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POTENTIAL THEORY FOR QUASILINIEAR ELLIPTIC EQUATIONS

A. BAALAL & A. BOUKRICHA

DEDICATED TO PROF. WOLFHARD HANSEN ON HIS 60TH BIRTHDAY

Abstract. We discuss the potential theory associated with the quasilinear elliptic equation

$$-\operatorname{div}(\mathcal{A}(x,\nabla u))+\mathcal{B}(x,u)=0.$$

We study the validity of Bauer convergence property, the Brelot convergence property. We discuss the validity of the Keller-Osserman property and the existence of Evans functions.

1. Introduction

This paper is devoted to a study of the quasilinear elliptic equation

$$-\operatorname{div}\left(\mathcal{A}(x,\nabla u)\right)+\mathcal{B}(x,u)=0\,, \tag{1.1}$$

where $\mathcal{A}:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ and $\mathcal{B}:\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying the structure conditions given in Assumptions (I), (A1), (A2), (A3), and (M) below. In particular we are interested in the potential theory, the degeneracy of the sheaf of continuous solutions and the existence of Evans functions for the equation (1.1).

Equation of the same type as (1.1) were investigated in earlier years in many interesting papers, [19, 20, 15, 18]. An axiomatic potential theory associated with the equation $\operatorname{div}(\mathcal{A}(x,\nabla u))=0$ was recently introduced and discussed in [10]. These axiomatic setting are illustrated by the study of the p-Laplace equation $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\,\nabla u)$ obtained by $\mathcal{A}(x,\xi)=|\xi|^{p-2}\xi$ for every $x\in\mathbb{R}^d$ and $\xi\in\mathbb{R}^d$. We have $\Delta_2=\Delta$ where Δ, the Laplace operator on $\mathbb{R}^d$.

Our paper is organized as follows: In the second section we introduce the basic notation. In the third section we present the structure conditions needed for the mappings $\mathcal{A}$ and $\mathcal{B}$ in order to consider the equation (1.1). We then use the variational inequality to prove the solvability of the variational Dirichlet problem related to (1.1). In section 4 we prove a comparison principle for supersolutions and subsolutions, existence and uniqueness of the Dirichlet problem related to the sheaf $\mathcal{H}$ of continuous solutions of (1.1), as well as the existence of a basis of regular sets

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stable by intersection. In the fifth section we discuss the potential theory associated with equation (1.1), prove that the harmonic sheaf $\mathcal{H}$ of solutions of (1.1) satisfies the Bauer convergence property, then introduce the presheaves of hyper-harmonic functions ${}^*\mathcal{H}$ and of hypoharmonic functions ${}_*\mathcal{H}$ and prove a comparison principle. In the sixth section we prove, using the obstacle problem, that ${}^*\mathcal{H}$ and ${}_*\mathcal{H}$ are sheaves. In the seventh section we study the degeneracy of the sheaf $\mathcal{H}$; we are not able to prove that the sheaf $\mathcal{H}$ is non degenerate even if we have the following Harnack inequality [19, 20, 18, 4]:

For every open domain U in $\mathbb{R}^d$ and every compact subset K of U there exists two non-negative constants c_1 and c_2 such that for every $h \in \mathcal{H}^+(U)$,

$$\sup_K h \leqslant c_1 \inf_K h + c_2 \,.$$

Let U be an open subset of $\mathbb{R}^d$, $d \geqslant 1$ and α a positive real number, let $0 < \epsilon < 1$ and b be a non-negative function in $L_{\mathrm{loc}}^{\frac{d}{p-\epsilon}}(\mathbb{R}^d)$. For every open U we consider the set $\mathcal{H}_\alpha(U)$ of all functions $u \in \mathcal{W}_{\mathrm{loc}}^{1,p}(U) \cap \mathcal{C}(U)$ which are solutions of the equation (1.1) with $\mathcal{B}(x,\zeta) = b(x)\operatorname{sgn}(\zeta)\left|\zeta\right|^\alpha$, then $(\mathbb{R}^d, \mathcal{H}_\alpha)$ is a nonlinear Bauer space. In particular $\mathcal{H}_\alpha$ is non degenerate on $\mathbb{R}^d$. For $\alpha < p-1$, the Harnack inequality and the Brelot convergence property are valid, but in contrast to the linear and quasilinear theory (see e.g. [10]) $(\mathbb{R}^d, \mathcal{H}_\alpha)$ is not elliptic in the sense of Definition 7.1. In the eighth section, we define, as in [5], regular Evans functions u tending to the infinity (or exploding) at the regular boundary points of U. We assume that $\mathcal{A}$ satisfies the following supplementary derivability and homogeneity conditions:

- For every $x_0 \in \mathbb{R}^d$, the function F from $\mathbb{R}^d$ to $\mathbb{R}^d$ defined by $F(x) = \mathcal{A}(x, x - x_0)$ is differentiable and $\operatorname{div} F$ is locally (essentially) bounded.
- $\mathcal{A}(x, \lambda\xi) = \lambda\left|\lambda\right|^{p-2}\mathcal{A}(x,\xi)$ for every $\lambda \in \mathbb{R}$ and every x, $\xi \in \mathbb{R}^d$.

These conditions are satisfied in the particular case of the p-Laplace operator with $p \geqslant 2$. We then prove that for every $\alpha > p-1$, the Keller-Osserman property in $(\mathbb{R}^d, \mathcal{H}_\alpha)$ is valid; i. e., every open ball admits a regular Evans function, which yields the validity of the Brelot convergence property. Among others, we prove for $\alpha > p-1$ a theorem of the Liouville type in the form $\mathcal{H}_\alpha(\mathbb{R}^d) = \{0\}$. Finally in the ninth section , we consider some applications of the previous results to the case of the p-Laplace operator, where we also prove the uniqueness of the regular Evans function for star domain and strict positive b and $\mathcal{H}_\alpha$ for $\alpha > p-1$.

Note that our methods are applicable to broader class of weighted equations (see [10]). The use of the constant weight $\equiv 1$ is only for sake of simplicity.

2. Notation

We introduce the basic notation which will be observed throughout this paper. $\mathbb{R}^d$ is the real Euclidean d-space, $d \geq 2$. For an open set U of $\mathbb{R}^d$ and an positive integer k, $\mathcal{C}^k(U)$ is the set of all k times continuously differentiable functions on an open set U. $\mathcal{C}^\infty(U) := \bigcap_{k\geq 1}\mathcal{C}^k(U)$ and $\mathcal{C}_c^\infty(U)$ the set of all functions in $\mathcal{C}^\infty(U)$ compactly supported by U. For a measurable set X, $\mathcal{B}(X)$ denotes the set of all Borel numerical functions on X and for $q \geq 1$, $L^q(X)$ is the $q^{th}-$power Lebesgue space defined on X. Given any set $\mathcal{Y}$ of functions $\mathcal{Y}_b$ ($\mathcal{Y}^+$ resp.) denote the set of all functions in $\mathcal{Y}$ which are bounded (positive resp.). $\mathcal{W}^{1,q}(U)$ is the $(1,q)$-Sobolev space on U. $\mathcal{W}_0^{1,q}(U)$ the closure of $\mathcal{C}_c^\infty(U)$ in $\mathcal{W}^{1,q}(U)$, relatively to its norm.

$\mathcal{W}^{-1,q'}(U)$ is the dual of $\mathcal{W}_0^{1,q}(U)$, $q' = q(q-1)^{-1}$. $u \wedge v$ (resp. $u \vee v$) is the infinimum (resp. the maximum) of u and v; $u^+ = u \vee 0$ and $u^- = u \wedge 0$.

3. Existence and Uniqueness of Solutions

Let Ω be a bounded open subset of $\mathbb{R}^d$ ($d \geqslant 1$). We will investigate the existence of solutions $u \in \mathcal{W}^{1,p}(\Omega)$, $1 < p \leqslant d$, of the variational Dirichlet problem associated with the quasilinear elliptic equation

$$-\operatorname{div}(\mathcal{A}(x, \nabla u)) + \mathcal{B}(x, u) = 0.$$

In this paper we suppose that the functions $\mathcal{A} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $\mathcal{B} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ are given Carathéodory functions and the following structure conditions are satisfied:

(I) $\zeta \to \mathcal{B}(x, \zeta)$ is increasing and $\mathcal{B}(x, 0) = 0$ for every $x \in \mathbb{R}^d$.

(A1) There exists $0 < \epsilon < 1$ such that for any $u \in L^\infty(\mathbb{R}^d)$,
$$\mathcal{B}(., u(.)) \in L_{\mathrm{loc}}^{\frac{d}{p-\epsilon}}(\mathbb{R}^d).$$

(A2) There exists $\nu > 0$ such that for every $\xi \in \mathbb{R}^d$,
$$|\mathcal{A}(x, \xi)| \leqslant \nu |\xi|^{p-1}.$$

(A3) There exists $\mu > 0$ such that for every $\xi \in \mathbb{R}^d$,
$$\mathcal{A}(x, \xi).\xi \geqslant \mu |\xi|^p.$$

(M) For all ξ, $\xi' \in \mathbb{R}^d$ with $\xi \neq \xi'$,
$$[\mathcal{A}(x, \xi) - \mathcal{A}(x, \xi')] \cdot (\xi - \xi') > 0.$$

We recall that assumptions (A2), (A3) and (M) are satisfied in the framework of [10] when the admissible weight is $\omega \equiv 1$.

Recall that $u \in \mathcal{W}_{\mathrm{loc}}^{1,p}(\Omega)$ is a *solution* of (1.1) in Ω provided that for all $\varphi \in \mathcal{W}_0^{1,p}(\Omega)$ and $\mathcal{B}(., u) \in L_{\mathrm{loc}}^{p^{*'}}(\Omega)$,

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx + \int_\Omega \mathcal{B}(x, u)\varphi dx = 0. \tag{3.1}$$

A function $u \in \mathcal{W}_{\mathrm{loc}}^{1,p}(\Omega)$ is termed *subsolutions* (resp. *supersolutions*) of (1.1) if for all non-negative functions $\varphi \in \mathcal{W}_0^{1,p}(\Omega)$ and $\mathcal{B}(., u) \in L_{\mathrm{loc}}^{p^{*'}}(\Omega)$,

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx + \int_\Omega \mathcal{B}(x, u)\varphi dx \leqslant 0 \quad (\text{resp. } \geqslant 0).$$

If u is a bounded subsolution (resp. bounded supersolution), then for every $k \geqslant 0$, $u - k$ (resp. $u + k$) is also subsolution (resp. supersolution) for (1.1).

