two elements. Let $X = \{a, b\}$.

The indiscrete topology on X is the set $\{X, \emptyset\} = \{\{a,b\}, \emptyset\}$.

For any metric on X, $d(a,b) = \varepsilon > 0$, and

$B(b, \frac{\varepsilon}{2}) = \{a\}$ and $B(b, \frac{\varepsilon}{2}) = \{b\}$ are open sets which are members of the metric topology on X.

But, $\{a\} \notin \{X, \emptyset\}$ and $\{b\} \notin \{X, \emptyset\}$, and thus the metric topology generated by d is not identical with the original indiscrete topology.

$\therefore$ The indiscrete topology on a set with more than one element is not metrizable.

Definition 63 *A topological space X is **disconnected** or **separated** if it is the union of two disjoint, non-empty open sets. Such a pair A, B of subsets of X is called a **separation** of X. A space X is **connected** if it is not disconnected.*

Chapter 4

PRODUCT SPACES

Chapter Four is dedicated to product spaces. Connectedness and separability of topological spaces and the properties of products of topological spaces are introduced and surveyed.

Definition 64 *Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2), \ldots, (X_n, \mathcal{T}_n)$ be a non-empty collection of topological spaces, and let X denote the Cartesian product*

$X = \prod X_i = X_1 \times X_2 \times \cdots \times X_n = \{x_1, x_2, \ldots, x_n : x_i \in X_i, i = 1, 2, \ldots n\}$.

Let $\mathcal{B}$ be a family of all subsets of X of the form $O = \prod_{i=1}^{n} O_i = O_1 \times O_2 \times \cdots \times O_n$ where each set O_i is an open set in the topology $\mathcal{T}_i$ for X_i. Then $\mathcal{B}$ is a basis for a topology T_X for X. This topology is called the ***product topology****, and the set (X, T_X) is a* ***product space****. The spaces $X_1, X_2, \ldots, X_n$ are the* ***coordinate spaces*** *or* ***factor spaces*** *of X. Since each point $x \in X$ is of the form $x = (x_1, x_2, \ldots, x_n)$, $x_i \in X_i$, $1 \leq i \leq n$, there exists a function $p_i : X \to X_i$ where $1 \leq i \leq n$, defined by $p_i(x_1, x_2, \ldots, x_n) = x_i$. This function p_i is called the* ***ith projection map****.*

Theorem 65 *The continuous image of a connected set is connected.*

Lemma 66 *The projection map* $p_i : X \rightarrow X_i$ *from a product space to the factor spaces is continuous.*

PROOF: The following is a summary of the proof given in [1].

Define $p_i : X \rightarrow X_i$ where $X = X_1 \times X_2 \times \cdots \times X_n$. Let $O_i \subset X_i$ be open.

$p_i^{-1}(O_i) = (X_1 \times X_2 \times \cdots X_{i-1} \times O_i \times \cdots \times X_n) \subset X$

$p_i^{-1}(O_i)$ is a finite product of open sets $\Rightarrow p_i^{-1}(O_i)$ is an open set

$\Rightarrow p_i$ is continuous.

$\therefore$ The projection map from a product space to the factor spaces is continuous.

Theorem 67 *The product of two Hausdorff spaces is a Hausdorff space.*

PROOF: Suppose each of X and Y is a Hausdorff space.

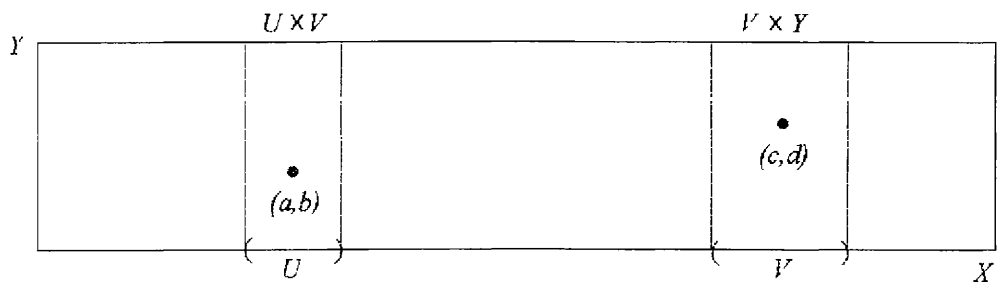
Let (a,b), $(c,d) \in X \times Y \ni (a,b) \neq (c,d)$.

$(a,b) \neq (c,d) \Rightarrow a \neq c$, or $b \neq d$. Assume $a \neq c$ (argument is similar for $b \neq d$).

X Hausdorff $\quad \Rightarrow \exists U, V \subset X \ni a \in U,\ c \in V,\ U \cap V = \emptyset$.

$(a,b) \in U \times Y, \quad (c,d) \in V \times Y, \quad (U \times Y) \cap (V \times Y) = \emptyset$.

$\therefore$ The product of two Hausdorff spaces is a Hausdorff space.



Product of Hausdorff Spaces

Theorem 68 *The product $X \times Y$ is connected if and only if each of X and Y is a connected space.*

PROOF:

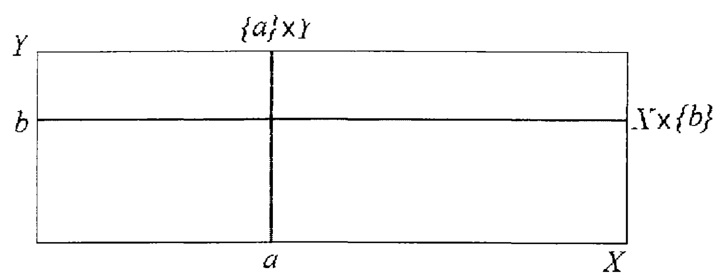
$\rightarrow$Let each of X and Y be connected, $a \in X$ and $b \in Y$.

Define $Z : X \times Y \to$ subspace of $X \times Y$ as follows.

$Z_{(a,b)} = X \times \{b\} \cup \{a\} \times Y = \{(p,q) : p \in X$ and $q = b$ or $p = a$ and $q \in Y\}$

Each of X and Y connected $\Rightarrow$ each of $X \times \{b\}$ and

$\{a\} \times Y$ is connected byThm.65.



Product of Connected Spaces

$$\bigcup Z_{(a,b)} = \bigcup (X \times \{b\}) \cup (\{a\} \times Y)$$

$\Rightarrow \quad \{a\}, \{b\} \in Z_{(a,b)} \, \forall a \in X$ and $\forall b \in Y$.

$\Rightarrow X \times Y$ is connected.

$\leftarrow$ Suppose $X \times Y$ is connected.

Define $p_X : X \times Y \to X, p_Y : X \times Y \to Y$, by $(x,y) \xrightarrow{p_X} x$ for $x \in X$ and $(x,y) \xrightarrow{p_Y} y$ for $y \in Y$.

The projection maps from a product space to the coordinate spaces are continuous by Lemma 66.

$\Rightarrow$ each of X and Y is connected by Thm.65

$\therefore\quad X \times Y$ is connected $\quad\Leftrightarrow\quad$ each of X and Y is connected.

Inductively, the product of a finite number of connected spaces is connected if and only if each of the factor spaces is itself connected.

Theorem 69 *Let X and Y be separable spaces. Then the product $X \times Y$ is separable.*

PROOF: Let $A \subset X$ be countable and dense, $B \subset Y$ be countable and dense.

Let Q be and open subset of $X \times Y$ where $Q = U \times V$ for $U \subset X$ is open and $V \subset Y$ is open.

X separable $\Rightarrow\ \exists\ a \in U \ni a \in A \subset X$

Y separable $\Rightarrow\ \exists\ b \in V \ni b \in B \subset Y$

Pick $a \in U \ni a \in A \subset X$ and $b \in V \ni b \in B \subset Y$.

$\Rightarrow \exists\,(a, b) \in U \times V \ni (a, b) \in A \times B \subset X \times Y$

$\Rightarrow A \times B$ is dense in $X \times Y$

each of A and B is countable $\Rightarrow\ A \times B \subset X \times Y$ is countable.

