ELEMENTARY PRESERVICE TEACHERS' CONCEPTIONS

OF AND REFLECTIONS ON STUDENT

AUTONOMOUS PROBLEM-SOLVING

AND MATHEMATICAL PRACTICES

by

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## DEDICATION

I would like to dedicate this to my elementary math teacher Mrs. Braniff, who helped me develop an understanding of mathematics that laid the foundation for me today. I would also like to dedicate this to my parents, whose support made my education possible.

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## LIST OF ABBREVIATIONS

| Abbreviation | Description |
|---|---|
| AMTE | Association of Mathematics Teacher Educators |
| APS | Autonomous Problem-Solving |
| CCSSM | Common Core State Standards Initiative for Mathematics |
| ePST | Elementary Preservice Teacher |
| LT | Lead Teacher |
| MEC | Mathematics Exploration Camp |
| NCTM | National Council of Teachers of Mathematics |
| NRC | National Research Council |
| PD | Professional Development |

# ABSTRACT

The findings of this dissertation study are from a single-site, qualitative, case study involving undergraduate elementary preservice teacher participants from a two-week summer mathematics camp for elementary and middle school-aged students. The purpose of this study was to investigate three elementary preservice teachers' conceptions of and reflections on student autonomous problem-solving and the mathematical practices of justification, mathematical language, mathematical troubles, perseverance, and visual representations. This study presents the elementary preservice teachers' conceptions, student interactions, and their reflections from the two-week camp. Additionally, I present how the preservice teachers' conceptions, interactions, and reflections align with each other and with what research tells us by using an analytic framework based on a corpus of literature that would accurately capture what the participants were saying. This study aided in answering researchers call to understand elementary preservice teachers' mathematical conceptions (Bransford, Brown, & Cocking, 1999; Conference Board of the Mathematical Sciences, 2001; Thanheiser, Browning, Edson, Kastberg, & Lo, 2013)

This study utilized data from written surveys, survey interviews, clinical interviews, interaction video observations, and stimulated recall interviews, all collected within the two weeks. The theoretical perspective of this study, which aligns with the setting's theoretical orientation, operationalized a social constructivist perspective of collaborative learning and teaching mathematics (Vygotsky, 1978, 1986). Additionally, the analytic framework used in the analysis of this data was based on literature using and

building on the camp's *governing precepts*, which formed the camp's foundational approach to teaching and learning mathematics. Three individual case reports, as well as a cross-analysis, are presented. The findings of this study indicate that

(1) All three elementary preservice teachers supported the elementary-aged students in was that surpassed their conceptions of the given practices. Moreover, the elementary preservice teachers were aware of the times in which their enactments did not match their conceptions or pointed out instances where their enactments supported the students in ways they did not conceptualize. For instance, the elementary preservice teachers' all conceptualized perseverance as something they could foster through questioning the students to focus on their strategies or to try other methods. However, the elementary preservice teachers also reflected on other ways of fostering perseverance that were not mentioned in their conceptions, such as changing the participation format to include group work or would praise the students for their effort or progress in problem-solving.

(2) The elementary preservice teachers tended to focus their enactments through supporting students' justifications. Thus, all their conceptions linked back to supporting justification; however, many of the links between the other practices were missing in the elementary preservice teachers' conceptions. Moreover, the elementary preservice teachers' views of justification did not match those of the mathematics education research community in that they did not differentiate between reasoning about a process and justifying why the process is true.

(3) The elementary preservice teachers reflected on the difficulties of supporting a student's mathematical trouble, particularly with troubles that required more than one repair cycle (Ingram, 2012). Moreover, these difficulties occurred most often in more complex interactions that involved the use of visual representations, troubles with mathematical language, and troubles with justifications of the students' work. This complex interaction involved multiple mathematical practices, consequently causing the elementary preservice teachers difficulty supporting the students' autonomous problem-solving.

(4) The elementary preservice teachers attributed several lived experiences to their conceptions and enactments of support for the mathematical practices. Most of these lived experiences involved some type of *decomposition*, *representation,* or *approximation of practice* (Grossman, Compton, Igra, Ronfeldt, Shahan, & Williamson, 2009a) centered around teaching; such as the camp experience, talking with teachers or their peers, the university coursework, or working with students. However, other lived experiences such as babysitting, working as a retail clothing salesperson, or playing volleyball were also mentioned as having an impact on their teaching.

Lastly, I include (i) a discussion of these key findings, and more, situated within the context of existing literature, (ii) implications for the strengthening and development of university coursework for elementary teacher preparation courses, and (iii) future research recommendations based on the finding from this study.

# I. RATIONALE

## The Importance of Elementary Preservice Teacher Conceptions

The Association of Mathematics Teacher Educators [AMTE] (2017) notes that preservice teacher coursework should model the ways we want the preservice teachers to teach. Additionally, policy documents from AMTE, the National Council for Teachers of Mathematics [NCTM], and the Common Core State Standards for Mathematics all mention that a students' learning should be built on prior knowledge. How can we as a mathematics education community improve upon the teacher preparation curriculum for elementary preservice teachers, if we do not understand what conceptions they bring to their teacher preparation programs, and how we can build on it (Thanheiser, Browning, Edson, Kastberg, & Lo, 2013). What are elementary preservice teachers' prior knowledge or conceptions of the mathematical practices that we want them to support, and how can we build on this knowledge in their preparation courses?

By conceptions, I mean "general notions or mental structures encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preference" (Philipp, 2007, p. 259). I use this definition because "to look at research on mathematics teachers' beliefs and conceptions in isolation from research on mathematics teachers' knowledge will necessarily result in an incomplete picture." (Thompson, 1992, p. 131) Thus, this definition broadly encompasses preservice teachers' general thoughts, which are always evolving as the preservice teachers acquire and assimilate new information.

Therefore, waiting until the classroom field experiences to survey conceptions and implementations may be too late. Future teachers require more than just field experience; they need meaningful experiences that align with their coursework. Having the opportunity to test what is taught in their university courses is vital towards establishing well-prepared teachers. Researchers have observed that the lack of quality classroom experiences that future teachers receive is the critical issue causing underprepared teachers (Ball & Forzani, 2009; Grossman et al., 2009a). It is imperative that preservice teachers are given ample opportunities to apply what they learn in their university courses, and that these courses "are structured with a focus on conceptual understanding to build meaning for procedures" (Stohlmann, Cramer, Moore, & Maiorca, 2014, p. 4).

Making connections between content and student strategies and thinking is essential for teaching mathematics (Ball, 2000; Bartell, Webel, Bowen, & Dyson, 2013; Shulman, 1986). Prospective teachers should not only have a conceptual understanding of the mathematics they are going to teach but be able to apply their knowledge to their students' thinking. Also, they should strive to teach according to important *processes and proficiencies* that are established as valuable in mathematics education (e.g., Common Core State Standards for Mathematics [CCSSM]). Although research indicates that skills develop over time, beginning teachers "must have an initial repertoire of effective and equitable teaching strategies; for example, in selecting tasks, orchestrating classroom discussion, building on prior knowledge, and connecting conceptual understanding and procedural fluency" (AMTE, 2017, p. 7).

All of these practices require time and experience to learn. Thus, considerable amounts of differing rehearsal time are needed to prepare future teachers to be as ready as possible for their profession (Hunter, Anthony, & Hunter, 2015). Classroom observations and student teaching are two primary ways that preservice teachers make these connections and apply what they have learned from their coursework. However, teacher education programs can help add to these experiences by incorporating teaching strategies and methods into the preservice teachers' coursework through different "pedagogies of practice" (Grossman et al., 2009a).

**Reducing the Complexity of Practice Through Deconstructing Practices**

Teaching has been considered a complex practice for several decades now because of its *multidimensionality*, *simultaneity*, and *unpredictability* (Doyle, 1977, pp. 8-9). However, due to preservice teachers' familiarity with teaching from being a student for over 12-plus years, it is often mistakenly viewed as easy by novices (Grossman et al., 2009a; Labaree, 2000). This mistaken view led to Grossman and colleagues' (2009a) work to address the misconception of teaching being easy in teacher education courses. They established a framework that depicts "the pedagogies of practice in professional education: representations, decomposition, and approximations of practice" (Grossman et al., 2009a, p. 2055).