For a positive constant M and $u \in L^p(\Omega)$, we define the truncated function

$$\tau_M(u)(x) = \begin{cases} -M & u(x) \leqslant -M \\ u(x) & -M < u(x) < M \\ M, & M \leqslant u(x) \end{cases}$$

(a.e. $x \in \Omega$). It is clear that the truncation mapping τ_M is bounded and continuous from $L^p(\Omega)$ to itself.

For $u \in \mathcal{W}^{1,p}(\Omega)$ and $\mathcal{B}(x, \tau_M(u)) \in L^{p^{*'}}_{\rm loc}(\Omega)$, we define $\mathcal{L}_M : \mathcal{W}^{1,p}(\Omega) \to \mathcal{W}^{-1,p'}(\Omega)$ as

$$\langle \mathcal{L}_M(u), \varphi\rangle := \int_\Omega \mathcal{A}(x, \nabla u)\cdot \nabla\varphi dx + \int_\Omega \mathcal{B}(x, \tau_M(u))\varphi dx\,, \quad \varphi \in \mathcal{W}^{1,p}_0(\Omega)$$

here $\langle .,.\rangle$ is the pairing between $\mathcal{W}^{-1,p'}(\Omega)$ and $\mathcal{W}^{1,p}(\Omega)$. It follows from Assumptions (A1), (A2), (A3), and the carathéodory conditions that $\mathcal{L}_M$ is well defined. We consider the variational inequality

$$\langle \mathcal{L}_M(u), v-u\rangle \geqslant 0, \quad \forall v \in \mathcal{K}, u \in \mathcal{K}, \tag{3.2}$$

where $\mathcal{K}$ is a given closed convex set in $\mathcal{W}^{1,p}(\not\leq)$ such that for given $f \in \mathcal{W}^{1,p}(\not\leq)$,

$$\mathcal{K} \subset f + \mathcal{W}^{1,p}_0(\not\leq).$$

Typical examples of closed convex sets $\mathcal{K}$ are as follows: for $f \in \mathcal{W}^{1,p}(\not\leq)$ and $\psi_1, \psi_2 : \Omega \to [-\infty, +\infty]$ let the convex set is

$$\mathcal{K}^f_{\psi_1,\psi_2} = \mathcal{K}^f_{\psi_1,\psi_2}(\Omega) = \left\{u \in \mathcal{W}^{1,p}(\Omega) : \psi_1 \le u \leqslant \psi_2 \text{ a.e. in } \Omega,\, u - f \in \mathcal{W}^{1,p}_0(\Omega)\right\}. \tag{3.3}$$

We write $\mathcal{K}^f_{\psi_1} = \mathcal{K}^f_{\psi_1,+\infty}(\Omega)$ and, if $f = \psi_1 \in \mathcal{W}^{1,p}(\Omega)$, $\mathcal{K}_f = \mathcal{K}^f_f$. A function u satisfying (3.2) with $M = +\infty$ and the closed convex sets $\mathcal{K}^f_{\psi_1}$ is called a *solution to the obstacle problem* in $\mathcal{K}^f_{\psi_1}$. For the notion of obstacle problem, the reader is referred to monograph [10, p. 60] or [18, Chap. 5]. We observe that any solution of the obstacle problem in $\mathcal{K}^f_{\psi_1}(\Omega)$ is always a supersolution of the equation (1.1) in Ω. Conversely, a supersolution u is always a solution to the obstacle problem in $\mathcal{K}^u_u(\omega)$ for all open $\omega \subset \overline{\omega} \subset \Omega$. Furthermore a solution u to equation (1.1) in an open set Ω is a solution to the obstacle problem in $\mathcal{K}^u_{-\infty}(\omega)$ for all open $\omega \subset \overline{\omega} \subset \Omega$. Similarly, a solution to the obstacle problem in $\mathcal{K}^u_{-\infty}(\Omega)$ is a solution to (1.1).

For the uniqueness of a solution to the obstacle problem we have following lemma [10, Lemma 3.22]:

Lemma 3.1. *Suppose that u is a solution to the obstacle problem in $\mathcal{K}^f_g(\Omega)$. If $v \in \mathcal{W}^{1,p}(\Omega)$ is a supersolution of (1.1) in Ω such that $u \wedge v \in \mathcal{K}^f_g(\Omega)$, then a.e. $u \leqslant v$ in Ω.*

Theorem 3.1. *Let ψ_1 and ψ_2 in $L^\infty(\not\leq)$, $f \in \mathcal{W}^{1,p}(\not\leq)$ and $\mathcal{K}^f_{\psi_1,\psi_2}$ as above assume that $\mathcal{K}^f_{\psi_1,\psi_2}$ is non empty. Then for every positive constant M, $\|\psi_1\|_\infty \vee \|\psi_2\|_\infty \leqslant M < +\infty$ the variational inequality* (3.2) *has a unique solution. Moreover, if $w \in \mathcal{W}^{1,p}(\Omega)$ is a supersolution (resp. subsolution) to the equation* (1.1) *such that $w \wedge u$ (resp. $w \vee u$) $\in \mathcal{K}^f_{\psi_1,\psi_2}$, then $u \leqslant w$ (resp. $w \leqslant u$).*

Proof. Let $\|\psi_1\|_\infty \vee \|\psi_2\|_\infty \leqslant M < +\infty$. If u, $v \in \mathcal{K}^f_{\psi_1,\psi_2}$ are solutions of (3.2), it follows from (**I**) and (**M**) that

$$\begin{aligned} 0 \quad \geqslant \quad & \int_\Omega [\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla v)]\cdot \nabla(v-u)dx \\ & + \int_\Omega [\mathcal{B}(x, \tau_M(u)) - \mathcal{B}(x, \tau_M(v))]\,(v-u)dx \\ = \quad & \langle \mathcal{L}_M(u) - \mathcal{L}_M(v), v-u\rangle \geqslant 0, \end{aligned}$$

then $v-u$ is constant on connected components of Ω. This, on the other hand, since $v-u\in\mathcal{W}_0^{1,p}(\Omega)$, implies that $v=u$.

To prove the existence we will use [12, Corollary III.1.8, p. 87]. Since $\mathcal{K}^f_{\psi_1,\psi_2}$ is a non empty closed convex subset of $\mathcal{W}^{1,p}(\Omega)$, it is enough to prove that $\mathcal{L}_M$ is monotone, coercive and weakly continuous on $\mathcal{K}^f_{\psi_1,\psi_2}$. We have

$$\begin{aligned}\langle\mathcal{L}_M(u)-\mathcal{L}_M(v),u-v\rangle &= \int_\Omega[\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla v)]\cdot\nabla(u-v)\,dx+\\ &\quad+\int_\Omega[\mathcal{B}(x,\tau_M(u))-\mathcal{B}(x,\tau_M(v))]\,.\,(u-v)\,dx\end{aligned}$$

for all v, $u\in\mathcal{K}^f_{\psi_1,\psi_2}$ and the structure conditions on $\mathcal{A}$ and $\mathcal{B}$ yield that $\mathcal{L}_M$ is monotone and coercive (for the definition of monotone or coercive operator the reader is referred to [14, 12]).

To show that $\mathcal{L}_M$ is weakly continuous on $\mathcal{K}^f_{\psi_1,\psi_2}$, let $(u_n)_n\subset\mathcal{K}^f_{\psi_1,\psi_2}$ be a sequence that converges to $u\in\mathcal{K}^f_{\psi_1,\psi_2}$. There is a subsequence $(u_{n_k})_k$ such that $u_{n_k}\to u$ and $\nabla u_{n_k}\to\nabla u$ pointwise a.e. in Ω. Since $\mathcal{A}$ and $\mathcal{B}$ are Carathéodory functions, $\mathcal{A}(.,\nabla u_{n_k})$ and $\mathcal{B}(.,\tau_M(u_{n_k}))$ converges in measure to $\mathcal{A}(.,\nabla u)$ and $\mathcal{B}(x,\tau_M(u))$ respectively [11]. Pick a subsequence, indexed also by n_k, such that $\mathcal{A}(.,\nabla u_{n_k})$ and $\mathcal{B}(.,\tau_M(u_{n_k}))$ converges pointwise a.e. in Ω to $\mathcal{A}(.,\nabla u)$ and $\mathcal{B}(x,\tau_M(u))$ respectively. Because $(u_{n_k})_{n_k}$ is bounded in $\mathcal{W}^{1,p}(\Omega)$, it follow that $(\mathcal{A}(.,\nabla u_{n_k}))_k$ is bounded in $\left(L^{\frac{p}{p-1}}(\Omega)\right)^d$ and that $\mathcal{A}(.,\nabla u_{n_k})\rightharpoonup\mathcal{A}(.,\nabla u)$ weakly in $\left(L^{\frac{p}{p-1}}(\Omega)\right)^d$. We have also $\mathcal{B}(.,\tau_M(u_{n_k}))\rightharpoonup\mathcal{B}(.,\tau_M(u))$ weakly in $L^{p^{*\prime}}(\Omega)$. Since the weak limits are independent of the choice of the subsequence, we have for all $\varphi\in\mathcal{W}_0^{1,p}(\Omega)$

$$\langle\mathcal{L}_M(u_n),\varphi\rangle\to\langle\mathcal{L}_M(u),\varphi\rangle$$

and hence $\mathcal{L}_M$ is weakly continuous on $\mathcal{K}^f_{\psi_1,\psi_2}$.

Let now $w\in\mathcal{W}^{1,p}(\Omega)$ be a supersolution of the equation (1.1) such that $u\wedge w\in\mathcal{K}^f_{\psi_1,\psi_2}$, then $u-(u\wedge w)\in\mathcal{W}_0^{1,p}(\Omega)$ and we have

$$\begin{aligned}0 &\leqslant \int_\Omega[\mathcal{A}(x,\nabla w)-\mathcal{A}(x,\nabla u)]\cdot\nabla(u-(u\wedge w))\,dx+\\ &\quad+\int_\Omega[\mathcal{B}(x,\tau_M(w))-\mathcal{B}(x,\tau_M(u))]\,.\,(u-(u\wedge w))\,dx\\ &= \int_{\{u>w\}}[\mathcal{A}(x,\nabla(u\wedge w))-\mathcal{A}(x,\nabla u)]\cdot\nabla(u-(u\wedge w))\,dx+\\ &\quad+\int_{\{u>w\}}[\mathcal{B}(x,\tau_M\,(u\wedge w))-\mathcal{B}(x,\tau_M(u))]\,.\,(u-(u\wedge w))\,dx\\ &\leqslant 0.\end{aligned}$$

It follow, by (**I**) and (**M**) , that $\nabla(u-(u\wedge w))=0$ a.e. in Ω and hence $u\leqslant w$ a.e. in Ω. The same proof is valid if w is a subsolution. $\square$

As an application of Theorem 3.1, we have the following two theorems.