$\therefore X \times Y$ is a separable space by Def.46.

Theorem 70 *(1) The product of a finite number of first countable spaces is first countable.*

(2) The product of a finite number of second countable spaces is second countable.

PROOF: It suffices to prove that the theorem holds for $n = 2$.

(1) Let A, B be first countable spaces, and $(a, b) \in A \times B \quad \Rightarrow a \in A,\, b \in B$.

Let $W \subset A \times B$ be open $\in (a, b) \in W$.

$\Rightarrow \exists$ open sets $U \subset A$ and $V \subset B \ni (U \times V) \subset W$ and $a \in U$, $b \in V$.

A first countable $\Rightarrow \exists$ a countable basis $\mathcal{A}$ at a, where $\mathcal{A}$ is the collection of open sets $\{O_i\}_{i=1}^{\infty}$ where $a \in O_i \subset U$ for some $i \in \mathbb{Z}^+$,

B first countable $\Rightarrow \exists$ a countable basis $\mathcal{B}$ at b, where $\mathcal{B}$ is the collection of open sets $\{P_j\}_{j=1}^{\infty}$ where $b \in P_j \subset V$ for some $j \in \mathbb{Z}^+$,

$\Rightarrow (a, b) \in (O_i \times P_j) \subset (U \times V) \subset W$

$\mathcal{A}$,$\mathcal{B}$ countable $\Rightarrow \mathcal{A} \times \mathcal{B}$ countable $\Rightarrow \mathcal{A} \times \mathcal{B}$ is a countable local basis at (a, b).

$\therefore \quad A \times B$ is a first countable space.

(2) Let each of X and Y be a second countable space.

X second countable $\quad \Rightarrow \exists$ countable basis $\{U_i\}_{i=1}^{\infty}$ for X

Y second countable $\quad \Rightarrow \exists$ countable basis $\{V_j\}_{j=1}^{\infty}$ for Y.

Define $\mathcal{B} = \left\{ U \times V : U \in \{U_i\}_{i=1}^{\infty}, V \in \{V_j\}_{j=1}^{\infty} \right\}$.

Since each of $\{U_i\}_{i=1}^{\infty}$ and $\{V_j\}_{j=1}^{\infty}$ is countable, $\mathcal{B}$ is countable.

To show that $\mathcal{B}$ is a basis for $X \times Y$, use Thm. 48:

(a) $\bigcup_{i=1}^{\infty} U_i \times \bigcup_{j=1}^{\infty} V_j = X \times Y$

(b) Let $(U_i \times V_j), (U_r \times V_s) \in \mathcal{B}$, and $(a, b) \in (U_i \times V_j) \cap (U_r \times V_s)$

$(U_i \times V_j) \cap (U_r \times V_s)$ is an open set, and

$(U_i \times V_j) \cap (U_r \times V_s) = (U_i \cap U_r) \times (V_j \cap V_s)$.

$(U_i \cap U_r)$ is an open set containing $a \Rightarrow \exists U_a \subset (U_i \cap U_r) \ni U_a \in \{U_i\}_{i=1}^{\infty}$ and $a \in U_a$

$(V_j \cap V_s)$ is an open set containing $b \Rightarrow \exists V_b \subset (V_j \cap V_s) \ni V_b \in \{V_j\}_{j=1}^{\infty}$ and $b \in V_b$

$\Rightarrow (a, b) \in (U_a \times V_b), \quad (U_a \times V_b) \subset (U_i \times V_j) \cap (U_r \times V_s), \quad$ and $\quad (U_a \times V_b) \in \mathcal{B}$

$\Rightarrow \mathcal{B}$ is a basis for a topology of $X \times Y$

$\Rightarrow \mathcal{B}$ is a countable basis for $X \times Y$

$\therefore \quad X \times Y$ is a second countable space.

Theorem 71 *The product of a finite number of compact spaces is compact.*

PROOF: Suppose each of X, Y is a compact space.

Let $\mathcal{B} = \{U \times V : U \subset X, V \subset Y$, and each of U,V is open$\}$ be a basis for $X \times Y$ and $\mathcal{B}$ be an open cover for $X \times Y$ where

$\{U_i\}_{i=1}^{\infty}$ is an open cover for X, and $\{V_j\}_{j=1}^{\infty}$ is an open cover for Y

X compact $\quad \Rightarrow \exists$ a finite subcover $\{U_i\}_{i=1}^{n}$ that covers X

Y compact $\quad \Rightarrow \exists$ a finite subcover $\{V_j\}_{j=1}^{m}$ that covers Y

$\Rightarrow \left\{U_i \times V_j : U_i \in \{U_i\}_{i=1}^{n} \text{ and } V_j \in \{V_j\}_{j=1}^{m} \text{ for } i,j \in \mathbb{Z}^+\right\}$ is a finite subcover of $\mathcal{B}$ that covers $X \times Y$. By Lemma 45, $X \times Y$ is compact.

$\therefore$ If each of X and Y is a compact space, then $X \times Y$ is a compact space. Inductively, the product of a finite number of compact spaces is compact.

Chapter 5

SPACES OF PARTICULAR TYPES

In this chapter, topological spaces will be studied and categorized according to the properties that arbitrary pairs of subsets of a topological space have in relation to each other. In particular, these are the properties determining whether a pair of subsets can be enclosed in a pair of disjoint open subsets of a space. These properties are called **separation properties**.

Definition 72 $\mathbf{T_0}$**−space** *A space X is a T_0-space if for each pair $a, b \in X$, there exists an open set $U \subset X$ such that $a \in U$, but $b \notin U$.*

$\mathbf{T_1}$**−space** *A space X is a T_1-space if for each pair $a, b \in X$, there exist open sets $U, V \subset X$ such that $a \in U$ but $b \notin U$, and $b \in V$ but $a \notin V$.*

$\mathbf{T_2}$**−space**, ***Hausdorff*** *space* *A space X is a T_2-space if for each pair $a, b \in X$, there exist open sets $U, V \subset X$ such that $a \in U$, $b \in V$, and $U \cap V = \emptyset$.*

T$_3$–space, ***regular*** *space is a T_1-space X such that for each closed set $C \subset X$ and each point $a \notin C$, there exist open sets $U, V \subset X$ such that $a \in U$, $C \subset V$, and $U \cap V = \emptyset$.*

T$_4$–space, ***normal*** *space is a T_1-space X with the property that for each pair of closed sets $A, B \subset X$, $A \cap B = \emptyset$ there exist open sets U,$V \subset X$ such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.*

Theorem 73 *A T_1 space X is regular if and only if for each $a \in X$ and for each open set U containing a, there exists an open set W containing a such that $\overline{W} \subset U$.*

PROOF:

$\rightharpoonup$ Suppose X is regular. Let $a \in X$ and $U \subset X$ such that $a \in U$.

$\Rightarrow X \backslash U$ is closed and $a \notin X \backslash U$

X regular $\Rightarrow \exists\, W, V \subset X$ are open and disjoint $\ni a \in W$ and $(X \backslash U) \subset V$

V open $\Rightarrow X \backslash V$ is closed

$W \subset (X \backslash V)$ and $X \backslash V$ closed $\Rightarrow \overline{W} \subset (X \backslash V)$

$W \subset \overline{W} \subset (X \backslash V) \subset X \backslash (X \backslash U) = U$

$\Rightarrow \overline{W} \subset U$

$\leftharpoondown$ Supppose $\forall a \in X$ and $\forall$ open set U containing a, $\exists\, W$ open $\ni a \in W$ and $\overline{W} \subset U$.

Let $a \in X$ and $C \subset X$ be a closed set $\ni a \notin C$.

C closed $\Rightarrow X \backslash C$ open

Define $U = X \backslash C$

$\Rightarrow a \in U$ and, by hypothesis, $\exists W$ open $\ni a \in W$ and $\overline{W} \subset U$.