The practice of observations, viewing videos, listening to narrative accounts of a classroom dilemma, and looking at student work are examples of *representations of practice* often incorporated into teacher preparation courses. These representations of practice are intended to make aspects of practice visible to novices and lend insight into the variety of ways the practice can be represented (Grossman et al., 2009a). Next, they defined *decomposition of practice* to involve deconstruction of practice into its integral parts to support teaching and learning. The breaking down of teaching into core practices is an example of the decomposition of practice. Other examples include focusing on the structure of a lesson plan or managing classroom transitions. By decomposing teaching practices, teacher educators can help foster preservice teachers' ability to attend to fundamental components of practice. Finally, the act of student teaching, tutoring, lesson planning, or any implementation of practices would be considered an *approximation of practice*, as it is an "opportunity to engage in practices that are more or less proximal to the practices of [the] profession" (Grossman et al., 2009a, p. 2058). The act of approximating practice can afford novices the chance to experiment with new ideas, skills, and ways of thinking.

Interacting with students often involves multiple practices that can be decomposed, represented, and approximated. Research suggests that a classroom setting may be too complex for preservice teachers to focus on the fundamentals of teaching (Doyle, 1977; Grossman et al., 2009a; Santagata, Zannoni, & Stigler, 2007; Star & Strickland, 2007; Stockero, Rupnow, & Pascoe, 2017). Thus, focusing preservice teachers on the decomposition of these practices along with approximations and representations, can be a beneficial component of teacher preparation.

**Practices Involved in Interacting with Students**

Reference policy documents such as the *CCSSM*, *Principles to Action* (National Council for Teachers of Mathematics [NCTM], 2014), and *Standards for Preparing Teachers of Mathematics* (AMTE, 2017) give a well-defined overview of what classroom teachers should be instilling in their K-12 students. Moreover, these documents describe what foundational skills, knowledge, and processes preservice teachers (PSTs) need to teach students to become mathematically proficient, what actions can be taken to support the students in this endeavor and why it is important, and what teacher preparation programs can do to model and support their PSTs' dispositions and skills for these practices. These documents focus PSTs toward essential teaching and mathematical practices and skills needed to be effective in the mathematics classroom. Across these documents and others, there is a push for teachers to support students' justification, to use multiple representations, to support student autonomy, to allow students to make and self-correct errors, to use precise language, and to persist in problem-solving.

**An Early Focus: What Can Be Learned?**

As mentioned previously, it is mostly during classroom observations and student teaching that PSTs are able to engage in the uses of teaching practices. However, teacher education programs should engage PSTs in the practices of recognizing students' conceptual understanding beginning in their content courses (Philipp, 2007, 2008; Thanheiser et al., 2013), as PSTs may not have developed all of the required content knowledge for teaching before interacting with students' understanding of mathematics (Bartell et al., 2013).

Elementary PSTs (ePSTs) often enter teaching programs with preconceived conceptions from their own experiences as learners (Stohlmann et al., 2014). However, these conceptions have been known to change through content courses that use artifacts of children's' mathematical thinking (Thanheiser et al., 2013) and are taught in ways that align with content standards for doing mathematics (Conference Board of Mathematical Sciences [CBMS], 2012). Thus, by changing ePSTs conceptions to align with teaching standards, there is reason to believe that these new-formed conceptions may influence teacher practice (Ambrose, Clement, Philipp, & Chauvot, 2004; Stohlmann et al., 2014; Thompson, 1984, 1992).

As a way to involve PSTs more with teaching earlier, some programs have incorporated early field experiences into their programs with great success (Jacobson, 2017). AMTE recommends early field experiences for PSTs as they can afford the opportunity to start focusing on teaching practices, develop their teacher identities, and determine what grade level they are most interested in teaching. Jacobson (2017) found that PSTs who participated in an early field experience had better *teacher education outcomes*, such as mathematical content knowledge and beliefs about the nature of mathematics and mathematics learning than those who did not participate in early field experience.

**Why Elementary Preservice Teachers?**

Elementary teachers "develop the foundation of mathematical understanding, beliefs, and attitudes among young learners that start children on their mathematical journeys." (Association of Mathematics Teacher Educators [AMTE], 2017, p. 48). This statement, coupled with the teacher educator and researcher communities call for strengthening preparation programs needed for teaching, is reason to help better elementary preservice teacher (ePST) education; specifically, generalist education, meaning the concentration is not on a core subject. Typical degree plans for ePSTs on a generalist track require only one to three mathematics content courses for teaching (Hiebert, Berk, DiNapoli, Mixel, & Young, 2017), which is below the recommended minimum (CBMS, 2012). Moreover, evidence from the literature suggests that the average ePST does not possess an adequate conceptual understanding of algorithms, numbers, and operations that are needed for teaching (Thanheiser et al., 2013), leading to the difficulties Thanheiser and associates (2013) reported from the literature regarding ePSTs difficulty in “carrying out teacher-like tasks, such as modeling operations with multiple representations (Luo, 2009; Rizvi & Lawson, 2007)” (p. 16), interpreting uncommon student algorithms, and pinpointing the origin of student errors.

Yet, ePSTs often don’t care about the mathematics content but do care about helping children (Philipp, 2008). In order to better focus ePSTs on the content, Philipp (2008) suggested centering the content around children’s thinking. However, certain positive conceptions regarding teaching and mathematics should be maintained to optimize the benefits of this focus, as conceptions “play a significant role in shaping the teachers’ characteristic patterns of instructional behavior.” (Thompson, 1992) Therefore, mathematics educators must understand what ePSTs conceptions are, how they and other experiences influence their teachings, and how we as a mathematics community can help the ePSTs develop.

**Introduction to the Camp**

MathKidz is a program whose goal is “to prepare students, undergraduate pre-service teachers, and in-service teachers from all backgrounds for success in learning, teaching and doing mathematics.” MathKidz hosts a two-week summer Mathematics Exploration Camp (MEC) for elementary and middle school students, which hires PSTs of all levels to help facilitate learning in classrooms led by experienced middle school teachers. MEC specifically looks to prepare students for advanced mathematics by focusing on the language of mathematics, persisting in problem-solving, and communication.

This study focuses on ePSTs engaged in interacting with elementary students working on addition and subtraction of integers. There were approximately 20 students in each classroom with demographics that matched the surrounding areas in which the ePSTs would be teaching. Moreover, MEC uses open enrollment of students, does not require a skills test, and has only one criterion for enrollment: the students want to be there. Although MEC includes five different content levels, two of which match the content from the ePSTs' first university mathematics content course, this study focuses on the first level with operations with integers as its primary subject matter. In general, ePSTs are comfortable with early number sense, and "the mathematics itself would not be a barrier to the examination of children's mathematical thinking" (Schack, Fisher, Thomas, Eisenhardt, Tassell, & Yoder, 2013, p. 384).

**Purpose Statement**

As noted above, it is critical to understand ePSTs' conceptions about teaching mathematics early in their education program. Although ePSTs may hold various conceptions about multiple mathematical topics, I limit this to six topics which were identified across policy documents, and clearly aligned with MEC's underlying standards (AMTE, 2017; CCSSM, 2010; Governing Precept of MEC, Appendix E; NCTM, 2014; National Research Council [NRC], 2002). Thus, I looked at the ePSTs' conceptions of students' justification, representations, student autonomy, errors, mathematical language, and perseverance. The purpose of this exploratory case study is to understand how ePSTs, who have completed at least one content course for prospective elementary and middle school teachers, conceptualized and supported these specific mathematical practices with elementary students in mathematical interactions during MEC, and what experiences they called upon when reflecting on their interactions. Additionally, this study looks at how the ePSTs connected these topics. In order to fulfill this purpose, I used the following research questions to guide my study.

**Research Questions**

1. What are the elementary preservice teachers' conceptions of
    a. Student *autonomous problem-solving* (APS), and
    b. the mathematical practices related to student autonomous problems-solving that are emphasized by the Governing Precepts of MEC?
        i. Fostering student *perseverance*
        i. Pressing for student *justifications*
        *ii.* Supporting students' development of *mathematical language*
        iii. Allowing students to work with *mathematical troubles*
        iv. Supporting students use of *visual representations*
2. What do the elementary preservice teachers notice about their fostering of student APS and their support of these five practices during camp classes at focused stimulated-recall reflection interviews?