Theorem 3.2. *Let $f\in\mathcal{W}^{1,p}(\nleqslant)\cap L^\infty(\nleqslant)$ and*

$$\mathcal{K}=\left\{u\in\mathcal{W}^{1,p}(\Omega):f\leq u\leqslant\ \|f\|_\infty\ a.\ e.,\ u-f\in\mathcal{W}_0^{1,p}(\Omega)\right\}.$$

Then there exists $u \in \mathcal{K}$ such that

$$\langle \mathcal{L}(u), v-u \rangle \geqslant 0 \quad \text{for all } v \in \mathcal{K}\,.$$

Moreover, u is a supersolution of (1.1) in Ω.

Proof. For $m>0$, by Theorem 3.1 there exists a unique function u_m in

$$\mathcal{K}^f_{f,\|f\|_\infty+m} = \left\{ u \in \mathcal{W}^{1,p}(\Omega) : f \leqslant u \leqslant \|f\|_\infty + m \text{ a. e., } u-f \in \mathcal{W}^{1,p}_0(\Omega) \right\}$$

such that

$$\left\langle \mathcal{L}_{\|f\|_\infty+m}(u_m), v-u_m \right\rangle \geqslant 0$$

for all $v \in \mathcal{K}^f_{f,\|f\|_\infty+m}$. Since $u_m - \|f\|_\infty = u_m - f + f - \|f\|_\infty \leqslant u_m - f$ and $(u_m-f)^+ \geqslant (u_m - \|f\|_\infty)^+$, we have $\eta := (u_m - \|f\|_\infty)^+ \in \mathcal{W}^{1,p}_0(\Omega)$ (see e. g. [10, Lemma1.25]). Moreover, since $u_m - \eta \in \mathcal{K}^f_{f,\|f\|_\infty+m}$ and $\|f\|_\infty$ is a supersolution of (1.1), we have

$$\begin{aligned}
0 &\leqslant -\int_\Omega \mathcal{A}(x,\nabla u_m)\cdot\nabla\eta dx - \int_\Omega \left[\mathcal{B}(x,u_m) - \mathcal{B}(x,\|f\|_\infty)\right]\eta dx \\
&= -\int_{\left\{u_m > \|f\|_\infty\right\}} \mathcal{A}(x,\nabla u_m)\cdot\nabla u_m dx + \\
&\quad -\int_{\left\{u_m > \|f\|_\infty\right\}} \left[\mathcal{B}(x,u_m) - \mathcal{B}(x,\|f\|_\infty)\right](u_m - \|f\|_\infty)\,dx \\
&\leqslant 0,
\end{aligned}$$

then $\nabla\eta = 0$ a.e. in Ω by (**M**). Because $\eta \in \mathcal{W}^{1,p}_0(\Omega)$, $\eta = 0$ a.e. in Ω. It follows that $u_m \leqslant \|f\|_\infty$ a.e. in Ω. It follows that $u_m \leqslant \|f\|_\infty$ a.e. in Ω, and therefore $f \leqslant u_m < \|f\|_\infty + m$ a.e. in Ω. Given a non-negative $\varphi \in \mathcal{C}^\infty_c(\Omega)$ and $\varepsilon > 0$ sufficiently small such that $u_m + \varepsilon\varphi \in \mathcal{K}^f_{f,\|f\|_\infty+m}$, consequently

$$\langle \mathcal{L}(u_m), \varphi \rangle \geqslant 0$$

which means that u_m is a supersolution of (1.1) in Ω. □

Theorem 3.3. *Let Ω be a bounded open set of $\mathbb{R}^d$, $f \in \mathcal{W}^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then there is a unique function $u \in \mathcal{W}^{1,p}(\Omega)$ with $u - f \in \mathcal{W}^{1,p}_0(\Omega)$ such that*

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi dx + \int_\Omega \mathcal{B}(x,u)\varphi dx = 0,$$

whenever $\varphi \in \mathcal{W}^{1,p}_0(\Omega)$.

Proof. For $m>0$, by Theorem 3.1, there exists a unique u_m in

$$\mathcal{K}_{f,m} := \left\{ u \in \mathcal{W}^{1,p}(\Omega) : |u| \leqslant \|f\|_\infty + m \text{ a. e., } u-f \in \mathcal{W}^{1,p}_0(\Omega) \right\},$$

such that

$$\left\langle \mathcal{L}_{\|f\|_\infty+m}(u_m), v-u_m \right\rangle \geqslant 0,$$

for all $v \in \mathcal{K}_{f,m}$. Since $u_m + \|f\|_\infty = u_m - f + f + \|f\|_\infty \geqslant u_m - f$ and $(u_m - f)^- \leqslant (u_m + \|f\|_\infty) \wedge 0$, we have $\eta := (u_m + \|f\|_\infty) \wedge 0 \in \mathcal{W}^{1,p}_0(\Omega)$ (see

e. g. [10, Lemma1.25]). Moreover, since $\eta + u_m \in \mathcal{K}_{f,m}$ and $-\|f\|_\infty$ is a subsolution of (1.1), we have

$$\begin{aligned} 0 &\leqslant \int_\Omega \mathcal{A}(x, \nabla u_m) \cdot \nabla \eta dx + \int_\Omega [\mathcal{B}(x, u_m) - \mathcal{B}(x, -\|f\|_\infty)] \eta dx \\ &= -\int_{\{u_m < -\|f\|_\infty\}} \mathcal{A}(x, \nabla u_m) \cdot \nabla u_m dx + \\ &\quad -\int_{\{u_m < -\|f\|_\infty\}} [\mathcal{B}(x, u_m) - \mathcal{B}(x, -\|f\|_\infty)] (u_m + \|f\|_\infty) dx \\ &\leqslant 0, \end{aligned}$$

then $\nabla \eta = 0$ a.e. in Ω by (**M**). Because $\eta \in \mathcal{W}_0^{1,p}(\Omega)$, $\eta = 0$ a.e. in Ω. It follows that $-\|f\|_\infty \leqslant u_m$ a.e. in Ω. Note that $-u_m$ is also a solution in $\mathcal{K}_{-f,m}$ of the following variational inequality

$$\begin{aligned} \left\langle \widetilde{\mathcal{L}}_{\|f\|_\infty + m}(u), v - u \right\rangle &= \int_\Omega \widetilde{\mathcal{A}}(x, \nabla u) \cdot \nabla (v - u) dx \\ &\quad + \int_\Omega \widetilde{\mathcal{B}}(x, \tau_{\|f\|_\infty + m}(u)) (v - u) dx \geqslant 0, \end{aligned}$$

where $\widetilde{\mathcal{A}}(., \xi) = -\mathcal{A}(., -\xi)$ and $\widetilde{\mathcal{B}}(., \zeta) = -\mathcal{B}(., -\zeta)$ which satisfy the same assumptions as $\mathcal{A}$ and $\mathcal{B}$. It follows that $u_m \leqslant \|f\|_\infty$ a.e. in Ω, and therefore $|u_m| < \|f\|_\infty + m$ a.e. in Ω. Given $\varphi \in \mathcal{C}_c^\infty(\Omega)$ and $\varepsilon > 0$ sufficiently small such that $u_m \pm \varepsilon\varphi \in \mathcal{K}_{f,m}$, consequently

$$\langle \mathcal{L}(u_m), \varphi \rangle = 0$$

which means that u_m is a desired function. □

By regularity theory (e.g. [18, Corollary 4.10]), any bounded solution of (1.1) can be redefined in a set of measure zero so that it becomes continuous.

Definition 3.1. A relatively compact open set U is called $p-$*regularity* if, for each function $f \in \mathcal{W}^{1,p}(U) \cap \mathcal{C}(\overline{U})$, the continuous solution u of (1.1) in U with $u - f \in \mathcal{W}^{1,p}(U)$ satisfies $\lim_{x \to y} u(x) = f(y)$ for all $y \in \partial U$.

A relatively compact open set U is called regular, if for every continuous function f on ∂U, there exists a unique continuous solution u of (1.1) on U such that $\lim_{x \to y} u(x) = f(y)$ for all $y \in \partial U$.

If U is p-regular and $f \in \mathcal{W}^{1,p}(U) \cap \mathcal{C}(\overline{U})$, then the solution u given by Theorem 3.3 satisfies

$$\lim_{x \in U, x \to z} u(x) = f(z)$$

for all $z \in \partial U$ [18, Corollary 4.18].

4. Comparison Principle and Dirichlet Problem

The following *comparison principle* is useful for the potential theory associated with equation (1.1):

Lemma 4.1. *Suppose that u is a supersolution and v is a subsolution on Ω such that*

$$\limsup_{x \to y} v(x) \leqslant \liminf_{x \to y} u(x)$$

for all $y \in \partial\Omega$ and if both sides of the inequality are not simultaneously $+\infty$ or $-\infty$, then $v \leqslant u$ in Ω.