$\Rightarrow X \backslash \overline{W}$ is open and $C \subset \left(X \backslash \overline{W}\right)$

$W \subset \overline{W}$ and $\overline{W} \cap (X \backslash \overline{W}) = \emptyset \quad \Rightarrow W \cap (X \backslash \overline{W}) = \emptyset$

$a \in W$, $C \subset (X \backslash \overline{W})$, and $W \cap (X \backslash \overline{W}) = \emptyset \quad \Rightarrow X$ is regular.

$\therefore$ X regular $\Leftrightarrow \forall a \in X$ and $\forall$ open set U containing a, $\exists W$ open set containing a and $\ni \overline{W} \subset U$.

Theorem 74 *A T_1 space X is regular if and only if for each $a \in X$ and each $C \subset X$ closed, with $a \notin C$, there exist open sets $U, V \subset X$ such that $a \in U$, $C \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.*

PROOF:

$\rightharpoonup$ Suppose X is regular.

Let $a \in X$, $C \subset X$ be closed, and $a \notin C$.

C closed $\Rightarrow X \backslash C$ open

$a \notin C \quad \Rightarrow a \in X \backslash C$

$\Rightarrow \exists W$ open $\ni a \in W$, $\overline{W} \subset (X \backslash C)$, and $\exists U$ open $\ni a \in U$, $\overline{U} \subset W$ by Thm.73

$\overline{U} \subset W \subset \overline{W} \subset (X \backslash C) \quad \Rightarrow C \subset X \backslash \overline{W}$ where $X \backslash \overline{W}$ is open

Define $V = X \backslash \overline{W}$. $\Rightarrow \overline{V} = \overline{X \backslash \overline{W}}$

$W \cap (X \backslash \overline{W}) = \emptyset$ and $\overline{U} \subset W \quad \Rightarrow \overline{U} \cap \overline{X \backslash \overline{W}} \subset W \cap \left(\overline{X \backslash \overline{W}}\right) = \emptyset$

$\Rightarrow \overline{U} \cap \overline{V} = \emptyset$

$\leftharpoonup$ Suppose for each $a \in X$ and closed set $C \subset X$, $a \notin C$, $\exists$ open sets $U, V \subset X \ni a \in U$, $C \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.

By definition, X is regular, and X is normal.

$\therefore$ X is regular $\Leftrightarrow$ $\forall a \in X$ and $\forall C \subset X$ closed, $\ni a \notin C$, there exist open sets $U, V \subset X \ni a \in U$, $C \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Theorem 75 *A T_1 space X is normal if and only if $\forall$closed $A \subset X$ and open U containing A, there exists an open set W containing A such that $\overline{W} \subset U$.*

PROOF: $\rightharpoonup$ Suppose X is normal. Let $A \subset X$ be closed and U be an open set containing A.

U open $\Rightarrow X \backslash U$ closed

X normal $\Rightarrow \exists W, V$ open $\ni A \subset W$, $(X \backslash U) \subset V$, and $W \cap V = \emptyset$

$(X \backslash U) \subset V \quad \Rightarrow (X \backslash V) \subset U$

$A \subset W \subset \overline{W} \subset (X \backslash V) \subset U \Rightarrow \exists W$ open, $A \subset W$, $\ni \overline{W} \subset U$.

$\leftharpoondown$ Let each of $A, B \subset X$ be closed and $A \cap B = \emptyset$.

Let $U \subset X$ be open $\ni A \subset U$ and W_1 be an open set $\ni A \subset W_1 \subset \overline{W_1} \subset U$.

U open $\Rightarrow X \backslash U$ closed

$\overline{W_1}$ closed $\Rightarrow X \backslash \overline{W_1}$ open

$\overline{W_1} \subset U \quad \Rightarrow X \backslash U \subset X \backslash \overline{W_1}$

$\overline{W_1} \cap X \backslash \overline{W_1} = \emptyset$ and $W_1 \subset \overline{W_1} \quad \Rightarrow \quad W_1 \cap X \backslash \overline{W_1} = \emptyset$

Let $V = X \backslash \overline{W_1}$. So, V is open and $B \subset V$.

$\exists W_2$ open $\ni B \subset W_2 \subset \overline{W_2} \subset V$

$\overline{W_2} \cap X \backslash \overline{W_2} = \emptyset$ and $W_2 \subset \overline{W_2} \quad \Rightarrow \quad W_2 \cap X \backslash \overline{W_2} = \emptyset$

$W_1 \cap V = \emptyset$ and $W_2 \subset V \quad \Rightarrow \quad W_1 \cap W_2 = \emptyset$

$W_1 \cap W_2 = \emptyset$ and $A \subset W_1$ and $B \subset W_2 \quad \Rightarrow X$ is normal by definition.

$\therefore$ X normal $\Leftrightarrow \forall A \subset X$ closed and $\forall U$ open $\ni A \subset U$, $\exists W$ open and $A \subset W$ $\ni \overline{W} \subset U$.

Theorem 76 *A T_1 space X is normal if and only if $\forall$ pair A, B of disjoint closed sets in X, there exist open sets U,V such that $A \subset U$ and $B \subset V$, $\overline{U} \cap \overline{V} = \emptyset$.*

PROOF:

$\rightarrow$ Suppose X is normal.

Let A,$B \subset X$ be closed and disjoint.

X normal $\Rightarrow \exists$ open S,$W \subset X \ni A \subset S$, $B \subset W$, $S \cap W = \emptyset$

$\Rightarrow \exists$ open $U \ni A \subset U$ and $\overline{U} \subset S$ by Thm.75

$\Rightarrow \exists$ open $V \ni B \subset V$ and $\overline{V} \subset W$ by Thm.75

$S \cap W = \emptyset$, $\overline{U} \subset S$, and $\overline{V} \subset W$

$\Rightarrow \overline{U} \cap \overline{V} = \emptyset$.

$\leftarrow$ Suppose $A, B \subset X$ are closed, the sets $U, V \subset X$ are open

$\ni A \subset U$, $B \subset V$, and $\overline{U} \cap \overline{V} = \emptyset$.

Thus, $A \subset U \subset \overline{U}$, and $B \subset V \subset \overline{V}$, and $\overline{U} \cap \overline{V} = \emptyset \quad \Rightarrow U \cap V = \emptyset$

$\Rightarrow X$ is normal.

$\therefore$ X is normal $\Leftrightarrow$ $\forall A, B$ closed in $X \ni A \cap B = \emptyset$, $\exists U, V$ open

$\ni A \subset U$, $B \subset V$, $\overline{U} \cap \overline{V} = \emptyset$.

Theorem 77 *Every regular Lindelöf space is normal. The proof of this theorem is given on page 238 in [1].*

Chapter 6

SEPARATION BY CONTINUOUS FUNCTIONS

In this chapter, one method of constructing the set of dyadic numbers is given, along with the proof of its denseness in the real number system. Furthermore, the instrumental tools for the proof of The Urysohn's Metrization Theorem–Urysohn's Lemma and The Tietze Extension Theorem–are proved here.

Definition 78 *Suppose* $f : X \to \mathbb{R}$ *is a continuous function on a space* X *and* $A,B \subset X$*. Then* f ***separates*** A *and* B *if there exist* $a,b \in \mathbb{R}$, $a \neq b$, *such that* $f(A) = a$ *and* $f(B) = b$.

Definition 79 *A function that separates a singleton set* $\{x\}$ *from a set* B ***separates the point*** x ***from the set*** B.

Definition 80 *A function that separates the singletons* $\{x\}$ *and* $\{y\}$ ***separates the points*** x *and* y.

Definition 81 *A **dyadic number** is a rational number $\frac{m}{2^k}$, $m, k \in \mathbb{Z}$, in its reduced form.*

Example 82 *The following is a method of constructing the set of dyadic numbers D_0 contained in the unit interval* $[0, 1]$.