Key teaching moves and mathematical practices that every teacher, including novices, should be familiar with before entering the classroom have been reported throughout policy documents, research papers, conference proceedings, teacher preparation programs, and more. The mathematical practices of focus in this study, therefore, align with and are valued by the national policy documents and the Governing Precept of the MathKidz curriculum. These documents will be discussed further in the next section.

This study can provide insight into the initial conceptions ePSTs' have towards these mathematical practices and how the ePSTs might go about supporting them based on their current experiences. Additionally, this study will inform mathematics teacher educators about what experiences ePSTs may draw upon, how they interpret and use these experiences, and which experiences they find valuable. This information can be used to enhance teacher preparation courses throughout the program to better customize the courses to the ePSTs' needs and focus.

# II. LITERATURE REVIEW

In this section, I address the theoretical framework that serves as the foundation of my work as a researcher, as well as a review of relevant literature for the study. First, I provide a brief review of the Governing Precepts of the math camp, which was used as the setting and its grounding in literature. Next, an overview is presented for the current influential policy documents on mathematical practices and student autonomy. This is followed by a description of the teacher education programs' preparation of future teachers and discussion of what is known about preservice teachers' initial tendencies when entering a classroom. Finally, I give a brief review of the literature on the five mathematical practices and student autonomy, which this study focuses on. This section concludes with an outline of the conceptual framework for the study.

## Theoretical Framework

I approached this study with a learning perspective that coincided with the settings learning perspective because this study is situated in a setting with elementary preservice teachers working together with a group of in-service teachers during a summer camp. Thus, I operationalized a social constructivist perspective of collaborative learning (Vygotsky, 1978). In using this approach, I conformed with the Vygotskian ideals of learning, meaning that people learn as they work to form understandings and create meaning through their shared experiences in any given situation. Therefore, I presumed that the participants in this study were learning due to a multitude of factors from the environment and the other teachers. I hypothesized that the social setting of the camp and the associated professional development (PD) created experiences the ePSTs called upon while they were interacting with students. The ePSTs were interacting with Professional

Development Teachers (PDTs) attending camp to learn more about MEC as well as experienced teachers with prior experiences at MEC. The ePSTs were also observing and aiding in the experienced teacher's camp classroom. I speculated that this social interaction and representation of practice influenced the views of the PSTs. Therefore, for this study, I examined how the group of ePSTs interacted with the students in this setting, and how the collaboration and other factors manifested in the ePSTs' rationale for their actions.

***Teaching and Learning Mathematics***

The actions the ePSTs took were in regard to supporting students. Thus, this study adopted a socioconstructivist view of teaching and learning mathematics, which was consistent with the view that learning happens on a social level first and builds upon students' already existing knowledge (Vygotsky, 1978, 1986). Therefore, teaching methods should aim to foster a learning environment where learners justify their thinking, have serious mathematical discussions, make conjectures, and use multiple problem-solving techniques (NCTM, 2000). In addition to fostering such an environment, the teacher should mediate and structure activities to accommodate for students' previous knowledge and past experiences as they influence students' interpretations (NCTM, 2000; Vygotsky, 1978). In building upon students' previous knowledge, the conversations and discussions should fall within the students' *zone of proximal development* (Vygotsky, 1978).

Vygotsky (1978) defined the *zone of proximal development* as "the distance between the actual developmental level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). Thus, it is through interactions with peers or more knowledgeable individuals that learning can take place. By teachers or aides mediating a discussion and guiding the students with purposeful questions, introducing or clarifying concepts, and/or referencing previous material students begin to assimilate the unknown to their knowledge bank (Berkeley GSI Teaching & Resource Center, 2018).

For this study, I examined how ePSTs support students' learning through one-on-one or small group interactions in a classroom. The setting was designed with the belief that learning occurs by making connections between old and new ideas through meaningful discussions with peers and facilitation from more knowledgeable individuals. The following *Governing Precepts* are how the camp facilitates these discussions.

**Governing Precepts**

The Governing Precepts form the foundation of the Mathematical Exploration Camp (MEC) curriculum, which is the setting for this study. These four precepts (and associated sub-precepts) were established based on research to integrate and support all students in learning mathematics (Appendix E).

The first precept promotes **doing mathematics**, which is about "making sense of and thinking deeply about fundamental concepts." This precept stresses the importance of having students think deeply about concepts and form connections between what they knew before to the new concepts being studied. This involves making connections using multiple representations, both visual and algebraic. The focus is on understanding and not just the answer. The goal is to know why something works, not just that it does.

Reflection on the problems to make sense of the mathematics and form those connections through justifications is an essential part of this precept.

In addition to focusing on the connections the students should be making, this precept guides teachers to focus on student errors and misconceptions as they provide valuable insight into students' thinking and make for great learning opportunities (Anghileri, 2006; Borasi, 1996). By including errors in the discussion, valuable discourse can take place with the appropriate scaffolding and mediation (Anghileri, 2006). However, learning from these errors comes from high-level problems and conversations. This first precept requires higher-level cognitively demanding questions that must be maintained throughout the implementation process (Henningsen & Stein, 1997; Smith & Stein, 1998). These problems and discussions require the students to persevere through the unknown and possible confusion.

The second precept is that "**[p]ersistence** is critical to success in problem solving and doing mathematics." This precept is related to Dweck's (2006) ideas about the importance of a growth mindset "[g]rowth mindset is based on the belief that your basic abilities can be developed through your efforts, your strategies, and help from others" (Dweck, 2006, p. 7). Students with a growth mindset enjoy challenging problems and will work longer on them before giving up than students with a fixed mindset, where students believe that one's abilities are fixed, and some problems are just too hard. A growth mindset supports the perseverance and resilience needed to focus on trying different approaches and understanding that this process will allow one's intelligence to "grow". Thus, in cultivating a growth mindset in students, children are encouraged to learn and further their understanding and focus less on finding the correct answer.

In addition to Dweck's ideas about a growth vs. fixed mindset, building persistence means the student must "be willing to take risks and understand that mistakes present opportunities for learning" (Governing Precepts, line 2b). In fact, mistakes and struggles can lead to deeper understandings through the process of productive struggle (Warshauer, 2015). When students struggle with a high cognitive demand task, and the teacher supports the students' understanding in such a way that does not reduce the cognitive level of the question, then the struggle is considered to be productive. In order to enable students to persist through the struggle, teachers must build confidence in their students. This is not something that can be done quickly or easily; it is something that teachers have to help their students develop for themselves. This can be done by "teaching them to value learning over the appearance of smartness, to relish challenge and effort, and to use errors as routes to mastery" (Dweck, 2006, p. 4). A positive and safe classroom environment must be created for students to feel comfortable making mistakes, to explore their thinking collectively, and to build confidence in their problem-solving abilities.

The third precept addresses the importance of establishing a classroom culture that is open and safe for fostering student curiosity. "Teachers need to establish a **classroom culture** that develops students' curiosity and imagination." This includes making math interesting and relevant, supporting productive struggle, allowing sufficient wait time, and using both individual and group activities. An established classroom culture "in which teachers model respectful interactions, focus on the success of every student and engage students in help-giving and help-seeking behaviors, can provide the safety net that students need to engage in autonomous, self-regulated behaviors" (Bozack, Vega, McCaslin, & Good, 2008, p. 2393).

Ball and Bass (2003) note that a classroom community should value differences, require students to care about and respect each other, and commit itself to foster the values of a just, democratic, and rational society. By caring for and respecting one another, this includes respecting, listening to, and taking other's ideas seriously. This also applies to the evaluation and critical appraisal of other's ideas. Establishing a classroom culture around respect and mathematical curiosity is of the utmost importance when one views learning through a Vygotskian perspective, where students learn in a social context and from their social setting.

The fourth and final precept is that "**Communication** between students and teachers is critical for learning". To aid students in their understanding, probing questions should be used to help and encourage students. Students should understand that they will be expected to defend their reasoning using precise mathematical language, whether they are right or wrong.

Thus, students should be expected to reason about their process and results as "the notion of mathematical understanding is meaningless without a serious emphasis on reasoning" (Ball & Bass, 2003, p. 28). In order for students to express their reasoning, they first must be able to construct mathematical arguments and give evidence that will expose and convey their thinking (Lampert & Cobb, 2003). Therefore, teachers should implement practices for orchestrating productive mathematics discussions (Smith & Stein, 2011).