Proof. By the regularity theory (see e.g. [18, Corollary 4.10]), we may assume that u is lower semicontinuous and v is upper semicontinuous on Ω. For fixed $\varepsilon > 0$, the set $K_\varepsilon = \{x \in \Omega : v(x) \geqslant u(x) + \varepsilon\}$ is a compact subset of Ω and therefore $\varphi = (v - u - \varepsilon)^+ \in \mathcal{W}_0^{1,p}(\mathbb{R}^d)$. Testing by φ, we obtain

$$\int_{\{v>u+\varepsilon\}} [\mathcal{A}(x, \nabla(u+\varepsilon)) - \mathcal{A}(x, \nabla v)] \cdot \nabla\varphi dx + \int_{\{v>u+\varepsilon\}} [\mathcal{B}(x, u+\varepsilon) - \mathcal{B}(x, v)]\, \varphi dx \geqslant 0 \quad (4.1)$$

Using Assumptions (I) and (M) we have

$$\int_{\{v>u+\varepsilon\}} [\mathcal{A}(x, \nabla u + \varepsilon) - \mathcal{A}(x, \nabla v)] \cdot \nabla(v - u - \varepsilon) dx = 0$$

and again by M we infer that $v \leqslant u + \varepsilon$ on Ω. Letting $\varepsilon \to 0$ we have $v \leqslant u$ on Ω. $\square$

Theorem 4.1. *Every p-regular set is regular in the sense of definition 3.1.*

Proof. Let Ω be a p-regular set in $\mathbb{R}^d$ and f be a continuous function on $\partial\Omega$. We shall prove that there exists a unique continuous solution u of (1.1) on Ω such that $\lim_{x\to y} u(x) = f(y)$ for all $y \in \partial\Omega$. The uniqueness is given by Lemma 4.1. By [18, Theorem 4.11] we have the continuity of u. For the existence, we may suppose that $f \in \mathcal{C}_c(\mathbb{R}^d)$ (Tietze's extension theorem). Let f_i be a sequence of functions from $\mathcal{C}_c^1(\mathbb{R}^d)$ such that $|f_i - f| \leqslant 2^{-i}$ and $|f_i| + |f| \leqslant M$ on $\overline{\Omega}$ for the same constant M and for all i. Let $u_i \in \mathcal{W}^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be the unique solution for the Dirichlet problem with boundary data f_i (Theorem 3.3). Then from Lemma 4.1 we deduce that $|u_i - u_j| \leqslant 2^{-i} + 2^{-j}$ and $|u_i| \leqslant M$ on Ω for all i and j. We denote by u the limit of the sequence $(u_i)_i$. We will show that u is a local solution of the equation. For this, we prove that the sequence $(\nabla u_i)_i$ is locally uniformly bounded in $(L^p(\Omega))^d$. Let $\varphi = -\eta^p u_i$, $\eta \in \mathcal{C}_c^\infty(\Omega)$, $0 \leqslant \eta \leqslant 1$ and $\eta = 1$ on $\omega \subset \overline{\omega} \subset \Omega$. Since $\varphi \in \mathcal{W}_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
0 &= \int_\Omega \mathcal{A}(x, \nabla u_i) \cdot \nabla\varphi dx + \int_\Omega \mathcal{B}(x, u_i)\varphi dx \\
&= \int_\Omega \mathcal{A}(x, \nabla u_i).(-\eta^p \nabla u_i - p u_i \eta^{p-1} \nabla\eta) dx - \int_\Omega \eta^p \mathcal{B}(x, u_i) u_i dx \\
&\leqslant -\mu \int_\Omega \eta^p |\nabla u_i|^p dx + p\nu \int_\Omega \eta^{p-1} |\nabla u_i|^{p-1} |u_i| |\nabla\eta| dx + C(M, \|\eta\|_\infty, |\Omega|),
\end{aligned}$$

and therefore, using the Young inequality, we obtain

$$\begin{aligned}
&\int_\Omega \eta^p |\nabla u_i|^p dx \\
&\leqslant p\frac{\varepsilon^{p'}\nu}{\mu} \int_\Omega \eta^p |\nabla u_i|^p dx + p\frac{\nu}{\varepsilon^p \mu} \int_\Omega |u_i|^p |\nabla\eta|^p dx + C(M, \|\eta\|_\infty, |\Omega|) \\
&\leqslant p\frac{\varepsilon^{p'}\nu}{\mu} \int_\Omega \eta^p |\nabla u_i|^p dx + C(M, \|\eta\|_\infty, |\Omega|, \|\nabla\eta\|_\infty, \varepsilon).
\end{aligned}$$

If $0<\varepsilon<\left(\frac{c_1}{pa_1}\right)^{\frac{p-1}{p}}$, then

$$\int_\omega |\nabla u_i|^p\,dx \leqslant \frac{\mu C(M,\|\eta\|_\infty,|\Omega|,\|\nabla\eta\|_\infty,\varepsilon)}{\mu-p\varepsilon^{p'}\nu} \text{ for all } i.$$

It follows that the sequence $(u_i)_i$ is locally uniformly bounded in $\mathcal{W}^{1,p}(\Omega)$. Fix $D \Subset G \Subset \Omega$. Since $(u_i)_i$ converges pointwise to u and by [10, Theorem 1.32], we obtain that $u\in\mathcal{W}^{1,p}(D)$ and $(u_i)_i$ converges weakly, in $\mathcal{W}^{1,p}(D)$, to u. Let $\eta\in\mathcal{C}_0^\infty(G)$ such that $0\leqslant\eta\leqslant 1$, $\eta=1$ in D and testing by $\varphi=\eta(u-u_i)$ for the solution u_i, we have

$$\begin{aligned}
&-\int_G \eta\mathcal{A}(x,\nabla u_i)\cdot\nabla(u-u_i)dx\\
&= \int_G (u-u_i)\mathcal{A}(x,\nabla u_i)\cdot\nabla\eta dx+\int_G \eta\mathcal{B}(x,u_i)(u-u_i)dx\\
&\leqslant \Big(\int_G |u-u_i|^p\,dx\Big)^{1/p}\Big[C+\nu\Big(\int_G|\nabla u_i|^p\,dx\Big)^{\frac{p-1}{p}}\Big]\\
&\leqslant C\Big(\int_G |u-u_i|^p\,dx\Big)^{1/p}.
\end{aligned}$$

Since

$$\begin{aligned}
0 &\leqslant \int_D[\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla u_i)]\cdot\nabla(u-u_i)dx\\
&\leqslant \int_G \eta\mathcal{A}(x,\nabla u)\cdot\nabla(u-u_i)dx + C\Big(\int_G|u-u_i|^p\,dx\Big)^{1/p}
\end{aligned}$$

and the weak convergence of $(\nabla u_i)_i$ to ∇u implies that

$$\lim_{i\to\infty}\int_G \eta\mathcal{A}(x,\nabla u)\cdot\nabla(u-u_i)dx=0,$$

we conclude

$$\lim_{i\to\infty}\int_D[\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla u_i)]\cdot\nabla(u-u_i)dx=0.$$

Now [10, Lemma 3.73] implies that $\mathcal{A}(x,\nabla u_i)$ converges to $\mathcal{A}(x,\nabla u)$ weakly in $\left(L^{p'}(D)\right)^n$.

Let $\psi\in\mathcal{C}_0^\infty(G)$. By the continuity in measure of the Carathéodory function $\mathcal{B}(x,z)$ [11] and by using the domination convergence theorem (in measure), we have

$$\lim_{i\to\infty}\int_\Omega \mathcal{B}(x,u_i)\psi dx=\int_\Omega\mathcal{B}(x,u)\psi dx.$$

Finally we obtain

$$\begin{aligned}
0 &= \lim_{i\to\infty}\Big[\int_\Omega \mathcal{A}(x,\nabla u_i)\cdot\nabla\psi dx+\int_\Omega\mathcal{B}(x,u_i)\psi dx\Big]\\
&= \int_\Omega\mathcal{A}(x,\nabla u)\cdot\nabla\psi dx+\int_\Omega\mathcal{B}(x,u)\psi dx.
\end{aligned}$$

By an application of [18, Corollay 4.18] for each u_i we obtain

$$\lim_{x\in\Omega,x\to z} u_i(x)=f_i(z)$$

for all $z \in \partial\Omega$. From the following estimation, of u on all Ω,

$$u_i - 2^{-i} \leqslant u \leqslant u_i + 2^{-i} \text{ for all } i$$

we deduce that for all i

$$f_i(z) - 2^{-i} \leqslant \liminf_{\substack{x \to z \\ x \in \Omega}} u(z) \leqslant \limsup_{\substack{x \to z \\ x \in \Omega}} u(z) \leqslant f_i(z) + 2^{-i}.$$

Letting $i \to \infty$ we obtain

$$\lim_{x \to z} u(x) = f(z)$$

for all $z \in \partial\Omega$ which finishes the proof. □

Corollary 4.1. *There exists a basis $\mathcal{V}$ of regular sets which is stable by intersection i.e. for every U and V in $\mathcal{V}$, we have $U \cap V \in \mathcal{V}$.*

The proof of this corollary can be found in Theorem 4.1 and [10, Corollary 6.32].

For every open set V and for every $f \in \mathcal{C}(\partial V)$ we shall denote by $\mathrm{H}_V f$ the *solution of the Dirichlet problem* for the equation (1.1) on V with the boundary data f.

5. Nonlinear Potential Theory associated with the equation (1.1)

For every open set U we shall denote by $\mathcal{U}(U)$ the set of all relatively compact open, regular subset V in U with $\overline{V} \subset U$.

By previous section and in order to obtain an axiomatic nonlinear potential theory, we shall investigate the harmonic sheaf associated with (1.1) and defined as follows: For every open subset U of $\mathbb{R}^d$ ($d \geqslant 1$), we set

$$\begin{aligned} \mathcal{H}(U) &= \big\{u \in \mathcal{C}(U) \cap \mathcal{W}^{1,p}_{\mathrm{loc}}(U) : u \text{ is a solution of (1.1)}\big\} \\ &= \big\{u \in \mathcal{C}(U) : \mathrm{H}_V u = u \text{ for every } V \in \mathcal{U}(U)\big\}. \end{aligned}$$

Element in the set $\mathcal{H}(U)$ are called *harmonic* on U.

We recall (see [4]) that $(X, \mathcal{H})$ satisfies the *Bauer convergence property* if for every subset U of X and every monotone sequence $(h_n)_n$ in $\mathcal{H}(U)$, we have $h = \lim_{n\to\infty} h_n \in \mathcal{H}(U)$ if it is locally bounded.