First, divide $[0, 1]$ into two subintervals of equal length. The midpoint $\left\{\frac{1}{2}\right\}$ is the first member of D_0. Next, divide each of the two resulting intervals into two subintervals of equal length. The set of midpoints obtained in this step $\left\{\frac{1}{2^2}, \frac{3}{2^2}\right\}$ is a subset of D_0. Continue this process of halving the intervals obtained in each step; union each set of midpoints of the intervals from the preceeding step with the set of dyadic numbers obtained in the preceeding steps as illustrated below.

Step	Sets formed
1	$\left\{\frac{1}{2}\right\}$
2	$\left\{\frac{1}{2}\right\} \cup \left\{\frac{1}{2^2}, \frac{3}{2^2}\right\}$
3	$\left\{\frac{1}{2}\right\} \cup \left\{\frac{1}{2^2}, \frac{3}{2^2}\right\} \cup \left\{\frac{1}{2^3}, \frac{3}{2^3}, \frac{5}{2^3}, \frac{7}{2^3}\right\}$
4	$\left\{\frac{1}{2}\right\} \cup \left\{\frac{1}{2^2}, \frac{3}{2^2}\right\} \cup \left\{\frac{1}{2^3}, \frac{3}{2^3}, \frac{5}{2^3}, \frac{7}{2^3}\right\} \cup \left\{\frac{1}{2^4}, \frac{3}{2^4}, \frac{5}{2^4}, \frac{7}{2^4}, \frac{9}{2^4}, \frac{11}{2^4}, \frac{13}{2^4}, \frac{15}{2^4}\right\}$
$\vdots$	$\vdots$
k	$\left\{\frac{1}{2}\right\} \cup \left\{\frac{1}{2^2}, \frac{3}{2^2}\right\} \cup \left\{\frac{1}{2^3}, \frac{3}{2^3}, \frac{5}{2^3}, \frac{7}{2^3}\right\} \cup \cdots \cup \left\{\frac{1}{2^{k-1}}, \ldots, \frac{2^{k-1}-1}{2^{k-1}}\right\} \cup \left\{\frac{1}{2^k}, \ldots, \frac{2^k-1}{2^k}\right\}$

After k iterations, $k \in \mathbb{Z}^+$, the interval $[0, 1]$ will be divided into 2^k subintervals of length $\frac{1}{2^k}$ whose endpoints are dyadic numbers and are members of $D_0 \subset [0, 1]$.

Note*:* Moving along the real number line and repeating the above procedure for any interval of the form $[n, n + 1]$, $n \in \mathbb{Z}$ will produce the set of dyadic numbers $D_n \subset [n, n + 1]$. Thus, the set of all dyadic numbers is $D = \bigcup\limits_{n \in \mathbb{Z}} D_n$.

Lemma 83 *The set of dyadic numbers is dense in* $\mathbb{R}$.

PROOF:

Define $D \subset \mathbb{R}$ to be the set of dyadic numbers.

Let $x \in \mathbb{R}$ and $\varepsilon > 0$.

D is dense if there exists a point $a \in D \ni a \in (x - \varepsilon, x + \varepsilon)$.

The length of the resulting intervals with dyadic endpoints in the construction of the set of dyadic numbers after k-iterations is $\frac{1}{2^k}$, $k \in \mathbb{Z}^+$ (Example82).

$\lim\limits_{k \to \infty} \frac{1}{2^k} = 0 \quad \Rightarrow \exists\ [a, b] \subset (x - \varepsilon, x + \varepsilon)$ with $a, b \in D$

$x \in \mathbb{R}$ and $\varepsilon > 0$ were arbitrary $\quad \Rightarrow \forall x \in \mathbb{R}$ and $\varepsilon > 0$, $\exists a \in D \ni x \in [a, b]$.

$\therefore$ The set of dyadic numbers is a dense subset of $\mathbb{R}$.

Lemma 84 *Given a space* X, $D \subset \mathbb{R}^+$ *dense such that* $\forall t \in D\ \exists$ *open* $U_t \subset X \ni$

1) if $t_1 < t_2$, *then* $\overline{U}_{t_1} \subset U_{t_2}$, *and*

2) $\cup_{t \in D} U_t = X$,

then $f : X \to \mathbb{R}$ *defined by* $f(x) = glb\{t \in D : x \in U_t\}$, $x \in X$, *is continuous.*

PROOF: We proceed by following the outline of proof in [1].

It suffices to show:

(1) f is well defined, and (2) $f^{-1}[I]$ is open $\forall I = (a, b) \subset \mathbb{R}$.

(1) $\quad x \in X \qquad \Rightarrow x \in U_t$ for some $t \in D$

$D \subset \mathbb{R}^+ \quad \Rightarrow f(x) = glb\{t \in D : x \in U_t\} \neq \emptyset \qquad \mathbb{R}^+$is bounded below

$\Rightarrow f(x)$ is unique $\qquad$ GLB is unique

$\therefore$ f is well defined.

(2) Define $I_1 = (-\infty, b)$ and $I_2 = (a, \infty)$ such that $a < b$. Then $I_1 \cap I_2 = (a, b) \subset \mathbb{R}$.

To prove $f^{-1}[(a,b)] \subset X$ is open $\forall\,(a,b) \subset \mathbb{R}$, we will first show that

(a) $f^{-1}[I_1]$ is open and (b) $f^{-1}[I_2]$ is open by showing that $X \backslash f^{-1}[I_2]$ is closed.

(a) $f^{-1}[I_1] = \{x \in X : f(x) < b\}$

$f(x) < b \qquad \Rightarrow x \in U_t$ for some $t \in D$

$\Rightarrow f[U_t] \subset I_1$

$\Rightarrow f^{-1}[I_1] = \{x \in X : f(x) < b\} = \bigcup_{t \in D} \{U_t : t < b\}$

U_t open $\forall t \in D \qquad \Rightarrow \bigcup_{t \in D} \{U_t : t < b\}$ open

$\therefore \quad f^{-1}[I_1]$ is open in X.

(b) $f^{-1}[I_2] = \{x \in X : f(x) > a\} \quad \Rightarrow \quad X \backslash f^{-1}[I_2] = \{x \in X : f(x) \leq a\}$

WTS: $X \backslash f^{-1}[I_2] = \{x \in X : f(x) \leq a\} = \bigcap_{t \in D} \{\overline{U}_t : t > a\}$

$x \in \{x \in X : f(x) \leq a\} \quad \Rightarrow f(x) = glb\{t \in D : x \in U_t\} \leq a$

D dense $\qquad \Rightarrow f(x) \leq a < t$

$f(x) < t \qquad \Rightarrow x \in U_t \forall t > a$

$U_t \subset \overline{U}_t \qquad \Rightarrow x \in \overline{U}_t \forall t > a$

$\Rightarrow x \in \bigcap_{t \in D} \{\overline{U}_t : t > a\}$

$\Rightarrow \{x \in X : f(x) \leq a\} \subset \bigcap_{t \in D} \{\overline{U}_t : t > a\}$

$x \in \bigcap_{t \in D} \{\overline{U}_t : t > a\} \quad \Rightarrow x \in \overline{U}_t \ \forall t > a,\ t \in D$

Suppose $x \in U_t \qquad$ since $U_t \subset \overline{U}_t$

$\Rightarrow f(x) = glb\{t \in D : x \in U_t\} \leq t \quad \forall t > a$

$a < t \qquad \Rightarrow f(x) \leq a$

$\Rightarrow x \in \{x \in X : f(x) \leq a\} = X \backslash f^{-1}[I_2]$

$\Rightarrow \bigcap_{t \in D} \{\overline{U}_t : t > a\} \subset X \backslash f^{-1}[I_2]$

$$\left.\begin{array}{l} X\backslash f^{-1}[I_2] \subset \bigcap_{t\in D}\left\{\overline{U}_t : t > a\right\} \\ \bigcap_{t\in D}\left\{\overline{U}_t : t > a\right\} \subset X\backslash f^{-1}[I_2] \end{array}\right\} \quad \Rightarrow \quad X\backslash f^{-1}[I_2] = \bigcap_{t\in D}\left\{\overline{U}_t : t > a\right\}$$

$$\begin{aligned} \overline{U}_t \text{ closed } \forall t \in D \quad &\Rightarrow \bigcap_{t\in D}\left\{\overline{U}_t : t > a\right\} \text{ closed} \\ &\Rightarrow X\backslash f^{-1}[I_2] \text{ closed} \\ &\Rightarrow f^{-1}[I_2] \subset X \text{ open} \end{aligned}$$