## Policy Documents

The following section will provide a brief overview of important and well-known policy documents that inform teacher education programs nationally. Within these documents, recommendations are made for the enhancement of content courses, methods courses, field experiences, overall degree requirements, and alignment throughout the program and its related courses to better prepare PSTs for their future careers.

### *The Common Core*

The Common Core State Standards for Mathematics (CCSSM) includes a section dedicated to standards for mathematical practice. The purpose of this section is to "describe varieties of expertise that mathematics educators at all levels should seek to develop in their students" (CCSSM, 2010, p. 6). These practices stem from central "processes and proficiencies" within mathematics education and are most visible in NCTM and *Adding It Up* by the National Research Council. CCSSM suggests that "designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction" (CCSSM, 2010, p. 8). Thus, it is only reasonable to ensure that our teachers and PSTs are aware of and support these practices. There are eight *Standards for Mathematical Practice*, which will be described in more detail below. These address standards of problem solving, reasoning and proof, communication, representation, connections, adaptive reasoning, strategic competence, conceptual understanding, procedural fluency, and productive dispositions (CCSSM, 2010; NRC 2002).

**Make Sense of Problems and Persevere in Solving Them.** Students should be able to follow a logical problem-solving process that starts with understanding the problem and looking for possible entry points. This means that students should examine the various constraints, relationships, given information, and the goal of the problem. From this, they can proceed to make conjectures and a planned pathway towards a solution. Depending on the problem, students may need to be able to consider similar problems or special cases but should always be evaluating and monitoring their pathways and change directions if necessary. Teachers should instill in their students that the problem-solving process is not finished once an answer is reached, rather students still need to check their answers using a variety of different methods to guarantee that the answer makes sense. Additionally, students should be able to understand the different approaches of their fellow students.

**Reason Abstractly and Quantitatively.** Students need to be able to make sense of quantitative relationships within problems and be able to decontextualize and contextualize as needed. These two abilities help create a coherent representation of the mathematics and allow students to reason with the quantities knowingly and to flexibly apply different operations and properties.

**Construct Viable Arguments and Critique the Reasoning of Others.** Students are expected to "understand and use stated assumptions, definitions, and previously established results in constructing arguments" (CCSSM, 2010, p. 6). They should also be able to build statements up by making conjectures and build logical progressing statements or break them down into cases. Students should be expected to justify conclusions, communicate their solution with others, and be able to listen to questions and arguments from others. Similarly, students must be able to ask questions to make sense, clarify, or improve peer arguments.

**Model with Mathematics.** Students ought to see the applications of mathematics to solve everyday problems in life, society, and the workplace. This may require students to make assumptions and simplify complicated problems to produce approximate results that may need adjustments later. Similarly, reflection on the results and model is a necessary part of drawing conclusions.

**Use Appropriate Tools Strategically.** Students are compelled to consider all available tools at their disposal when solving a problem, as well as make sound decisions as to the appropriateness of a tool towards the application of the problem. These tools may also include external content and technology.

**Attend to Precision.** In communicating with each other, students need to be clear and use appropriate terminology and precise definitions if possible. Symbols used need to be made explicit in meaning.

**Look for and Make Use of Structure.** Similar to drawing conclusions and making connections, students should always seek patterns or structures that could be generalized. This means they can depersonalize a given situation to see an overview or generalization. As in the case of *constructing viable arguments* and breaking down problems, students should be encouraged to view complicated structures as being composed of smaller ones.

**Look for and Express Regularity in Repeated Reasoning.** Like discerning patterns, calculations can repeat. Students should look for ways of dealing with such repetitions, such as general methods or shortcuts. As expressed earlier, students should be paying attention to their overall solution pathway. Moreover, students should also be keeping a watchful eye on their immediate results and calculations. In other words, students should be self-monitoring at all times.

**CCSSM Conclusion.** This concludes the standards Common Core recommends educators to instill in their students to be proficient mathematical thinkers. As noted previously, these practices stem from other bodies of mathematics education research and are present in many of the suggestions and recommendations in policy documents that follow. These standards, however, do not lend insight into how teachers might support such practices. Thus, this might account for the issue found by Mortimer (2018) in her dissertation study. An issue to be raised with such an influential document for policy is that ePSTs views and use of these standards differ from the mathematics education field (Mortimer, 2018).

In her dissertation, Mortimer (2018) found that ePSTs overgeneralized and broadly assigned such practices as *make sense of problems and persevere in solving them, attend to precision, construct viable arguments and critique the reasoning of others,* and *use appropriate tools*. In contrast, the ePSTs in the study also considered a more restrictive definition when it came to the practice of *model with mathematics*. Here the ePSTs did not consider real-world problems to include realistic problems not all students would encounter, for instance building a rabbit pen. Thus, Mortimer notes that much can be done to clarify these standards for mathematical practices.

Next, we look at NCTM's (2014) *Principles to Action* to see how a teacher might support these standards. NCTM highlights the need for teachers to use equitable teaching practices to allow all students to learn and form a conceptual understanding of mathematics. We can see the practices from this document within their *Mathematics*

*Teaching Practices,* and how NCTM recommends implementing them in an effective and equitable manner. Thus, it is essential for my study to account for equitable ways for the ePSTs to support these practices.

***Principles to Action***

The National Council of Teachers of Mathematics released their *Principles to Actions: Ensuring Mathematical Success for All* in 2014. Within this policy document, they state eight mathematics teaching practices that create a framework of teaching and learning mathematics based upon the last two decades of research. The essential and core teaching skills needed to instill a conceptual understanding of mathematics are:

- Establish mathematics goals to focus learning;
- Implement tasks that promote reasoning and problem-solving;
- Use and connect mathematical representations;
- Facilitate meaningful mathematical discourse;
- Pose purposeful questions;
- Build procedural fluency from conceptual understanding;
- Support productive struggle in learning mathematics; and
- Elicit and use evidence of student thinking. (NCTM, 2014, p. 10)

Each of these practices is a product of a corpus of research and is seen as an essential skill and a "high-leverage practice" (NCTM, 2014, p. 9). Thus, it is important that these practices be added to our PSTs' skill sets by their respective teaching preparation programs. Therefore, I explore each practice to make visible what NCTM distills into each of these practices. I will now elaborate on each of the essential teaching practices given by NCTM (2014) in their *Principles to Action* report.

**Establish Mathematics Goals to Focus Learning.** "Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions" (NCTM, 2014, p. 12). These goals should be clear, standards-based, and more than just a reiteration of the standards; they should be presented in a way that is specific to the curriculum and class needs. NCTM notes that students should not be left guessing what the mathematical purpose of a lesson is, rather students are more focused and perform better when goals are made clear and referred to throughout the lesson. Establishing mathematical goals indicates what the students will be learning, and the initial steps towards building the foundation for effective teaching. NCTM suggests the following actions for teachers in order to establish and use goals to guide and focus mathematical learning:

- Establishing clear goals that articulate the mathematics that students are learning as a result of instruction in a lesson, over a series of lessons, or throughout a unit;
- Identifying how the goals fit within a mathematics learning progression;
- Discussing and referring to the mathematical purpose and goal of a lesson during instruction to ensure that students understand how the current work contributes to their learning; and
- Using the mathematics goals to guide lesson planning and reflection and to make in-the-moment decisions during instruction. (NCTM, 2014, p. 16)

**Implement Tasks That Promote Reasoning and Problem Solving.** "Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies" (NCTM, 2014, p. 17). Teachers should engage students in higher-level cognitive demands tasks (Smith & Stein, 1998) which allow for the use of different representations and tools, and can be created based on shared experiences, culture, contexts, conditions, and language. However, this does not mean that every task must be of such a demanding nature; tasks that promote procedural fluency are still an important part of the curriculum. The key here is that tasks should lend themselves to active student engagement and showcase the need to reason through a problem and emerge with a better, more complete mathematical understanding. Potential actions to implement such tasks are:

- Motivating students' learning of mathematics through opportunities for exploring and solving problems that build on and extend their current mathematical understanding;
- Selecting tasks that provide multiple entry points through the use of varied tools and representations;
- Posing tasks on a regular basis that require a high level of cognitive demand;
- Supporting students in exploring tasks without taking over student thinking; and
- Encouraging students to use varied approaches and strategies to make sense of and solve tasks. (NCTM, 2014, p. 24)