Proposition 5.1. *Let be U an open subset of $\mathbb{R}^d$. Then every family $\mathcal{F} \subset \mathcal{H}(U)$ of locally uniformly bounded harmonic functions is equicontinuous.*

Proof. Let $V \subset \overline{V} \subset U$ and a family $\mathcal{F} \subset \mathcal{H}(U)$ of locally uniformly bounded harmonic functions. Then $\sup\big\{|u(x)| : x \in \overline{V} \text{ and } u \in \mathcal{F}\big\} < \infty$ and by [18], is equicontinuous on $\overline{V}$. □

Corollary 5.1. *We have the Bauer convergence properties and moreover every locally bounded family of harmonic functions on an open set is relatively compact.*

Proof. Let U be an open set and $\mathcal{F}$ a locally bounded subfamily of $\mathcal{H}(U)$. By Proposition 5.1, there exist a sequence $(u_n)_n$ in $\mathcal{F}$ which converge to u on U locally uniformly. Let now $V \in \mathcal{U}(U)$. For every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $u - \varepsilon \leqslant u_n \leqslant u + \varepsilon$ for every $n \geqslant n_0$. The comparison principle yields therefore $(\mathrm{H}_V u) - \varepsilon \leqslant u_n \leqslant (\mathrm{H}_V u) + \varepsilon$, thus $(\mathrm{H}_V u) - \varepsilon \leqslant u \leqslant (\mathrm{H}_V u) + \varepsilon$. Letting $\varepsilon \to 0$, we get $u = \mathrm{H}_V u$. □

Proposition 5.2. [4] *Let V a regular subset of $\mathbb{R}^d$ and let $(f_n)_n$ and f in $\mathcal{C}(\partial V)$ such that $(f_n)_n$ is a monotone sequence converging to f. Then* $\sup_n \mathrm{H}_V f_n$ *converge to* $\mathrm{H}_V f$.

Proof. Let V a regular subset of $\mathbb{R}^d$ and let $(f_n)_n$ and f in $\mathcal{C}(\partial V)$ such that $(f_n)_n$ is increasing to f. Then, by Lemma 4.1, we have

$$\sup_n \mathrm{H}_V f_n \leqslant \mathrm{H}_V f$$

and, by Corollary 5.1 $\sup_n \mathrm{H}_V f_n \in \mathcal{H}(V)$. Moreover, For every n and every $z \in \partial V$ we have

$$f_n(z) \leq \liminf_{x\to z}(\sup_n \mathrm{H}_V f_n(x)) \leq \limsup_{x\to z}(\sup_n \mathrm{H}_V f_n(x)) \leqslant f(z).$$

Letting n tend to infinity we obtain that

$$f(z) = \lim_{x\to z}(\sup_n \mathrm{H}_V f_n)(x).$$

By Lemma 4.1, this shows that in fact $\mathrm{H}_V f = \sup_n \mathrm{H}_V f_n$. An analogous proof can be given if $(f_n)_n$ is decreasing.

□

Corollary 5.2. [4] *Let V be a regular subset of $\mathbb{R}^d$ and $(f_n)_n$ and $(g_n)_n$ to sequences in $\mathcal{C}(\partial V)$ which are monotone in the same sense such that* $\lim_n f_n = \lim_n g_n$. *Then* $\lim_n \mathrm{H}_V f_n = \lim_n \mathrm{H}_V g_n$.

Proof. We assume without loss the generality that (f_n) and (g_n) are both increasing. Obviously, $\mathrm{H}_V(g_n \wedge f_m) \leqslant \mathrm{H}_V g_n$ for every n and m in $\mathbb{N}$, hence $\sup_n \mathrm{H}_V(g_n \wedge f_m) \leqslant \sup_n \mathrm{H}_V g_n$ for every m. Since the sequence $(g_n \wedge f_m)_n$ is increasing to f_m, the previous proposition implies that $\mathrm{H}_V f_m \leqslant \sup_n \mathrm{H}_V g_n$. We then have $\sup_n \mathrm{H}_V f_n \leqslant \sup_n \mathrm{H}_V g_n$. Permuting (f_n) and (g_n) we obtain the converse inequality. □

Let V be a regular subset of $\mathbb{R}^d$. For every lower bounded and lower semicontinuous function v on ∂V we define the set

$$\mathrm{H}_V v = \sup_n \left\{\mathrm{H}_V f_n : (f_n)_n \text{ in } \mathcal{C}(\partial V) \text{ and increasing to } v\right\}.$$

For every upper bounded and upper semicontinuous function u on ∂V we define

$$\mathrm{H}_V u = \inf_n \left\{\mathrm{H}_V f_n : (f_n)_n \text{ in } \mathcal{C}(\partial V) \text{ and decreasing to } u\right\}.$$

Let be U an open set of $\mathbb{R}^d$. A lower semicontinuous and locally lower bounded function u from U to $\overline{\mathbb{R}}$ is termed *hyperharmonic* on U if $\mathrm{H}_V u \leqslant u$ on V for all V in $\mathcal{U}(U)$. A upper semicontinuous and locally upper bounded function v from U to $\overline{\mathbb{R}}$ is termed *hypoharmonic* on U if $\mathrm{H}_V u \geqslant u$ on V for all V in $\mathcal{U}(U)$. We will denote by ${}^*\mathcal{H}(U)$ (resp. ${}_*\mathcal{H}(U)$) the set of all hyperharmonic (resp. hypoharmonic) functions on U.

For $u \in {}^*\mathcal{H}(U)$, $v \in {}_*\mathcal{H}(U)$ and $k \geqslant 0$ we have $u + k \in {}^*\mathcal{H}(U)$ and $v - k \in {}_*\mathcal{H}(U)$. Indeed, let $V \in \mathcal{U}(U)$ and a continuous function such that $g \leqslant u + k$ on ∂V, then $\mathrm{H}_V(g-k) \leqslant \mathrm{H}_V u \leqslant u$. Since $(\mathrm{H}_V g) - k \leqslant \mathrm{H}_V(g-k)$, we therefore get $\mathrm{H}_V g \leqslant u + k$ and thus $u + k \in {}^*\mathcal{H}(U)$.

We have the following comparison principle:

Lemma 5.1. *Suppose that u is hyperharmonic and v is hypoharmonic on an open set U. If*

$$\limsup_{U\ni x\to y} v(x) \leqslant \liminf_{U\ni x\to y} u(x)$$

for all $y \in \partial U$ and if both sides of the previous inequality are not simultaneously $+\infty$ or $-\infty$, then $v \leqslant u$ in U.

The proof is the same as in [10, p. 133].

6. Sheaf Property for Hyperharmonic and Hypoharmonic Functions

For open subsets U of $\mathbb{R}^d$, we denote by $\overline{\mathcal{S}}(U)$ (resp. by $\underline{\mathcal{S}}(U)$) the set of all supersolutions (resp. subsolutions) of the equation (1.1) on U.

Recall that a map $\mathfrak{F}$ which to each open subset U of $\mathbb{R}^d$ assigns a subset $\mathfrak{F}(U)$ of $\mathfrak{B}(U)$ is called sheaf if we have the following two properties:
(*Presheaf Property*) For every two open subsets U, V of $\mathbb{R}^d$ such that $U \subset V$, $\mathfrak{F}(V)_{|U} \subset \mathfrak{F}(U)$
(*Localization Property*) For any family $(U_i)_{i\in I}$ of open subsets and any numerical function h on $U = \bigcup_{i\in I} U_i$, $h \in \mathfrak{F}(U)$ if $h_{|U_i} \in \mathfrak{F}(U_i)$ for every $i \in I$.

An easy verification gives that $\overline{\mathcal{S}}$ and $\underline{\mathcal{S}}$ are sheaves. Furthermore, we have the following results which generalize many earlier [17, 2, 7, 10].

Theorem 6.1. *Let U be a non empty open subset in $\mathbb{R}^d$ and $u \in {}^*\mathcal{H}(U) \cap \mathfrak{B}_b(U)$. Then u is a supersolution on U.*

Proof. First, we shall prove that for every open $O \subset \overline{O} \subset U$, there exists an increasing sequence $(u_i)_i$ in in O of supersolutions such that $u = \lim_{i\to\infty} u_i$ on O. Let $(\varphi_i)_i$ be an increasing sequence in $\mathcal{C}_c^\infty(U)$ such that $u = \sup_i \varphi_i$ on O. Let u_i be the solution of the obstacle problem in the non empty convex set

$$\mathcal{K}_i := \Big\{ v \in \mathcal{W}^{1,p}(O) : \varphi_i \leqslant v \leqslant \|\varphi_i\|_\infty + \|\varphi_{i+1}\|_\infty \text{ and } v - \varphi_i \in \mathcal{W}_0^{1,p}(O) \Big\}.$$

The existence and the uniqueness are given respectively by Theorem 3.1; moreover is a supersolution (Theorem3.2). Since u_{i+1} is a supersolution and $u_i \wedge u_{i+1} \in \mathcal{K}_i$, we have $u_i \leqslant u_{i+1}$ in O.We have to prove that the sequence $(u_i)_i$ is increasing to u. Let x_0 be an element of the open subset $G_i := \{x \in O : \varphi_i(x) < u_i(x)\}$ and ω be a domain such that $x_0 \in \omega \subset \overline{\omega} \subset G_i$. Since for every $\psi \in \mathcal{C}_c^\infty(\omega)$ and for sufficiently small $|\varepsilon|$ $u_i \pm \varepsilon\psi \in \mathcal{K}_i$,

$$\int_\omega \mathcal{A}(x, \nabla u_i) \cdot \nabla \psi dx + \int_\omega \mathcal{B}(x, u_i)\psi dx = 0\,.$$

Then u_i is a solution of the equation (1.1) on ω and by the sheaf property of $\mathcal{H}$, u_i is a solution of the equation (1.1) on G_i. Now the comparison principle implies that $u_i \leqslant u$ on G_i, hence $\varphi_i \leqslant u_i \leqslant u$ on O and therefore $u = \sup_i u_i$. Finally, the boundedness of the sequence $(u_i)_i$ and the same techniques in the proof of Theorem 4.1 yield that $(u_i)_i$ is locally bounded in $\mathcal{W}^{1,p}(O)$ and that u is a supersolution of the equation (1.1) in O. □

Corollary 6.1. *Let U be a non empty open subset in $\mathbb{R}^d$ and $u \in \mathcal{W}_{loc}^{1,p}(U) \cap {}^*\mathcal{H}(U)$. Then u is a supersolution on U. Moreover the infinimum of two supersolutions is also a supersolution.*

Proof. Let $u \in \mathcal{W}^{1,p}_{\mathrm{loc}}(U) \cap {}^*\mathcal{H}(U)$. The Theorem 6.1 implies that $u \wedge n$ is a supersolution for all $n \in \mathbb{N}$, consequently we have for every positive $\varphi \in \mathcal{C}_c^\infty(U)$

$$\begin{aligned} 0 &\leqslant \int_U \mathcal{A}(x, \nabla (u \wedge n)) \cdot \nabla \varphi dx + \int_U \mathcal{B}(x, u \wedge n) \varphi dx \\ &= \int_{\{u<n\}} \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx + \int_U \mathcal{B}(x, u \wedge n) \varphi dx. \end{aligned}$$