Each of $f^{-1}[I_1] \subset X$ and $f^{-1}[I_2] \subset X$ open $\Rightarrow$ $f^{-1}[I_1] \cap f^{-1}[I_2]$ open

$$\begin{aligned} f^{-1}[I_1] \cap f^{-1}[I_2] \quad &= f^{-1}[I_1 \cap I_2] && \text{Prop.14} \\ &= f^{-1}[(-\infty, b) \cap (a, \infty)] \\ &= f^{-1}[(a, b)] \text{ open} \end{aligned}$$

$\therefore$ $f^{-1}[(a,b)] \subset X$ is open $\forall\, (a,b) \subset \mathbb{R}$, $a < b$.

WTS: $f^{-1}[O]$ open when $O \subset \mathbb{R}$, O open.

$O = \bigcup_{j\in J} I_j$ where I_j is an open interval

$f^{-1}[O] = f^{-1}\left[\bigcup_{j\in J} I_j\right] = \bigcup_{j\in J} f^{-1}[I_j]$ open $\qquad$ Prop.14

since $f^{-1}[I_j]$ was proved to be open.

$\therefore$ f is a continuous function.

Lemma 85 ***Urysohn's Lemma***

Suppose X is a T_1-space. The following statements are equivalent.

(a) X is normal.

(b) $\forall$ pair $A, B \subset X$, $A \cap B = \emptyset$, A, B closed, $\exists\ f : X \to [0,1]$ continuous

$\ni f(A) = \{0\}$ and $f(B) = \{1\}$.

PROOF: The following is based on the proof given in [1].

(b)⇒(a) Suppose each of $A, B \subset X$ is closed, $A \cap B = \emptyset$, and $f : X \to [0,1]$ is continuous $\ni f(A) = \{0\}$ and $f(B) = \{1\}$.

$\Rightarrow\ \exists\ U, V$ open, disjoint in $[0,1]\ \ni 0 \in U$ and $1 \in V$

f continuous $\Rightarrow\ f^{-1}(U),\ f^{-1}(V)$ open in X, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$

$\ni A \subset f^{-1}(U)$ and $B \subset f^{-1}(V)$

$\therefore$ X is normal.

(a)⇒(b) Suppose X is normal.

Let $D \subset [0,1]$ contain dyadic numbers, $A, B \subset X$ be closed and $A \cap B = \emptyset$.

B closed	$\Rightarrow$	$X \backslash B$ open and $A \subset X \backslash B$	
X normal, $A \subset X \backslash B$	$\Rightarrow$	$\exists W_{\frac{1}{2}} \supset A \ni \overline{W}_{\frac{1}{2}} \subset X \backslash B$	Thm.75
$A \subset W_{\frac{1}{2}}$ and $\overline{W}_{\frac{1}{2}} \subset X \backslash B$	$\Rightarrow$	$\exists W_{\frac{1}{4}} \supset A \ni \overline{W}_{\frac{1}{4}} \subset W_{\frac{1}{2}}$ and	Thm.75
		$\exists W_{\frac{3}{4}} \supset \overline{W}_{\frac{1}{2}} \ni \overline{W}_{\frac{3}{4}} \subset X \backslash B$	
$\vdots$		$\vdots$	

Continue the indicated process. Define $W_t = X$ for $t = 1 \in D$.

By Lemma 83, the set of dyadic numbers $D \subset [0,1]$ is dense.

$\forall t \in D\ \exists W_t \subset X\ \ni t_1 < t_2 \Rightarrow \overline{W}_{t_1} \subset W_{t_2}$ and $\bigcup_{t \in D} W_t = X$.

Define $f : X \to [0,1]$ by $f(x) = glb\{t \in D : x \in W_t\}$

As shown in Lemma 84, f is continuous.

$A \subset W_t \; \forall t \in D \qquad \Rightarrow f(x) = 0 \; \forall x \in A \quad \Rightarrow f(A) = \{0\}$

$B \subset W_t \Leftrightarrow t = 1 \qquad \Rightarrow f(y) = 1 \; \forall y \in B \quad \Rightarrow f(B) = \{1\}$

$\therefore$ $\exists$ a continuous function $f : X \to [0,1] \ni f(A) = \{0\}$ and $f(B) = \{1\}$.

$\therefore$ The conclusion of the Urysohn's Lemma is a true statement.

From this point on, we will refer to functions that satisfy the conditions stated in Urysohn's Lemma simply as Urysohn's functions.

Lemma 86 *Suppose $A \subset B \subset X$, A is closed in B, and B is closed in X. Then, A is closed in X.*

PROOF:

WTS: The closure of A in X, $\overline{A} = A$.

$A \subset \overline{A}$ by definition, we will show that $\overline{A} = A \cup A' \subset A$.

t suffices to show that $A' \subset A$.

Let $a \in A'$.

$a \in A' \quad \Rightarrow \quad a \in B' \quad \Rightarrow \quad a \in B$ since B is closed and $\overline{B} = B$.

Either $a \in A$ or $a \notin A$.

Assume $a \notin A$. This will lead to a contradiction.

Define $O = B \backslash A$.

A closed $\quad \Rightarrow \quad O$ open and $a \in O$.

$O \cap A = \emptyset$, but $a \in A' \quad \Rightarrow \quad A$ is not closed in B contradicts the hypothesis

$\Rightarrow \quad A' \subset A \quad \Rightarrow \quad \overline{A} = A$

$\therefore$ A is closed in X.

Theorem 87 ***The Tietze Extension Theorem***

If a space X is normal, the set $A \subset X$ is closed, and if the function $f : A \to \mathbb{R}$ is continuous, then f has a continuous extension $F : X \to \mathbb{R}$.

PROOF: The following is an elaboration on the proof in [1]:

Suppose X is a normal space, $A \subset X$ is closed, and $f : A \to \mathbb{R}$ is continuous.

First, we indicate a breakdown of the proof by steps using diagrams to illustrate the mapping of the given closed set $A \subset X$ under f and the procedure leading to the definition of the extension $F : X \to \mathbb{R}$ whose existence and continuity we are to prove.

I. Define a homeomorphism $\phi : \mathbb{R} \to (-1,1)$, and $f^* = \phi \circ f$. An open interval $(a,b) \subset \mathbb{R}$, $a < b$, is homeomorphic to the set $\mathbb{R}$ since there is a homeomorphism $\phi : \mathbb{R} \to (a,b)$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "f^* = \phi \circ f"'] & \mathbb{R} \arrow[r, "\phi"] & (-1,1) \end{tikzcd}$$

II. Show: $\exists F^* : X \to [-1,1] \ni$ for $a \in A$, $F^*(a) = f^*(a)$ and F^* is continuous.

$$\begin{tikzcd} A \arrow[r, "f^*"] \arrow[d] & (-1,1) \arrow[r, "\phi^{-1} \circ f^*"] & \mathbb{R} \\ X \arrow[ur, "F^*"'] & & \end{tikzcd}$$

III. Show: $\exists F : X \to [-1,1] \ni$ for $a \in A$, $F(a) = f^*(a)$ and F is continuous.

$$\begin{tikzcd} A \arrow[r, "f^*"] \arrow[d] & (-1,1) \arrow[r, "\phi^{-1} \circ f^*"] & \mathbb{R} \\ X \arrow[urr, "F = \phi^{-1} \circ F^*"'] & & \end{tikzcd}$$

In summary, an element $a \in A$ is mapped as follows:

$$a \longmapsto f(a) \longmapsto \phi \circ f(a) \longmapsto \phi \circ \phi^{-1}[f(a)] = f(a) = F(a)$$

I. Suppose $\phi : \mathbb{R} \to (-1,1)$ is a homeomorphism, define $f^*(a) = \phi[f(a)]$, and initially assume that $f^*[A] \subset [-1,1]$.