**Use and Connect Mathematical Representations.** "Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving" (NTCM, 2014, p. 24). NCTM (2014) notes five different representations: contextual, visual, verbal, physical, and symbolic. These different representations, although all interconnected, provide a variety of perspectives that aid in student learning and help form a clearer picture of the mathematics. Teachers should encourage their students to use different representations as tools to aid in their problem-solving approaches and to switch between representations until one produces a solution that they are able to understand. In understanding one representational approach, other representations may become clearer, and connections between the representations will be strengthened. Suggestions from NCTM (2014) to aid teachers in this practice are:

- Selecting tasks that allow students to decide which representations to use in making sense of the problems;
- Allocating substantial instructional time for students to use, discuss, and make connections among representations;
- Introducing forms of representations that can be useful to students;
- Asking students to make math drawings or use other visual supports to explain and justify their reasoning;
- Focusing students' attention on the structure or essential features of mathematical ideas that appear, regardless of the representation; and
- Designing ways to elicit and assess students' abilities to use representations meaningfully to solve problems. (NCTM, 2014, p. 29)

**Facilitate Meaningful Mathematical Discourse.** "Effective teaching of mathematics facilitates discourse among students to build a shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments" (NCTM, 2014, p. 29). Discourse can happen through discussion or other verbal, visual, or written forms of communication. The purpose of facilitating discourse is to allow students the opportunity to share and clarify their ideas and understandings. Additionally, students learn to develop the language needed to express their ideas, construct arguments to justify their thinking, and learn to see things from other students' and teachers' perspectives (NCTM, 2000, 2014).

There is no one clear path for teachers to facilitate meaningful discourse in the classroom; rather teachers need to understand their students' thinking so that they, the teachers, can create opportunities for students to build on their own ideas while the lesson's core objectives remain intact (NCTM, 2014; Engle & Conant, 2002). NCTM (2014). Smith and Stein (2011) note five practices for creating meaningful discourse through the use of students' thinking and responses during whole-class discussions: *anticipating, monitoring, selecting, sequencing, and connecting*. This means that teachers must create a classroom environment that allows students to have conversations with each other so that they can respond to one another's ideas, ask questions, and work together to formulate ideas. Teachers should also consider the framework presented by Hufferd-Ackles, Fuson, and Sherin (2004) that describes how to create a discourse-centered classroom community. This framework asks the teachers to consider how they support engagement, what questions are asked and by whom, what explanations are given and by whom, how mathematical representations are used, and what level of responsibility do the students accept for their and their peers' learning (Hufferd-Ackles et al., 2004; NCTM, 2014).

Discourse in the classroom is vital for learning mathematics. Teachers must carefully consider how to facilitate discourse to build on students' thinking and guide the students in a meaningful direction. Some suggested actions on how to create such engagement in mathematics discourse are:

- Engaging students in purposeful sharing of mathematical ideas, reasoning, and approaches, using varied representation;
- Selecting and sequencing student approaches and solution strategies for whole-class analysis and discussion;
- Facilitating discourse among students by positioning them as authors of ideas, who explain and defend their approaches; and
- Ensuring progress toward mathematical goals by making explicit connections to student approaches and reasoning. (NCTM, 2014, p. 35)

**Pose Purposeful Questions.** "Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense-making about important mathematical ideas and relationships" (NCTM, 2014, p. 35). Although a variety of questions can be used, and all are necessary for learning, these questions should be used to inform the teacher of what the student knows so they can adjust the lesson to better foster understanding, connections, and provide support for students' self-questioning (NCTM, 2014). NCTM (2014) notes four types of questioning: *gathering information, probing thinking, making the mathematics visible, and encouraging reflection and justification*. Additionally, the patterns of questions used are just as important as the questions being implemented.

It is important for teachers to note the difference between *focusing and funneling* questions (Herbel-Eisenmann & Breyfogle, 2005; Wood, 1998) as this affects student's autonomy in their learning. Funneling patterns of questioning limit the potential solution paths a student can take, to a chosen path, generally by the teacher, that leads to the desired response. This limitation eliminates the student's choice in the mathematics and therefore does not allow the student the opportunity to make sense of the mathematics themselves or build connections based on their own understanding. Focusing patterns are based on the student's understanding, and therefore requires the teacher to attend to the student's thinking and press them to communicate their thoughts clearly. This allows the student to be in control of the solution pathway, and therefore gives the student more autonomy and allows the student to have more control over their own learning.

Additionally, the use of appropriate wait-time (Rowe, 2003) is highly related to students' sense-making. If an insufficient amount of time is given to the students to think and respond about questions, then this not only limits the opportunities for students to make connections but also restricts the information gained by an assessment of the students' thinking. By giving the students ample time to think in combination with appropriate question types, teachers can extend students' ideas so that students can advance their current knowledge to make sense of new ideas and make mathematical connections. Possible actions include:

- Advancing student understanding by asking questions that build on, but do not take over or funnel, student thinking;
- Making certain to ask questions that go beyond gathering information to probing thinking and requiring explanation and justification;

- Asking intentional questions that make the mathematics more visible and accessible for student examination and discussion; and
- Allowing sufficient wait time so that more students can formulate and offer responses. (NCTM, 2014, p. 41)

**Build Procedural Fluency from Conceptual Understanding.** "Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems" (NCTM, 2014, p. 42). NCTM notes the work done by Fuson, Kalchman, and Bransford (2005) in which students' retention of procedures improved when these procedures are connected to the foundational concepts. Additionally, when students become fluent in a concept, they can work flexibly through problems and are able to produce and explain their solutions. Thus, teachers must be aware of the significance of both conceptual understanding and procedural fluency, and that procedural fluency is learned over time and requires a strong understanding of the underlying foundational ideas. Teachers can help build their students' fluency by:

- Providing students with opportunities to use their own reasoning strategies and methods for solving problems;
- Asking students to discuss and explain why the procedures that they are using work to solve particular problems;
- Connecting student-generated strategies and methods to more efficient procedures as appropriate;
- Using visual models to support students' understanding of general methods; and

- Providing students with opportunities for distributed practice of procedures. (NCTM, 2014, pp. 47-48)

**Support Productive Struggle in Learning Mathematics.** "Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships" (NCTM, 2014, p. 48). When teachers embrace students' struggles and view them as occasions to further students' understandings, teachers provide their students with opportunities that will lead to long-term benefits (Hiebert & Grouws, 2007; Kapur, 2010; Warshauer, 2011).

Struggle and exploring errors, although avoided by many teachers (Ingram, Pitt, & Baldry, 2015), is essential for student learning and should be an integral part of the classroom dynamic. When teachers allow their students to struggle, they allow their students the chance to persevere through and resolve their uncertainties. In doing so, students can develop a growth mindset (Dweck, 2006), which is a key aspect of mathematical development and an underlying assumption of educational literature (AMTE, 2017). Teachers must praise and value their students' trials and perseverance in their attempts to make sense of the mathematics. NCTM (2014) reports that teachers can embrace and support student struggle in the mathematics classroom by:

- Anticipating what students might struggle with during a lesson and being prepared to support them productively through the struggle;
- Giving students time to struggle with tasks, and asking questions that scaffold students' thinking without stepping in to do the work for them;

- Helping students realize that confusion and errors are a natural part of learning, by facilitating discussions on mistakes, misconceptions, and struggles; and
- Praising students for their efforts in making sense of mathematical ideas and perseverance in reasoning through problems. (NCTM, 2014, p. 52)

**Elicit and Use Evidence of Student Thinking.** "Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning" (NCTM, 2014, p. 53). Evidence of students' thinking comes from teachers identifying and noticing what students are thinking, planning ways to elicit students' thoughts, interpreting the evidence, and deciding how to respond the evidence (Jacobs, Lamb, & Philipp, 2010; NCTM, 2014; Sleep & Boerst, 2012; van Es, 2010). This means that evidence should be elicited and interpreted during instruction, and not be left until the formal assessments when misconceptions and errors may already be solidified in the students' processes. In gathering and using evidence of student thinking during the lesson, teachers can plan and adapt future lessons to accommodate students better. Teachers can assess, support, and extend their students' learning in the mathematics classroom by:

- Identifying what counts as evidence of student progress towards mathematics learning goals;
- Eliciting and gathering evidence of student understanding at strategic points during instruction; interpreting student thinking to assess mathematical understanding, reasoning, and methods;

- Making in-the-moment decisions on how to respond to students with questions and prompts that probe, scaffold, and extend; and
- Reflecting on evidence of student learning to inform the planning of next instructional steps. (NCTM, 2014, p. 56)

**Summary of NCTM and CCSSM.** These eight-core practices represent the essential skills teachers need to effectively teach mathematics, so their students will become proficient in the subject. Table 1 below elaborates on how the Governing Precepts, CCSSM, and Principles to Action align.