Letting $n \to +\infty$ we obtain

$$0 \leqslant \int_U \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx + \int_U \mathcal{B}(x, u) \varphi dx$$

for all positive $\varphi \in \mathcal{C}_c^\infty(U)$, thus u is a supersolution. Moreover, if u and v are two supersolutions then $u \wedge v \in \mathcal{W}^{1,p}_{\mathrm{loc}}(U) \cap {}^*\mathcal{H}(U)$ so $u \wedge v$ is a supersolution. □

Theorem 6.2. *${}^*\mathcal{H}$ is a sheaf.*

Proof. Let $(U_i)_{i \in I}$ be a family of open subsets of $\mathbb{R}^d$, $U = \bigcup_{i \in I} U_i$ and $h \in {}^*\mathcal{H}(U_i)$ for every $i \in I$. Then by the definition of hyperharmonic function, we have $h \wedge n \in {}^*\mathcal{H}(U_i)$ for every $(i, n) \in I \times \mathbb{N}$ and by Theorem 6.1, $h \wedge n$ is a supersolution on each U_i. Since $\overline{\mathcal{S}}$ is a sheaf, we get $h \wedge n \in \overline{\mathcal{S}}(U) \subset {}^*\mathcal{H}(U)$. Thus $h = \sup_n h \wedge n \in {}^*\mathcal{H}(U)$ and ${}^*\mathcal{H}$ is a sheaf. □

Remark 6.1. For every open subset U of $\mathbb{R}^d$, let $\widetilde{\mathcal{H}}(U)$ denote the set of all $u \in \mathcal{W}^{1,p}(U) \cap \mathcal{C}(U)$ such that $\widetilde{\mathcal{B}}(x, u) \in L^{p^{*'}}_{\mathrm{loc}}(U)$ and

$$\int_U \mathcal{A}(x, \nabla u) \cdot \nabla \varphi dx + \int_U \widetilde{\mathcal{B}}(x, u) \varphi dx = 0$$

for every $\varphi \in \mathcal{W}^{1,p}_0(U)$, where $\widetilde{\mathcal{B}}(x, \zeta) = -\widetilde{\mathcal{B}}(x, -\zeta)$. It is easy to see that the mapping $\zeta \to \widetilde{\mathcal{B}}(x, \zeta)$ is increasing and that $u \in \mathcal{H}(U)$ if and only if $-u \in \widetilde{\mathcal{H}}(U)$. Furthermore $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ have the same regular sets and for every $V \in \mathcal{U}(U)$ and $f \in \mathcal{C}(\partial V)$ we have $\mathrm{H}_V f = -\widetilde{\mathrm{H}}_V(-f)$. It follows that $u \in {}_*\mathcal{H}(U)$ if and only if $-u \in {}^*\widetilde{\mathcal{H}}(U)$ and therefore ${}_*\mathcal{H}$ is a sheaf.

7. The degeneracy of the sheaf $\mathcal{H}$

As in the previous section we consider the sheaf $\mathcal{H}$ defined by (1.1). Recall that the *Harnack inequality* or the *Harnack principle* is satisfied by $\mathcal{H}$ if for every domain U of $\mathbb{R}^d$ and every compact subset K in U, there exists two constants $c_1 \geqslant 0$ and $c_2 \geqslant 0$ such that for every $h \in \mathcal{H}^+(U)$,

$$\sup_{x \in K} h(x) \leqslant c_1 \inf_{x \in K} h(x) + c_2 \tag{HI}$$

We remark that, if for every $\lambda > 0$ and $h \in \mathcal{H}^+(U)$ we have $\lambda h \in \mathcal{H}^+(U)$, then we can choose $c_2 = 0$ and we obtain the classical Harnack inequality.

The Harnack inequality, for quasilinear elliptic equation, is proved in the fundamental tools of Serrin [19], see also [20, 13]. For the linear case see [9, 3, 1, 8].

In the rest of this section, we assume that $\mathcal{B}$ satisfy the following supplementary condition.

(∗) There exists $b \in L^{\frac{d}{p-\epsilon}}_{\mathrm{loc}}(\mathbb{R}^d)$, $0 < \epsilon < 1$, such that $|\mathcal{B}(x, \zeta)| \leqslant b(x) |\zeta|^\alpha$ for every $x \in \mathbb{R}^d$ and $\zeta \in \mathbb{R}$.

Small powers $(0 < \alpha < p-1)$**.** We have the validity of Harnack principle given by the following proposition.

Proposition 7.1. *Let $\mathcal{H}$ be the sheaf of the continuous solutions of the equation* (1.1)*. Assume that the condition* $(*)$ *is satisfies with* $0 < \alpha < p-1$*. Then the Harnack principle is satisfied by* $\mathcal{H}$*.*

The proof of this proposition can be found in [18, p. 178] or [19]

Definition 7.1. The sheaf $\mathcal{H}$ is called elliptic if for every regular domain V in $\mathbb{R}^d$, $x \in V$ and $f \in \mathcal{C}^+(\partial V)$, $\mathrm{H}_V f(x) = 0$ if and only if $f = 0$.

In the following example, we have the Harnack inequality but not the ellipticity. This is in contrast to the linear theory or quasilinear setting of nonlinear potential theory given by the $\mathcal{A}$-harmonic functions in [10].

Example 7.1. We assume that $\mathcal{B}(x,\zeta) = \operatorname{sgn}(\zeta)\,|\zeta|^{\alpha}$ with $0 < \alpha < p-1$ and $\mathcal{A}(x,\xi) = |\xi|^{p-2}\,\xi$. Let $u = cr^{\beta}$ with $\beta = p(p-1-\alpha)^{-1}$ and

$$c = p^{\frac{p-1}{p-1-\alpha}}\,(p-1-\alpha)^{\frac{p}{p-1-\alpha}}\,[d(p-1-\alpha)+\alpha p]^{\frac{1}{p-1-\alpha}}\,.$$

With an easy verification, we will find that for every $x_0 \in \mathbb{R}^d$ and ball $\mathrm{B}(x_0,\rho)$, there exists a solution u (in the form $c\,\|x-x_0\|^{\beta}$) on $\mathrm{B}(x_0,\rho)$ such that $\Delta_p u = u^{\alpha}$ with $u(x_0) = 0$ and $u(x) > 0$ for every $x \in \mathrm{B}(x_0,\rho)\setminus\{x_0\}$. We therefore obtain that the sheaf $\mathcal{H}$ is not elliptic and curiously we have the existence of a basis of regular set $\mathcal{V}$ such that for every $V \in \mathcal{V}$, there exist $x_0 \in V$ and $f \in \mathcal{C}(\partial V)$ with $f > 0$ on ∂V and $\mathrm{H}_V f(x_0) = 0$.

We will prove that the sheaf given in the previous example is non-degenerate in the following sense:

Definition 7.2. A sheaf $\mathcal{H}$ is called non-degenerate on an open U if for every $x \in U$, there exists a neighborhood V of x and $h \in \mathcal{H}(V)$ with $h(x) \neq 0$.

Proposition 7.2. *Assume that the condition* $(*)$ *is satisfies with* $0 < \alpha < p-1$ *and* $\mathcal{A}(x,\lambda\xi) = \lambda\,|\lambda|^{p-2}\,\mathcal{A}(x,\xi)$ *for all* $x, \xi \in \mathbb{R}^d$ *and for all* $\lambda \in \mathbb{R}$*. Then the sheaf* $\mathcal{H}$ *is non degenerate and more we have: for every regular set* V *and* $x \in V$*,* $\sup_{h\in\mathcal{H}(V)} h(x) = +\infty$*.*

Proof. It is sufficient to prove that for every $x_0 \in \mathbb{R}^d$, $\rho > 0$, $n \in \mathbb{N}$ and $u_n = \mathrm{H}_{\mathrm{B}(x_0,\rho)} n$ we have u_n converges to infinity at any point of $\mathrm{B}(x_0,\rho)$. The comparison principle yields that $0 \leqslant u_n \leqslant n$ on $\mathrm{B}(x_0,\rho)$. Put $u_n = nv_n$, we then obtain:

$$\int \mathcal{A}(x,\nabla v_n)\nabla\varphi dx + n^{1-p}\int \mathcal{B}(x,nv_n)\varphi dx = 0$$

for every $\varphi \in \mathcal{C}_c^{\infty}(\mathrm{B}(x_0,\rho))$ and for every $n \in \mathbb{N}^*$. The assumptions on $\mathcal{B}$ yields

$$\lim_{n\to\infty}\int \mathcal{A}(x,\nabla v_n)\nabla\varphi dx = 0;$$

since $0 \leqslant v_n \leqslant 1$, we have

$$\left|n^{1-p}\mathcal{B}(x,nv_n)\right| \leqslant n^{\alpha-p+1}b(x) \leqslant b(x)$$

and by [18, Theorem 4.19], v_n are equicontinuous on the closure $\overline{\mathrm{B}}_{x_0,\rho}$ of the ball B (x_0,ρ), then by the Ascoli's theorem, $(v_n)_n$ admits a subsequence which is uniformly convergent on $\overline{\mathrm{B}}_{x_0,\rho}$ to a continuous function v on $\overline{\mathrm{B}}_{x_0,\rho}$. Further we can easily verify that $v \in \mathcal{W}^{1,p}_{\mathrm{loc}}(\mathrm{B}(x_0,\rho))$ and

$$\int \mathcal{A}(x,\nabla v)\nabla\varphi dx = 0$$

for every $\varphi \in \mathcal{W}^{1,p}_0(\mathrm{B}(x_0,\rho))$. Since $v = 1$ on $\partial\mathrm{B}(x_0,\rho)$, $v = 1$ on $\overline{\mathrm{B}}_{x_0,\rho}$. The relation $u_n = nv_n$ yields the desired result. □

Big Powers $(\alpha \geqslant p-1)$**.** We shall investigate (1.1) in the case $\alpha \geqslant p-1$. Let $\mathcal{H}$ be the sheaf of the continuous solutions of (1.1). In [18] or [19], we find the following form of the Harnack inequality.