Let $A_1 = \left\{x \in A : f^*(x) \leq -\frac{1}{3}\right\}$ and $B_1 = \left\{x \in A : f^*(x) \geq \frac{1}{3}\right\}$.

A_1 and B_1 are closed subsets of A since f is continuous, and

$A_1 = f^{-1}\left[\left[-1, -\frac{1}{3}\right]\right]$ and $B_1 = f^{-1}\left[\left[\frac{1}{3}, 1\right]\right]$.

A_1, B_1 are closed in A, and since A is closed in X, then A_1, B_1 are closed in X (L.86).

$$A_1, B_1 \subset X \text{ closed, } A_1 \cap B_1 = \emptyset \quad \Rightarrow \quad \exists f_1 : X \to \left[-\tfrac{1}{3}, \tfrac{1}{3}\right] \text{ continuous}$$

$$\ni f_1(A_1) = \left\{-\tfrac{1}{3}\right\},\ f_1(B_1) = \left\{\tfrac{1}{3}\right\} \quad \text{L.85}$$

Observe that $\forall x \in A$, $|f^*(x) - f_1(x)| \leq \frac{2}{3}$

Define $f^1 : A \to \left[-\frac{2}{3}, \frac{2}{3}\right]$ by $f^1(x) = f^*(x) - f_1(x)$, $x \in A$. Then f^1 is continuous.

Construct a sequence of continuous functions $\{f_n\}_{n=1}^{\infty}$ as follows.

There exist:			$\forall x \in A$
$f_1 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$	$\ni f_1[A_1] = \left\{-\frac{1}{3}\right\}$	$f_1[B_1] = \left\{\frac{1}{3}\right\}$	$\lvert f_1(x)\rvert \leq \frac{1}{3}$
$f_2 : X \to \left[-\frac{2}{9}, \frac{2}{9}\right]$	$\ni f_2[A_2] = \left\{-\frac{2}{9}\right\}$	$f_2[B_2] = \left\{\frac{2}{9}\right\}$	$\lvert f_2(x)\rvert \leq \frac{2}{9}$
$f_3 : X \to \left[-\frac{4}{27}, \frac{4}{27}\right]$	$\ni f_3[A_3] = \left\{-\frac{4}{27}\right\}$	$f_3[B_3] = \left\{\frac{4}{27}\right\}$	$\lvert f_3(x)\rvert \leq \frac{4}{27}$
$f_4 : X \to \left[-\frac{8}{81}, \frac{8}{81}\right]$	$\ni f_4[A_4] = \left\{-\frac{8}{81}\right\}$	$f_4[B_4] = \left\{\frac{8}{81}\right\}$	$\lvert f_4(x)\rvert \leq \frac{8}{81}$
$f_5 : X \to \left[-\frac{16}{243}, \frac{16}{243}\right]$	$\ni f_5[A_5] = \left\{-\frac{16}{243}\right\}$	$f_5[B_5] = \left\{\frac{16}{243}\right\}$	$\lvert f_5(x)\rvert \leq \frac{16}{243}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f_n : X \to \left[-\frac{2^{n-1}}{3^n}, \frac{2^{n-1}}{3^n}\right]$	$\ni f_n[A_n] = \left\{-\frac{2^{n-1}}{3^n}\right\}$	$f_n[B_n] = \left\{\frac{2^{n-1}}{3^n}\right\}$	$\lvert f_n(x)\rvert \leq \frac{2^{n-1}}{3^n}$

Define:

$$f^1 : A \to \left[-\tfrac{2}{3}, \tfrac{2}{3}\right] \qquad f^1(x) = f^*(x) - f_1(x)$$

$$f^2 : A \to \left[-\tfrac{4}{9}, \tfrac{4}{9}\right] \qquad f^2(x) = f^*(x) - [f_1(x) + f_2(x)]$$

$$f^3 : A \to \left[-\tfrac{8}{27}, \tfrac{8}{27}\right] \qquad f^3(x) = f^*(x) - [f_1(x) + f_2(x) + f_3]$$

$$f^4 : A \to \left[-\tfrac{16}{81}, \tfrac{16}{81}\right] \qquad f^4(x) = f^*(x) - [f_1(x) + f_2(x) + f_3 - f_4(x)]$$

$$f^5 : A \to \left[-\tfrac{32}{243}, \tfrac{32}{243}\right] \qquad f^5(x) = f^*(x) - [f_1(x) + f_2(x) + f_3 + f_4(x) + f_5(x)]$$

$$\vdots \qquad\qquad \vdots$$

$$f^n : A \to \left[-\tfrac{2^n}{3^n}, \tfrac{2^n}{3^n}\right] \qquad f^n(x) = f^*(x) - \sum_{n=1}^{\infty} f_n(x)$$

$$\forall x \in A, \left| f^*(x) - \sum_{n=1}^{k} f_n(x) \right| \le \left(\tfrac{2}{3}\right)^n \quad \text{and} \quad \forall x \in X, |f_n(x)| \le \tfrac{2^{n-1}}{3^n} = \tfrac{1}{2}\left(\tfrac{2}{3}\right)^n$$

and since $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}$ exists, and

$$\forall n \in \mathbb{N}, |f_n(x)| \le \tfrac{2^{n-1}}{3^n} \quad \Rightarrow \quad \sum_{n=1}^{\infty} |f_n(x)| \le \sum_{n=1}^{\infty} \tfrac{2^{n-1}}{3^n} = 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} |f_n(x)| \quad \text{exists}$$

$$\left| \sum_{n=1}^{\infty} f_n(x) \right| \le \sum_{n=1}^{\infty} |f_n(x)| \quad \Rightarrow \quad -1 \le \sum_{n=1}^{\infty} f_n(x) \le 1$$

$$\text{Put } F^*(x) = \sum_{n=1}^{\infty} f_n(x) \quad \Rightarrow \quad F^*(x) \in [-1, 1]$$

Next, we show that F^* is continuous.

Let $\varepsilon > 0$, $x \in X$.

Let $O \subset \mathbb{R}$ be open $\ni (F^*(x) - \varepsilon, F^*(x) + \varepsilon) \subset O$.

Pick $N \in \mathbb{N} \ni \sum_{n=N+1}^{\infty} \left(\frac{2}{3}\right)^n < \frac{\varepsilon}{3}$.

$$\forall n \in \mathbb{N}, |f_n(x)| \le \tfrac{2^{n-1}}{3^n} < \left(\tfrac{2}{3}\right)^n \quad \Rightarrow \quad \sum_{n=N+1}^{\infty} |f_n(x)| < \tfrac{\varepsilon}{3}$$

$$f_n \text{ continuous } \forall n \in \mathbb{N} \quad \Rightarrow \quad \exists U_x \subset X, x \in U_x \ni$$

$$y \ne x \text{ and } y \in U_x \Rightarrow |f_n(x) - f_n(y)| < \tfrac{\varepsilon}{3N}$$

Choose $y \in U_x$, $y \ne x$. Then,

$$\begin{aligned}
|F^*(x) - F^*(y)| &= \left|\sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{\infty} f_n(y)\right| \\
&= \left|\sum_{n=1}^{N} f_n(x) - \sum_{n=1}^{N} f_n(y) + \sum_{n=N+1}^{\infty} f_n(x) - \sum_{n=N+1}^{\infty} f_n(y)\right| \\
&\leq \left|\sum_{n=1}^{N} [f_n(x) - f_n(y)]\right| + \left|\sum_{n=N+1}^{\infty} f_n(x)\right| + \left|\sum_{n=N+1}^{\infty} f_n(y)\right| \\
&< N \cdot \tfrac{\varepsilon}{3N} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

$$\begin{aligned}
U_x \subset X \text{ open}, x \in U_x, y \in U_x &\Rightarrow F^*(y) \in (F^*(x) - \varepsilon, F^*(x) + \varepsilon) \subset O \\
&\Rightarrow F^*[U_x] \subset O \text{ and } U_x \subset (F^*)^{-1}[O]
\end{aligned}$$

$\therefore$ F^* is continuous.