*Table 1.* Connecting the governing precepts of MEC to policy documents

| | **Principles to Action** | **CCSSM** | **Governing Precepts** |
|---|---|---|---|
| **Student Autonomy** | (2) Implement tasks that promote reasoning and problem solving<br>(4) Facilitate meaningful mathematical discourse<br>(5) Pose purposeful questions<br>(6) Build procedural fluency from conceptual understanding | *Ambitious teaching using these standards | (3) Classroom Culture<br>(4) Communication |
| **Perseverance** | Making sense of problems and persevere in solving them | Making sense of problems and persevere in solving them | (2) Persistence |
| **Justification** | (3) Construct viable arguments and critique the reasoning of others<br>(7) Look for and make use of structure<br>(8) Look for and express regularity in repeated reasoning | (3) Construct viable arguments and critique the reasoning of others<br>(7) Look for and make use of structure<br>(8) Look for and express regularity in repeated reasoning | (1) Doing Mathematics<br>(4) Communication |
| **Mathematical Language** | (6) Attend to precision | (1) Making sense of problems and persevere in solving them<br>(3) Construct viable arguments and critique the reasoning of others | (1) Doing Mathematics<br>(4) Communication |
| **Mathematical Troubles** | (4) Facilitate meaningful mathematical discourse | (7) Support productive struggle in learning mathematics | (1) Doing Mathematics<br>(4) Communication |
| **Visual Representations** | (6) Attend to precision | (1) Making sense of problems and persevere in solving them<br>(3) Construct viable arguments and critique the reasoning of others | (1) Doing Mathematics<br>(4) Communication |

The guidance from these documents on their own, although helpful, is not enough for PSTs to become effective teachers. How can teacher preparation programs use this knowledge to educate and facilitate courses and experiences that allow PSTs to model these skills and center their teaching practices around them? To answer this question, we next look at AMTE's (2017) *Standards for Preparing Teachers of Mathematics*.

***Standards for Preparing Teachers of Mathematics.***

In 2017, AMTE produced this policy document, which "describes a set of proficiencies for well-prepared beginners and programs preparing mathematics teachers" (AMTE, 2017, p.1). These proficiencies are based on five main assumptions:

1. Ensuring the success of each and every learner requires a deep, integrated focus on equity in every program that prepares teachers of mathematics.
2. Teaching mathematics effectively requires career-long learning.
3. Learning to teach mathematics requires a central focus on mathematics.
4. Multiple stakeholders must be responsible for and invested in preparing teachers of mathematics.
5. Those involved in mathematics teacher preparation must be committed to improving their effectiveness in preparing future teachers of mathematics. (AMTE, 2017, pp. 1-2)

These assumptions underly the needs of the PSTs, as well as their future students. The standards that follow are created upon the foundation of these assumptions.

**Knowledge, Skills, and Dispositions.** Beginning teachers should be proficient in their understanding of the mathematics they teach as well as the content both preceding and following the content they teach. This proficiency should go beyond what the curriculum, standards, or textbooks state so as to understand different and deeper levels of the mathematics that goes beyond the surface level (AMTE, 2017; CBMS, 2012). In addition, beginning teachers must be prepared to teach every student, have a collection of working and equitable teaching approaches, and hold a productive disposition that can be instilled in every student that they too can be proficient in mathematics.

Effective teachers must have knowledge of their students and their students' abilities (AMTE, 2017). Moreover, the teachers must be able to use this knowledge of their students to be able to assess and modify instruction for their students accurately. AMTE highlights four standards for teachers' initial preparation regarding their knowledge, skills, and dispositions. These four standards encompass the mathematical practices of this study and are related to the policy documents that were discussed above. AMTE's standards for well-prepared beginning teachers of mathematics are

> (C.1.) Mathematics Concepts, Practices, and Curriculum
>
> Well-prepared beginning teachers of mathematics possess robust knowledge of mathematical and statistical concepts that underlie what they encounter in teaching. They engage in appropriate mathematical and statistical practices and support their students in doing the same. They can read, analyze, and discuss curriculum, assessment, and standards documents as well as students' mathematical productions.
>
> (C.2.) Pedagogical Knowledge and Practices for Teaching Mathematics
>
> Well-prepared beginning teachers of mathematics have foundations of pedagogical knowledge, effective and equitable mathematics teaching practices,

and positive and productive dispositions toward teaching mathematics to support students' sense making, understanding, and reasoning.

(C.3.) Students as Learners of Mathematics

Well-prepared beginning teachers of mathematics have foundational understandings of students' mathematical knowledge, skills, and dispositions. They also know how these understandings can contribute to effective teaching and are committed to expanding and deepening their knowledge of students as learners of mathematics.

(C.4.) Social Contexts of Mathematics Teaching and Learning

Well-prepared beginning teachers of mathematics realize that the social, historical, and institutional contexts of mathematics affect teaching and learning and know about and are committed to their critical roles as advocates for each and every student. (AMTE, 2017, p. 5)

Thus, AMTE's suggestions and recommendations for PSTs' skill sets align with the theoretical framework for this study and provides evidence towards the importance of investigating ePSTs' initial knowledge, implementations, and justifications for these practices.

**Program Characteristics for Learning.** AMTE (2017) offers a set of standards for both the opportunities to learn mathematics and to learn to teach mathematics. AMTE states that an effective teacher preparation program will develop their PSTs' knowledge of mathematics and teaching practices. They also agree with *The Mathematical Education of Teachers II (MET II)* (CBMS, 2012) report that ePSTs need to take 12-hours of mathematics and statistics courses focused on the mathematics they will be teaching as well as the preceding and following courses mentioned earlier. "Opportunities to learn mathematics content needs to be **required of all who are preparing to teach** mathematics or to support student learning of mathematics" (AMTE, 2017, p. 30). In helping PSTs develop positive mathematical identities and ensuring that they have strong content knowledge and understanding of mathematical practices, programs can provide teacher preparation that will support equitable education for every learner (AMTE, 2017).

In order to accomplish such an engagement of mathematics, AMTE (2017) recommends that "programs ensure that candidates are immersed in mathematical practices and processes of reasoning, sense making, and problem solving" (p. 31). In order to make mathematical connections, candidates should be provided with opportunities to make links between mathematics and other subjects. PSTs should not only be taught such practices, but their coursework should consistently model these methods too. Thus, it is important to better align the trajectory between the content courses, the methods courses, and the field experience and observations (induction or exploration-focused experiences), as these courses should not be superficial or isolated. AMTE reports that methods courses, which lie at the intersection of content and pedagogy, should integrate a focus on deep and meaningful mathematics content knowledge, provide knowledge about mathematical learners, address teaching and learning's social contexts, incorporate practice-based experiences, and be taught by effective mathematics methods instructors (AMTE, 2017).

**Partnerships and Clinical Settings.** Partnerships should be made so that all partners are productively engaged. These partners can include mathematics teacher educators, faculty who teach mathematics and/or statistics, faculty from education and other disciplines who work with the teacher preparation programs, Pre-K-12 teachers and personnel, families and community leaders/programs, and business and industry representatives who can help with and support real-world applications and contexts (AMTE, 2017). “An effective mathematics teacher preparation program provides clinical experiences that are developed mutually with school partners, are scaffolded to build in complexity, include opportunities to work in diverse settings and with a range of learners, and are supervised by qualified mentors” (AMTE, 2017, p. 37).