Theorem 7.1. *Assume that the condition* $(*)$ *is satisfies with* $\alpha \geqslant p-1$. *Then For every non empty open set* U *in* $\mathbb{R}^d$, *for every constant* $M > 0$ *and every compact* K *in* U, *there exists a constant* $C = C(K,M) > 0$ *such that for every* $u \in \mathcal{H}^+(U)$ *with* $u \leqslant M$,

$$\sup_K u \leqslant C \inf_K u\,.$$

Corollary 7.1. *If the condition* $(*)$ *is satisfies with* $\alpha \geqslant p-1$, *then* $\mathcal{H}$ *is non-degenerate and elliptic. Moreover, for every domain* U *in* $\mathbb{R}^d$ *and* $u \in \mathcal{H}^+(U)$, *we have either* $u > 0$ *on* U *or* $u = 0$ *on* U.

Remark 7.1. If $\alpha = p-1$, the constant in *Theorem 7.1* does not depend on M and we have the classical form of the Harnack inequality.

We recall that a sheaf $\mathcal{H}$ satisfies the *Brelot convergence property* if for every domain U in $\mathbb{R}^d$ and for every monotone sequence $(h_n)_n \subset \mathcal{H}(U)$ we have $\lim_n h_n \in \mathcal{H}(U)$ if it is not identically $+\infty$ on U.

Using the same proof as in [4], we have the following proposition.

Proposition 7.3. *If the Harnack inequality is satisfied by* $\mathcal{H}$, *then the convergence property of Brelot is fulfilled by* $\mathcal{H}$.

Remark 7.2. In contrast to the linear case *(see* [16]*)* the converse of *Proposition 7.3* is not true *(see* [5]*)* and hence the validity of the convergence property of Brelot does not imply the validity of the Harnack inequality.

An Application. Let $\mathcal{H}_\alpha$ be the sheaf of all continuous solution of the equation

$$-\operatorname{div}\mathcal{A}(x,\nabla u) + b(x)\operatorname{sgn}(u)\,|u|^\alpha = 0$$

where $b \in L^{\frac{d}{d-\epsilon}}_{\mathrm{loc}}(\mathbb{R}^d)$, $b \geqslant 0$ and $0 < \epsilon < 1$.

Theorem 7.2. *a) For each* $0 < \alpha < p-1$, $(\mathbb{R}^d, \mathcal{H}_\alpha)$ *is a Bauer harmonic space satisfying the Brelot convergence property, but it is not elliptic in the sense of Definition 7.1.*
b) For each $\alpha \geqslant p-1$, $(\mathbb{R}^d, \mathcal{H}_\alpha)$ *is a Bauer harmonic space elliptic in the sense of Definition 7.1 and the convergence property of Brelot is fulfilled by* $\mathcal{H}_{p-1}$.

8. Keller-Osserman Property

Let $\mathcal{H}$ be the sheaf of continuous solutions related to the equation (1.1).

Definition 8.1. Let U be a relatively compact open subset of $\mathbb{R}^d$. A function $u \in \mathcal{H}^+(U)$ is called regular Evans function for $\mathcal{H}$ and U if $\lim\limits_{U\ni x\to z} u(x) = +\infty$ for every regular point z in the boundary of U.

For an investigation of regular Evans functions see [5].

Definition 8.2. We shall say that $\mathcal{H}$ satisfies the Keller-Osserman property, denoted (KO), if every ball admits a regular Evans function for $\mathcal{H}$.

As in [5, Proposition 1.3], we have the following proposition.

Proposition 8.1. *$\mathcal{H}$ satisfies the (KO) condition if and only if $\mathcal{H}^+$ is locally uniformly bounded (i.e. for every non empty open set U in $\mathbb{R}^d$ and for every compact $K \subset U$, there exists a constant $C > 0$ such that $\sup_K u \leqslant C$ for every $u \in \mathcal{H}^+(U)$).*

Corollary 8.1. *If $\mathcal{H}$ fulfills the (KO) property, then $\mathcal{H}$ satisfies the Brelot convergence property.*

Theorem 8.1. *Assume that $\mathcal{A}$ and $\mathcal{B}$ satisfies the following supplementary conditions*

i) *For every $x_0 \in \mathbb{R}^d$, the function F from $\mathbb{R}^d$ to $\mathbb{R}^d$ defined by $F(x) = \mathcal{A}(x, x - x_0)$ is differentiable and $\operatorname{div} F$ is locally (essentially) bounded.*
ii) *$\mathcal{A}(x, \lambda\xi) = \lambda\,|\lambda|^{p-2}\,\mathcal{A}(x,\xi)$ for every $\lambda \in \mathbb{R}$ and every $x, \xi \in \mathbb{R}^d$.*
iii) *$|\mathcal{B}(x,\zeta)| \geqslant b(x)\,|\zeta|^\alpha$, $\alpha > p-1$ where $b \in L_{loc}^{\frac{d}{d-\epsilon}}(\mathbb{R}^d)$, $0 < \epsilon < 1$, with $\operatorname{ess\,inf}_U b(x) > 0$ for every relatively compact U in $\mathbb{R}^d$.*

Then the (KO) property is valid by $\mathcal{H}$.

Proof. Let U be the ball with center $x_0 \in \mathbb{R}^d$ and radius R. Put $f(x) = R^2 - \|x - x_0\|^2$ and $g = cf^{-\beta}$, we obtain the desired property if we find a constant $c > 0$ such that g is a supersolution of the equation (1.1). We have $\nabla f(x) = -2(x - x_0)$ and $\nabla g(x) = 2c\beta\,(f(x))^{-(\beta+1)}\,(x - x_0)$ and then

$$\mathcal{A}(x, \nabla g(x)) = (2c\beta)^{p-1}\,(f(x))^{-(\beta+1)(p-1)}\,\mathcal{A}(x, x - x_0).$$

Let $\varphi \in \mathcal{C}_c^\infty(U)$, $\varphi \geqslant 0$ and we set $I_\varphi = \int \mathcal{A}(x,\nabla g)\nabla\varphi dx + \int \mathcal{B}(x,g)\varphi dx$, then

$$\begin{aligned}
I_\varphi &= -\int \operatorname{div}\mathcal{A}(x,\nabla g)\varphi dx + \int \mathcal{B}(x,g)\varphi dx \\
&= -\int \Big[2(\beta+1)(p-1)(2c\beta)^{p-1} f^{-(\beta+1)(p-1)-1}\mathcal{A}(x, x-x_0).(x-x_0) \\
&\quad +(2c\beta)^{p-1} f^{-(\beta+1)(p-1)} \operatorname{div}\mathcal{A}(x, x-x_0) - \mathcal{B}(x,g)\Big]\varphi dx \\
&\geqslant -\int \Big[2(\beta+1)(p-1)(2c\beta)^{p-1} f^{-(\beta+1)(p-1)-1}\mathcal{A}(x, x-x_0).(x-x_0) \\
&\quad +(2c\beta)^{p-1} f^{-(\beta+1)(p-1)} \operatorname{div}\mathcal{A}(x, x-x_0) - c^\alpha b f^{-\alpha\beta}\Big]\varphi dx \\
&= -\int \Big[2c^{p-1-\alpha}(2\beta)^{p-1}(\beta+1)(p-1)\mathcal{A}(x, x-x_0).(x-x_0) \\
&\quad +c^{p-1-\alpha}(2\beta)^{p-1} f \operatorname{div}\mathcal{A}(x, x-x_0) - b f^{\beta(p-1-\alpha)+p}\Big] c^\alpha f^{-(\beta+1)(p-1)-1}\varphi dx.
\end{aligned}$$

Putting $\beta = p(\alpha - p + 1)^{-1}$ we obtain

$$\begin{aligned}
I_\varphi &\geqslant -\int \Big[2(\tfrac{2p}{\alpha-p+1})^{p-1}(\tfrac{\alpha+1}{\alpha-p+1})(p-1)\mathcal{A}(x, x-x_0).(x-x_0) \\
&\quad +(\tfrac{2p}{\alpha-p+1})^{p-1} f \operatorname{div}\mathcal{A}(x, x-x_0) - c^{\alpha-p+1} b\Big] c^{p-1} f^{\frac{\alpha p}{p-1-\alpha}}\varphi dx.
\end{aligned}$$

It follows from A2 that $\mathcal{A}(x, x-x_0).(x-x_0)$ is locally bounded. Hence if we take c so that$\frac{p-1}{\alpha-p+1}$

$$c \geqslant \Big[\sup_{x\in U}\Big\{\tfrac{2(\alpha+1)(p-1)}{\alpha-p+1}\frac{|\mathcal{A}(x,x-x_0).(x-x_0)|}{b(x)} + R^2\frac{|\operatorname{div}\mathcal{A}(x,x-x_0)|}{b(x)}\Big\}\Big]^{\frac{1}{\alpha-p+1}} \times \Big(\frac{2p}{\alpha-p+1}\Big)^{\frac{p-1}{\alpha-p+1}},$$

then $I_\varphi \geqslant 0$ holds for every $\varphi \in \mathcal{C}_c^\infty(U)$ with $\varphi \geqslant 0$. Thus the function $g(x) = c(R^2 - \|x-x_0\|^2)^{p(p-1-\alpha)}$ is a supersolution satisfying $\lim\limits_{x\to z} g(x) = +\infty$ for every $z \in \partial U$. By the comparison principle we have $\mathrm{H}\ _U n \leqslant g$ for every $n \in \mathbb{N}$ and therefore, the increasing sequence $(\mathrm{H}_U n)_n$ of harmonic functions is locally uniformly bounded on U. The Bauer convergence property implies that $u = \sup\limits_n \mathrm{H}_U n \in \mathcal{H}(U)$, therefore we have $\liminf\limits_{x\to z} u(x) \geqslant n$ for every z in ∂U, thus $\lim\limits_{x\to z} u(x) = +\infty$ for every z in ∂U and u is a regular Evans function. Since U is an arbitrary ball, we get the desired property. □

Corollary 8.2. *Under the assumptions in Theorem 8.1, for every ball* B=B(x_0, R) *with center* x_0 *and radius* R *and for every* $u \in \mathcal{H}(U)$,

$$|u(x_0)| \leqslant cR^{\frac{2p}{p-1-\alpha}}$$

where

$$c = \big[\sup_{x\in \mathrm{B}}\Big\{\tfrac{2(\alpha+1)(p-1)}{\alpha-p+1}\frac{|\mathcal{A}(x,x-x_0).(x-x_0)|}{b(x)} + R^2\frac{|\operatorname{div}\mathcal{A}(x,x-x_0)|}{b(x)}\Big\}\Big]^{\frac{1}{\alpha-p+1}} \times \Big(\frac{2p}{\alpha-p+1}\Big)^{\frac{p-1}{\alpha-p+1}}.$$