III. Assume $f^*[A] \subset [-1,1]$ and $\exists y \in A \ni |F^*(y)| = \{1\}$.

At this point $\phi^{-1} \circ F^*(y)$ is undefined. To provide a solution for this discontinuity, we apply Lemma 85 once more.

Define $B = \{x \in X : |F^*(x)| = \{1\}\}$

$$\begin{aligned}
&F^*\text{continuous}, \{-1\}, \{1\} \text{ closed} &&\Rightarrow B \text{ closed} && \text{(inv. image)} \\
&A, B \subset X \text{ closed}, A \cap B = \emptyset &&\Rightarrow \exists g : X \to [0,1] \ni \\
& && \quad g(x) = \begin{cases} 0 & \text{for } x \in B \\ 1 & \text{for } x \in A \end{cases} && \text{Lemma 85} \\
&\text{Each of } g \text{ and } F^* \text{ continuous} &&\Rightarrow g \cdot F^* \text{ continuous.}
\end{aligned}$$

$$\begin{aligned}
x \in B &\Rightarrow g(x) \cdot F^*(x) = 0 \cdot F^*(x) = 0 \\
x \in A &\Rightarrow g(x) \cdot F^*(x) = 1 \cdot F^*(x) = F^*(x) \in (-1,1) \\
&\Rightarrow g \cdot F^*[X] \subset (-1,1) \text{ and } \phi^{-1} \circ [g \cdot F^*] \subset X
\end{aligned}$$

Define $F : X \to \mathbb{R}$ by $F(x) = \phi^{-1}[g(x) \cdot F^*(x)]$, $x \in X$.

$$
\begin{aligned}
a \in A \quad \Rightarrow \quad F(a) &= \phi^{-1}\left[g(a) \cdot F^*(a)\right] \qquad \text{since } F^* \text{ is continuous} \\
&= \phi^{-1}\left[1 \cdot F^*(a)\right] \\
&= \phi^{-1}\left[F^*(a)\right] \\
&= \phi^{-1}\left[f^*(a)\right] \\
&= f(a)
\end{aligned}
$$

$F(a) = f(a) \quad \Rightarrow \quad F : X \to \mathbb{R}$ is a continuous extension of f.

$\therefore$ The function $f : A \to \mathbb{R}$ has a continuous extension $F : X \to \mathbb{R}$.

Theorem 88 *The product of a countable collection of metric spaces is metrizable.The proof of this theorem can be found in[1].*

Theorem 89 ***The Urysohn Metrization Theorem.***

Every second countable regular space is metrizable.

PROOF: The following is an elaboration of the proof in [1].

Suppose X is second countable and regular.

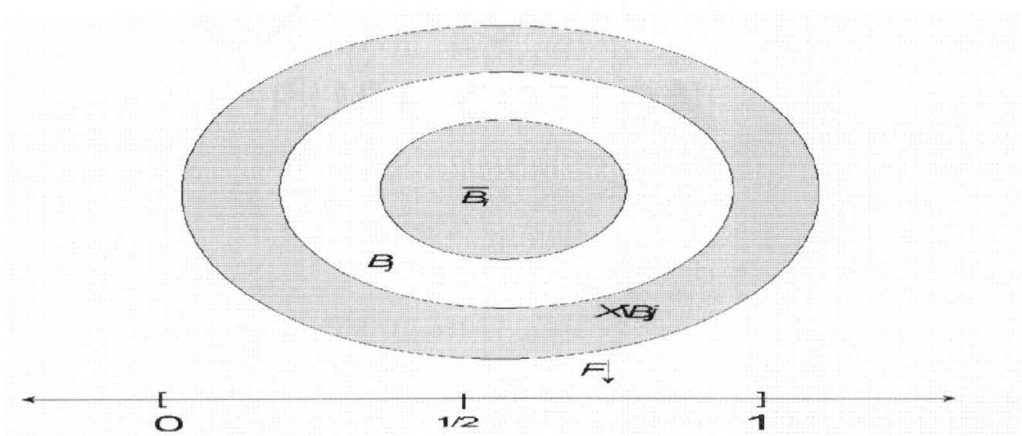
Let $x \in X$.

X regular	$\Rightarrow$	X is a T_1-space by definition
X second countable	$\Rightarrow$	X is Lindelöf
X regular and Lindelöf	$\Rightarrow$	X is normal.

Let $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$ be a countable basis for X.

Let $i, j \in \mathbb{N}$ and consider the collection of ordered pairs $(i, j) \ni \overline{B}_i \subset B_j$.

Note, $\overline{B}_i \cap (X \backslash B_j) = \emptyset$.



By Lemma 85, there exists a Urysohn function $f : X \to [0, 1] \ni$

$f\left[\overline{B}_i\right] = \{0\}$ and $f\left[(X \backslash B_j\right] = \{1\}$.

Let $\{f_n\}_{n=1}^{\infty}$ be the collection of functions such that $\forall (i, j), \overline{B}_i \subset B_j$.

Since B is countable, $\{f_n\}_{n=1}^{\infty}$ is countable.

Define $F : X \to \mathbb{H}$ by $F(x) = \left(f_1(x), \frac{f_2(x)}{2}, \frac{f_3(x)}{3}, \ldots, \frac{f_n(x)}{n}, \ldots\right)$ where

$\mathbb{H} = \left\{x \in X : x = (x_1, x_2, x_3, \ldots, x_n, \ldots) \text{ for } x_n \in \mathbb{R}\, \forall n \in \mathbb{N} \ni \sum\limits_{n=1}^{\infty}(x_n)^2 < \infty\right\}$.

For each n, $0 \leq f_n \leq 1$, thus $\sum\limits_{n=1}^{\infty} \left(\frac{f_n(x)}{n}\right)^2 \leq \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$.

Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ exists (p-series with $p > 1$), $\sum\limits_{n=1}^{\infty} \left(\frac{f_n(x)}{n}\right)^2$ is bounded by the convergent series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$, and thus $\sum\limits_{n=1}^{\infty} \left(\frac{f_n(x)}{n}\right)^2$ converges (term-by-term comparison).

Then, $F : X \to \mathbb{H}$ is well defined, and $F(x) \in \mathbb{H} \, \forall x \in X$.

$\therefore F(X) \subset \mathbb{H}$ and F is well defined.

To complete the proof, we will show that

(1) F is one-to-one,

(2) F is continuous,

(3) the restricted map $F : X \to F(X)$ is open, and

(4) the function $F^{-1} : F(X) \to X$ is continuous.

Clearly, F is onto its image $F(X)$.

From these properties it will follow that X is homeomorphic to $F(X) \subset \mathbb{H}$.

(1) X regular $\Rightarrow$ X is a T_1-space by definition

$\Rightarrow$ X is a T_0-space

$\Rightarrow$ $\exists B_j \in B \ni x \in B_j$, but $y \notin B_j$

$\Rightarrow$ $\exists B_i \in B \ni x \in B_i$ and $\overline{B}_i \subset B_j$ Thm.73

$\Rightarrow$ $\exists f_n \in \{f_n\}_{n=1}^{\infty}$ corresponding to (i, j) such that

$f_n\left[\overline{B_i}\right] = \{0\}$ and $f_n[X \backslash B_j] = \{1\}$ and thus

$f_n(x) = 0$ while $f_n(y) = 1$

since $x \in \overline{B}_i$, and $y \notin \overline{B}_i$, but $y \in X \backslash B_j$.

Because $F(x)$ and $F(y)$ differ in their n-th coordinate, then $F(x) \neq F(y)$.