These clinical experiences should be developed in collaboration with the schools as to “enact a shared vision of effective mathematics teaching” (AMTE, 2017, p. 37) so that the practices modeled in the methods courses are mirrored in the field experiences. Thus, the quality of the field-based experience is just as important as the amount of time spent in them. Observations and field experiences should be done with a critical lens to give the PST insight into teaching practices and student thinking. They also should be scaffolded in such a way as to support a trajectory of increasingly more complex teaching practices that eventually lead to classroom independence (AMTE, 2017). AMTE recommends that programs include an early field-experience to start shifting PSTs’ mathematics-teaching identities, focus their attention on students’ thinking, introduce effective instructional practices, and enable the PSTs to gain insight into what grade level(s) they want to teach.

**Lived Experiences**

Research has found that preservice teachers’ conceptions and supports (teacher moves) are influenced by many things and are multifaceted. As mentioned earlier, by conceptions, I refer to Philipp’s (2007) definitions; “general notions or mental structures encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preference" (Philipp, 2007, p. 259). A teachers' practice is informed by their conceptions; however, practice often differs from these conceptions due to outside factors such as the curriculum, standards, testing, and administration (Cohen, 1990; Philipp, 2007; Thompson, 1984; Thompson, 1992).

Moreover, the conceptions concerning teaching and learning mathematics are formed in various ways and through different experiences, most of which are related to a teacher's own mathematical experiences (Valentine & Bolyard, 2019). Research has cited that influential factors stem from a preservice teacher's own K-12 and university experiences (CBMS, 2012; Stohlmann et al., 2014), family role models (Pavlovich, 2020; Valentine & Bolyard, 2019), relationships with teachers, peers, and learning communities (Valentine & Bolyard, 2019), and their own work with students (Ambrose et al., 2004; Daniel, 2020).

However, I have found that research does not report on any non-mathematical experiences the elementary preservice teachers engage in that may affect their conceptions. I refer mainly to experiences of interaction with children, since preservice teachers' primary concern has been found to be the protection, safety, and happiness of children (Philipp, 2008).

**Teacher Education**

There are different routes to becoming an elementary teacher, and there is not a set degree requirement when it comes to the type and amount of content, methods, and practicum courses required in the U.S. Although there is research regarding recommendations on the type and number of courses needed, each university sets its own degree requirements for future elementary teachers, which seldom meet the standards suggested by research. For example, the Conference Board of Mathematical Sciences (2012) recommends that ePSTs should be taking a minimum of 12 hours of content related mathematics that provide an in-depth look into the mathematics they will teach, as well as make connections to the material in the surrounding curriculum. This recommendation for content courses is one that is seldom met as most universities have only one to three content courses (Hiebert et al., 2017)

In addition to having fewer than recommended courses, these courses are often disjoint and separated from the methods courses and school-based practicums (Grossman, Hammerness, & McDonald., 2009b; Hollins, 2011). Upon a review of the literature regarding ePSTs' mathematical content knowledge, Thanheiser, Browning, Edson, Kastberg, and Lo (2013) found that the ePSTs' knowledge of algorithms was similar to those reported by Ma (1999). ePSTs were found to have an inadequate conceptual understanding needed for teaching, difficulty with modeling teacher-like situations, problems in understanding students' unique thinking, and were challenged to be able to pinpoint students' errors (Thanheiser et al., 2013). Thanheiser and colleagues (2013) noted that only recently have studies been conducted to aid in the development of ePSTs' conceptual understanding. They suggest that this development, along with a more fluid transition between content courses, typically held in mathematics departments, and methods courses, which are held in education departments, be considered. For example, methods courses have been known to use artifacts of children's thinking for the purposes of exploring supportive teaching moves, but these artifacts can also be implemented sooner as they have been shown to increase PSTs' content knowledge (Philipp, 2007; Philipp, Thanheiser, & Clement, 2002).

In order to resolve this divide between content and methods courses, and better prepare teachers, Grossman, Hammerness, and McDonald (2009b) suggest that "teacher education be organized around a core set of practices in which knowledge, skill, and professional identity are developed in the process of learning to practice during professional education" (p. 273). Similarly, there has been a call to reform teacher education programs to provide more opportunities for practice and to integrate knowledge and skills into this practice (Ball, 2000; Ball & Forzani, 2009; Brown, Collins, & Duguid, 1989). One suggestion for this has been to incorporate children's mathematical thinking into the ePSTs' mathematics courses (Philipp et al., 2002). However, this still does not lend itself to the practice of interpreting or interacting with student ideas in-the-moment but rather presents children's ideas as static.

Another consideration has been towards early field experiences. Early instruction-focused field experiences, as compared to no early instruction-focused field experiences, accounted for a significant increase in ePSTs' content knowledge, active-learning beliefs, and math-as-inquiry beliefs (Jacobson, 2017). Additionally, although "all U.S. elementary teacher preparation programs reported exploration-focused field experience at the beginning and instruction-focused field experience at the end of the program" (Jacobson, 2017, p. 180) the duration of the exploration-focused field experience did not correlate to an increase in ePSTs' content knowledge or beliefs. It is thought that this may result from the fact that the PSTs are not likely engaged in situations that is problematic with their current practices and beliefs (Jacobson, 2017; Philipp et al., 2007). Thus, a push could be made for shortened (less than 13 contact weeks) early instruction-focused field experiences as they have been proven to increase the teacher education outcomes (Jacobson, 2017). However, it is unclear from the studies conducted whether an early field experience would improve PSTs' support of research-based mathematical practices for teaching.

**Complex Classrooms**

When teaching began to be viewed as a complex task, researchers tried to decompose the different practices that future teachers needed to know. For instance, Bellack, Kliebard, Hyman, and Smith (1966) discussed the teaching skills of structuring, eliciting, and reacting "moves". Doyle (1977) found that student teachers can become overwhelmed when entering a classroom and discussed five strategies for adapting to the complex environment, one of which was "rapid judgment, or the ability to interpret events with a minimum of delay" (p. 15). Doyle found merit in decomposing the practice of questioning, and although using a different verbiage, found benefits in student-teacher noticing of students. Similarly, AMTE (2017) notes the complexity of teaching and how PSTs should have their teaching experiences scaffolded to incorporate practices and slowly increase in complexity. Thus, researchers are focusing on the identification of important teaching situations and how PSTs respond to them.

Considering the complexity of the classrooms, and how little time PSTs spend in a classroom observing or working with students, it becomes imperative that PSTs are able to notice and interpret such situations. PSTs, however, tend to focus on sequencing events chronologically unless instructed to focus on salient moments (Sherin & Han, 2004; Sherin & van Es, 2005). Similarly, Star and Strickland (2007) found that PSTs focus on classroom management and the teacher's moves to maintain control of the classroom environment, and not the mathematics content or the teacher moves to support student thinking. However, these researchers and others (Santagata & Yeh, 2016; Stockero, Rupnow, & Pascoe, 2017; Stockero & Stenzelbarton, 2017) have found that PSTs can be taught to focus on such details.

**Mathematical Practices**

As expressed in the literature, teaching is a complex endeavor that needs to be decomposed into key practices for PSTs. These skills and practices, as denoted in the above policy documents and Governing Precepts, all encompass practices of supporting students' justification, the use of multiple representations, allowing students to make and self-correct errors, use precise language, and develop perseverance in problem-solving.

This study focuses on how elementary preservice teachers support their students' learning of mathematics; specifically, the students' learning involving the mathematical practices mentioned here. I use the term *mathematical practices*, since it is the most current verbiage in literature and refers to "what students are doing as they learn mathematics" (NCTM, 2014, p. 7). The mathematical practices that are seen here are only part of what literature notes as important, and are more visible as the mathematical *reasoning habits* seen from previous literature (NCTM, 2009) used in the creation of the new and more broad meaning of *mathematical practices*. Thus, *reasoning habits*, "a productive way of thinking that becomes common in the processes of mathematical inquiry and sense making" (NCTM, 2009, p. 9), more clearly highlight the specific mathematical practices of this study.

Therefore, it is important that teachers instill these practices in their students, but how do they do that? In order to consider this question, first we will look at a brief literature review for each of these practices.

***Justification***

NCTM (2000) stresses that students at all grade levels should be justifying and exploring phenomena and results to make sense of the mathematics. "The role of the teacher is to promote mathematical understandings through the orchestration of small group and whole class discussions where students actively participate by making explicit their thinking, by listening to contributions made by classmates and indicating when they do not understand an explanation, and by asking clarifying questions" (Anghileri, 2006, p. 44). Teachers found value in seeking out students' justification as it served as formative assessment of the students' understanding and knowledge (Staples, Bartlo, & Thanheiser, 2012).