Proof. From the proof of the previous theorem, if B_n =B $(x_0, R(1-n^{-1}))$, $n \geqslant 2$, we have

$$u(x_0) \leqslant c_n \Big(\tfrac{R(n-1)}{n}\Big)^{\frac{2p}{p-1-\alpha}}$$

for every $n \geqslant 2$ and

$$\begin{aligned} c_n &= \Big[\sup_{x\in \mathrm{B}_n}\Big\{\tfrac{2(\alpha+1)(p-1)}{\alpha-p+1}\frac{|\mathcal{A}(x,-x_0).(x-x_0)|}{b(x)} \\ &\quad + \Big(\tfrac{R(n-1)}{n}\Big)^2\frac{|\operatorname{div}\mathcal{A}(x,x-x_0)|}{b(x)}\Big\}\Big]^{\frac{1}{\alpha-p+1}} \Big(\tfrac{2p}{\alpha-p+1}\Big)^{\frac{p-1}{\alpha-p+1}} \\ &\leqslant \Big[\sup_{x\in \mathrm{B}}\Big\{\tfrac{2(\alpha+1)(p-1)}{\alpha-p+1}\frac{|\mathcal{A}(x,x-x_0).(x-x_0)|}{b(x)} \\ &\quad + R^2\frac{|\operatorname{div}\mathcal{A}(x,x-x_0)|}{b(x)}\Big\}\Big]^{\frac{1}{\alpha-p+1}} \Big(\tfrac{2p}{\alpha-p+1}\Big)^{\frac{p-1}{\alpha-p+1}}. \end{aligned}$$

Then we obtain the inequality

$$u(x_0) \leqslant cR^{\frac{2p}{p-1-\alpha}}.$$

Since $-u$ is a solution of similarly equation, we get

$$-u(x_0) \leqslant cR^{\frac{2p}{p-1-\alpha}}$$

with the same constant c as before. Then we have the desired inequality. $\square$

We now have a Liouville like theorem.

Theorem 8.2. *Assume that the conditions in* Theorem 8.1 *are satisfied and that*

$$\liminf_{R\to\infty}\left(R^{-2p}M(R)\right)=0$$

where

$$M(R)=\sup_{\|x-x_0\|\leqslant R}\left\{\tfrac{2(\alpha+1)(p-1)}{\alpha-p+1}\frac{|\mathcal{A}(x,x-x_0).(x-x_0)|}{b(x)}+R^2\frac{|\operatorname{div}\mathcal{A}(x,x-x_0)|}{b(x)}\right\}.$$

Then $u\equiv 0$ is the unique solution of the equation (1.1) on $\mathbb{R}^d$.

Proof. Let u be a solution of the equation (1.1) on $\mathbb{R}^d$. By the previous corollary, we have for every $x_0\in\mathbb{R}^d$ and every $R>0$

$$\begin{aligned}|u(x_0)| \quad\leqslant\quad & \Bigg[\sup_{\|x-x_0\|\leqslant R}\Bigg\{\tfrac{2(\alpha+1)(p-1)}{\alpha-p+1}\frac{|\mathcal{A}(x,x-x_0).(x-x_0)|}{b(x)}\\ &\qquad +R^2\frac{|\operatorname{div}\mathcal{A}(x,x-x_0)|}{b(x)}\Bigg\}R^{-2p}\Bigg]^{\frac{1}{\alpha-p+1}}\left(\tfrac{2p}{\alpha-p+1}\right)^{\frac{p-1}{\alpha-p+1}}.\end{aligned}$$

Hence $u(x_0)=0$ and $u\equiv 0$. $\square$

9. Applications

We shall use the previous results for the investigation of the p–Laplace Δ_p, $p\geqslant 2$ which is the Laplace operator if $p=2$. Δ_p is associated with $\mathcal{A}(x,\xi)=|\xi|^{p-2}\,\xi$, an easy calculation gives $\operatorname{div}\mathcal{A}(x,x-x_0)=(d+p-2)\,\|x-x_0\|^{p-2}$. Let, for every $\alpha>0$, $\mathcal{H}_\alpha$ denote the sheaf of all continuous solution of the equation

$$-\Delta_p u+b(x)\operatorname{sgn}(u)\,|u|^\alpha=0 \tag{9.1}$$

where $b\in L_{\text{loc}}^{\frac{d}{d-\epsilon}}(\mathbb{R}^d)$, $b\geqslant 0$ and $0<\epsilon<1$.

Theorem 9.1. *Assume that $p\geqslant 2$. For $\alpha>0$, let $\mathcal{H}_\alpha$ denote the sheaf of all continuous solution of the equation*

$$-\Delta_p u+b(x)\operatorname{sgn}(u)\,|u|^\alpha=0\,.$$

where $b\in L_{loc}^{\frac{d}{d-\epsilon}}(\mathbb{R}^d)$, $b\geqslant 0$ and $0<\epsilon<1$. Then

1. *For every $\alpha>0$, $(\mathbb{R}^d,\mathcal{H}_\alpha)$ is a nonlinear Bauer harmonic space with the Brelot convergence Property.*
2. *$\mathcal{H}_\alpha$ is elliptic for every $\alpha\geqslant p-1$.*
3. *If $\alpha>p-1$ and $\inf_U b>0$ for every relatively compact open U in $\mathbb{R}^d$, then the property (KO) is satisfied by $\mathcal{H}_\alpha$.*
4. *If $\alpha>p-1$ and , $\inf_{\mathbb{R}^d} b>0$, then $\mathcal{H}_\alpha(\mathbb{R}^d)=\{0\}$.*

Theorem 9.2. *Let $U\subset\mathbb{R}^d$ be an bounded open set whose boundary, ∂U, can be represented locally as a graph of function with Hölder continuous derivatives. Assume that $\alpha>p-1$. Then U admits a regular Evans function for $\mathcal{H}$.*

Proof. We first prove the existence of a continuous supersolution v on U such that $\lim_{x\to z} v(x) = +\infty$, for every $z \in \partial U$.

Let f in $\mathcal{C}_c^\infty(U)$ be a positive function ($f \neq 0$) and $w \in \mathcal{W}_0^{1,p}(U)$ be the solution of the problem

$$\int_U |\nabla w|^{p-2}\, \nabla w \cdot \nabla\varphi dx = \int_U f\varphi dx, \quad \varphi \in \mathcal{W}_0^{1,p}(U)$$
$$w = 0 \quad \text{on } \partial U$$

By the regularity theory, w has a Hölder continuous gradient, w is continuous supersolution $w > 0$ in U, $\lim_{x\to z} w(x) = 0$ for every $z \in \partial U$ and $\|w\|_\infty + \|\nabla w\|_\infty \to 0$ as $\|f\|_\infty \to 0$. Then we set $v = w^{-\beta}$ and look for $\beta > 0$ and f such that

$$\int_U |\nabla v|^{p-2}\, \nabla v \cdot \nabla\varphi dx + \int_U b(x) v^\alpha \varphi dx \geqslant 0 \quad \varphi \geqslant 0, \varphi \in \mathcal{W}_0^{1,p}(U).$$

For every $\varphi \geqslant 0, \in \mathcal{W}_0^{1,p}(U)$, we have

$$\begin{aligned}
\int_U |\nabla v|^{p-2}\nabla v \cdot \nabla\varphi dx &= -\,\beta^{p-1}\int_U w^{-(\beta+1)(p-1)}|\nabla w|^{p-2}\nabla w \cdot \nabla\varphi dx \\
&= -\,\beta^{p-1}\int_U |\nabla w|^{p-2}\nabla w \cdot \nabla(w^{-(\beta+1)(p-1)}\varphi) dx \\
&\quad -\beta^{p-1}(\beta+1)(p-1)\int_U w^{-(\beta+1)(p-1)-1}\varphi|\nabla w|^p dx \\
&= -\,\beta^{p-1}\int_U w^{-(\beta+1)(p-1)-1}[wf + (\beta+1)(p-1)|\nabla w|^p]\varphi dx;
\end{aligned}$$

thus

$$\begin{aligned}
&\int_U |\nabla v|^{p-2}\, \nabla v \cdot \nabla\varphi dx \\
&\qquad + \beta^{p-1}\int_U b v^{\frac{(\beta+1)(p-1)+1}{\beta}} \left[b^{-1} wf + (\beta+1)(p-1) b^{-1}\, |\nabla w|^p\right] \varphi dx = 0\,.
\end{aligned}$$

Put $\beta = \frac{p}{\alpha-p+1}$ and choose f such that $wf + (\beta+1)(p-1)\, |\nabla w|^p \leqslant b\beta^{1-p}$. Then

$$\int_U |\nabla v|^{p-2}\, \nabla v \cdot \nabla\varphi dx + \int_U b v^\alpha \varphi dx \geqslant 0, \text{ for every } \varphi \geqslant 0, \varphi \in \mathcal{W}_0^{1,p}(U);$$

therefore, v is a continuous supersolution of (9.1) such that $\lim_{x\to z} v(x) = +\infty$, for every $z \in \partial U$.

Let u_n denote the continuous solution of the problem

$$\int_U |\nabla u|^{p-2}\, \nabla u \cdot \nabla\varphi dx + \int_U b u^\alpha \varphi dx = 0, \quad \varphi \in \mathcal{W}_0^{1,p}(U)$$
$$u = n \in \mathbb{N} \quad \text{on } \partial U$$

By the comparison principle we have $0 \leqslant u_n \leqslant v$ for all n and by the convergence property, the function $u = \sup_n u_n$ is a regular Evans function for $\mathcal{H}$ and U. □

Theorem 9.3. *Let $\alpha > p - 1$ and let U be a star domain and b continuous and strictly positive function on $\mathbb{R}^d$. Assume that the conditions in Theorem 9.1 are satisfied. If there exists a regular Evans function u associated with U and $\mathcal{H}_\alpha$, then u is unique.*

The proof is the same as in [4] and [6] when $b \equiv 1$.

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Azeddine Baalal
Département de Mathématiques et d'Informatique, Faculté des Sciences Aïn Chock, Km 8 Route El Jadida B.P. 5366 Mâarif, Casablanca - Maroc
E-mail address: `baalal@facsc-achok.ac.ma`

Abderahman Boukricha
Département de Mathématiques, Faculté des Sciences de Tunis,, Campus Universitaire 1060 Tunis - Tunisie.
E-mail address: `aboukricha@fst.rnu.tn`