Thus, $x \neq y \Rightarrow F(x) \neq F(y)$

$\therefore$ The function $F : X \to F(X)$ is one-to-one.

(2) F is continuous if and only if for every open $U \subset \mathbb{H}$, $F^{-1}(U) \subset X$ is open.

Let $x \in X, \varepsilon > 0$, and $B(F(x), \varepsilon)$ be an open ball in $\mathbb{H}$. Choose $N \in \mathbb{N} \ni$

$\sum\limits_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{\varepsilon^2}{2}$. Such a number N exists since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Since f_n is continuous $\forall n \in \mathbb{N}$ and continuous for $1 < n \leq \mathbb{N}$, there exists a collection of open sets $O_1, O_2, O_3, \ldots, O_n, O_N$, such that for any $y \in O_n$ and $y \neq x$,

$$|f_n(x) - f_n(y)| < \tfrac{\varepsilon}{\sqrt{2N}} \quad \text{for } 1 < n \leq N.$$

$$x \in O_n \ \forall n = 1, 2, 3, \ldots, N \quad \Rightarrow \quad x \in \cap_{n=1}^{N} O_n = O$$

Since O is the intersection of finitely many open sets, O is open in X.

Next, we show that $O \subset F^{-1}[B(F(x), \varepsilon)] \subset X$ by first showing that $F[O] \subset B(F(x), \varepsilon)$.

Let $y \in O$. Recall,

$F(x) = \left(f_1(x), \frac{f_2(x)}{2}, \frac{f_3(x)}{3}, \ldots, \frac{f_n(x)}{n}, \ldots\right)$ and $F(y) = \left(f_1(y), \frac{f_2(y)}{2}, \frac{f_3(y)}{3}, \ldots, \frac{f_n(y)}{n}, \ldots\right)$.

Then,

$$\begin{aligned}
|F(x) - F(y)| &= \left|\sum_{n=1}^{\infty} \left|\frac{f_n(x) - f_n(y)}{n}\right|^2\right|^{\frac{1}{2}} \\
&= \left|\sum_{n=1}^{N} \left|\frac{f_n(x) - f_n(y)}{n}\right|^2 + \sum_{n=N+1}^{\infty} \left|\frac{f_n(x) - f_n(y)}{n}\right|^2\right|^{\frac{1}{2}} \\
&\leq \left|\sum_{n=1}^{N} \left|\frac{|f_n(x) - f_n(y)|^2}{n^2}\right|^2 + \sum_{n=N+1}^{\infty} \left|\frac{1}{n^2}\right|^2\right|^{\frac{1}{2}} \\
&< \left|\sum_{n=1}^{N} |f_n(x) - f_n(y)|^2 + \frac{\varepsilon^2}{2}\right|^{\frac{1}{2}} \\
&< \left|N\left(\frac{\varepsilon}{\sqrt{2N}}\right)^2 + \frac{\varepsilon^2}{2}\right|^{\frac{1}{2}} \\
&= \varepsilon
\end{aligned}$$

$$
\begin{aligned}
|F(x) - F(y)| < \varepsilon \quad &\Rightarrow \quad F(y) \in B[F(x), \varepsilon] \\
&\Rightarrow \quad F[O] \subset B(F(x), \varepsilon) \subset \mathbb{H} && F[O] \text{ is open} \\
&\Rightarrow \quad O \subset F^{-1}[B(F(x), \varepsilon)] \subset X && \text{Prop.16} \\
&\Rightarrow \quad F^{-1}[B(F(x), \varepsilon)] \subset X \ \text{ is open} && \cup \text{ of open sets}
\end{aligned}
$$

$\therefore \quad F : X \to F(X) \subset \mathbb{H}$ is continuous.

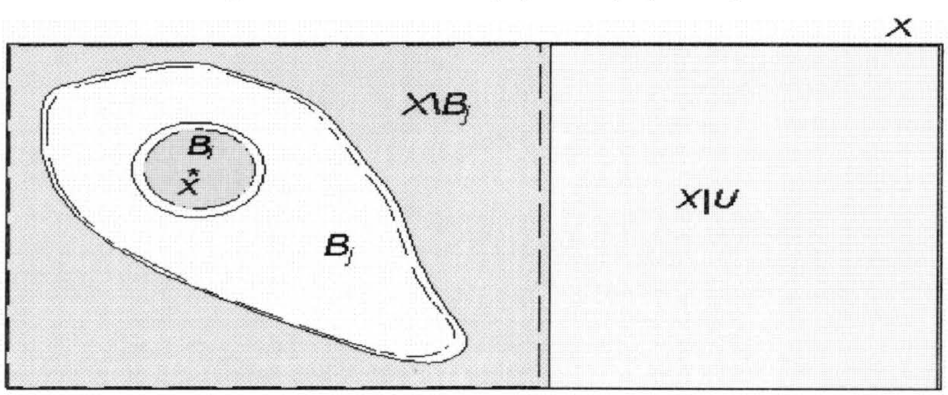
(3) The function $F : X \to F(X)$ is an open function if for each open set $U \subset X$, $F(U)$ is open in $F(X)$.

Let $U \subset X$ be open and $x \in U$.

$$
\begin{aligned}
X \text{ regular} \quad &\Rightarrow \quad \exists\, B_i, B_j \in B \ni x \in B_i \subset \overline{B}_i \subset B_j \subset U \\
&\Rightarrow \quad \exists\, f_n \in \{f_n\}_{n=1}^{\infty} \ni f_n\left[\overline{B}_i\right] = 0, \ \ f_n[X \backslash B_j] = 1 \\
x \in \overline{B}_i,\ (X \backslash U) \subset (X \backslash B_j) \quad &\Rightarrow \quad f_n(x) = 0 \ \text{ and } \ f_n(X \backslash U) = 1.
\end{aligned}
$$

Let $y \in X$ such that $F(y) \in B\left(F(x), \frac{1}{n}\right) \cap F(X)$.

$$
\begin{aligned}
\text{Then,} \quad d(F(x), F(y) < \tfrac{1}{n} \quad &\Rightarrow \quad f_n(y) \neq 1 \Rightarrow y \notin X \backslash U && \Rightarrow y \in U \\
&\Rightarrow \quad B\left(F(x), \tfrac{1}{n}\right) \cap F(X) \subset F(U) \\
&\Rightarrow \quad F(U) \ \text{ is a union of open sets} \\
&\Rightarrow \quad F(U) \ \text{ is an open set} \\
U \subset X \text{ open} \quad &\Rightarrow \quad F(U) \subset F(X) \ \text{ is open.}
\end{aligned}
$$



$\therefore \quad F : X \to F(X)$ is an open function.

(4) To show that F is a homeomorphism, we show that F^{-1} is a continuous function.

Let $O \subset X$ be an open set.

F open function $\Rightarrow$ $F(O) \subset F(X)$ is open

F onto $\Rightarrow$ $F(F^{-1}(F(O))) = F(O)$ by Prop.16

$\Rightarrow$ $F(F^{-1}(F(O)))$ is open

F one-to-one $\Rightarrow$ $F^{-1}(F(O)) = O$ by Prop.16

$\Rightarrow$ $F^{-1}(F(O))$ is open in X

$\Rightarrow$ F^{-1} is continuous.

$\therefore$ $F : X \to F(X) \subset \mathbb{H}$ is a homeomorphism by Definition 57.

By Definition 58, F is an embedding.

Observe: $F(X)$ is metrizable since it is a subspace of the metric space $\mathbb{H}$.

But, X is topologically equivalent to $F(X)$ (Def.57) implies that X is metrizable.

As an example, the metric on X can be derived from the metric on $F(X)$ as follows.

$F : X \to Y = F(X)$ where Y is metrizable, defined by $d_X(a,b) = d_Y(F(a), F(b))$.

$\therefore$ **The Urysohn Metrization Theorem:** *Every second countable regular space is metrizable* is a true statement.

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