Justification can take on many forms: argumentation, reasoning, conjectures, or proofs. Proofs are essentially a formal type of justification (NCTM, 2000), and argumentations, which lead to proofs, are developed by justification (Hoffman, Breyfogle, & Dressler, 2009). However, these skills must be developed early on and established in the culture of the classroom, which is no easy feat (Anghileri, 2006; Hoffman et al., 2009). Lo and McCrory (2010) found four elements that are an essential part of teaching mathematics with a focus on justification:

1) Knowing what counts as a valid justification for a given answer;
2) Familiarizing oneself with the struggles elementary school students may have;

3) Understanding how mathematical topics connect across operations and number systems; and
4) Knowing how to teach in a way that supports mathematical reasoning. (p. 150).

All four of these elements rely on the definition of justification.

Like the many forms it can take on, justification has various definitions amongst teachers and researchers. The overarching ideas behind many of these definitions is a focus on a progression of logical statements that form a true argument. However, students' justification may also be incomplete or incorrect (Melhuish, Thanheiser, Fasteen, & Fredericks, 2015). Thus, a definition of justification should encompass those ideas also. Melhuish and colleagues (2015) emphasize that justification should focus on why a procedure is being used, and not just how the student used it. A more specific definition of justifying used by Melhuish, Thanheiser, and Guyot (2020) was

> Reason[ing] with meaning off ideas, definitions, mathematical properties, established generalizations to
>
> - Show why an idea/solution is true;
> - Refute the validity of an idea;
> - Give mathematical defense of an idea that was challenged.
>
> (Melhuish, Thanheiser, & Guyot, 2020, p. 37)

Additionally, Melhuish and colleagues (2020) defined an explanation, which was not considered a justification, but as using procedures and facts in two different categories.

*No evidence of reasoning*

- Short answer to a direct question,
- Restating facts/statements/rules, and
- Showing or asking for procedures.

*Using meanings, definitions, properties, and known mathematical ideas to describe reasoning when*

- Explaining ideas and methods,
- Asking questions to clarify, and
- Noticing relationships/connections.

(Melhuish et al., 2020, pp. 40-41)

In their study, they found that elementary teachers' definitions of justifying varied vastly from researchers, and teachers tended to overreport the amount of justifying that happened in their classrooms. Additionally, the elementary teachers were not noticing justification at a deep level and would focus more on surface-level dialog (Melhuish et al., 2020). Similarly, when studying ePSTs abilities to justify in a mathematics content course, Lo, Grant, and Flowers (2004) found that the majority, but not all, of the ePSTs were able to develop relatively clear and valid justifications. Moreover, "[t]heir arguments were explanatory in nature and often supported with diagrams, story line, or a combination or both" (Lo, Grant, & Flowers, 2004, p.7).

***Mathematical Troubles***

Mathematical troubles have been a topic of much debate and research over the past several decades. Differentiations have been made between *errors*- "systematic, persistent and pervasive mistakes performed by learners across a range of contexts (Nesher, 1987)" (Brodie, 2011, p.28), *slips*-easily corrected mistakes (Brodie, 2011), and *misconceptions*-"the result of a lack of understanding or in many cases misapplication of a 'rule' or mathematical generalization (Spooner, 2002)" (Mohyuddin & Khalil, 2016). For this study, I will not differentiate errors from slips or misconceptions, as the purpose is to see how ePST's support students in correcting their troubles of any kind.

Although addressing errors is an often feared and avoided teaching practice, research has shown it to be extremely beneficial towards students' learning (Anghileri, 2006; Borasi, 1996; Brodie, 2011, 2014). In addition to being avoided, when mistakes are recognized, they are often teacher-directed (Rach, Ufer, & Heinze, 2013). This could be due to a goal or answer focused orientation, which would treat mathematical troubles as lacking abilities and a danger to self-esteem (Steuer, Rosentritt-Brunn, & Dresel, 2013). This orientation is related to maladaptive reaction patterns, which decreases learning motivation, reduces effort, and increases feelings of shame, hopelessness, and avoidance of challenges. However, an adaptive reaction pattern would align with a strong orientation on mastery, which means that when encountering mathematical troubles would maintain learning motivation, continue with adequate effort, and sustain functional effects such as joy (Steuer et al., 2013).

Therefore, recent research has focused on professional development, learning communities, and teacher education that elaborates on the nature and support of students' errors (Brodie, 2014; Ingram, Pitt, & Baldry, 2015; Jacobs, Lamb, & Philipp, 2010; Rach et al., 2013). Self-correction of mistakes is ideal for students to learn mathematics, but there are also several other ways a mistake can be recognized, and the correction process be initiated (Ingram, 2012). Ingram (2012) notates several paths that can take place when a *mathematical trouble* occurs. When a *mathematical trouble* occurs, the *trouble* is first recognized and then repaired. This process of initiate repair and repair can repeat multiple times and can happen on the part of the student, a peer, or the teacher.

Since ePSTs do not get to interact within a classroom setting until closer to the end of their program, little to no work is done in understanding how ePSTs work with student mistakes. However, ePSTs may be exposed to artifacts that involve children's mathematical thinking and errors during methods and/or content courses as a way to enhance their mathematical and specialized content knowledge (Thanheiser et al., 2013). Through this work, researchers have found that ePSTs "tended to use procedures, algorithms, and memorized rules to address problem situations…[but] struggled when asked to explain why the algorithms work" (Thanheiser et al., 2013, p. 13).

### ***Visual Representations***

NCTM (2000) defines representation as both a process and a product, or "the act of capturing a mathematical concept or relationship in some form and to the form itself" (p. 67). NCTM's (2000) *Principles and Standards for School Mathematics* cites representations as one of its process standards, stating that all grades should enable students to:

- create and use representations to organize, record, and communicate mathematical ideas;
- select, apply, and translate among mathematical representations to solve problems;
- use representations to model and interpret physical, social, and mathematical phenomena. (NCTM, 2000, p. 67)

Much of the research involving representation revolves specifically around secondary or tertiary schooling, mainly in the subjects of algebra and calculus. The following literature, however, is based on either elementary schooling or more general principles. "In general, multiple representations allow students to experience a variety of modes to communicate mathematics. The multimodality of multiple representations (visual, auditory, kinesthetic, and tactile) allows for multisensory experiences" (Jao, 2013, p. 11). Thus, regardless of ability, students need to have access to the mathematics and the opportunity to form their own understanding (Jao, 2013).

However, Stylianou (2010) found that for some teacher's "representation seemed to be a topic of study rather than a means of coming to understand mathematics and a tool in doing mathematics" (p 337). Thus, these teachers expressed concern regarding the addition of representations to an already overwhelmed curriculum. The same teachers also mentioned that representations were not realistic, were not required for the tests, and were an enrichment activity for advanced kids. Additionally, some teachers in Stylianou's study said that multiple representations were confusing for low-performing students.

The flaws that researchers have found in the use of multiple representations reside in the use and sequencing of representations (Jao, 2013). Professional development and/or teacher preparation programs have done little to support teachers and PSTs in integrating multiple representations successfully (Ge, 2012; Stylianou, 2010). The main suggestion made by researchers is to develop a way for teachers to sequence representations to better connect ideas (Ge, 2012; Smith & Stein, 2011; Stylianou, 2010; Witzel, Riccomini, & Schneider, 2008). The consensus is that teachers should sequence the use of multiple representations from the most meaningful and least abstract representation to the most simple and abstract (Ge, 2012; Jao, 2013).

In considering the current state of teachers' views and implementation of multiple representations, Stylianou (2010) states the following:

> It is clear that more needs to be done in expanding teachers' views on representation and providing them with instructional tools related to representation. A starting point can be to engage teachers in doing mathematics while making the role of representation explicit and in discussions about representation so that, in their turn, teachers can make their students explicitly aware of this…Hence, the importance of undergraduate mathematics courses (and, in the case of several alternative certification programs for teachers, graduate level mathematics courses) in shaping the conceptions of representation teachers develop cannot be understated. These courses need to provide prospective teachers with the opportunity to experience the use of representation as a rich and flexible tool in problem solving and in developing mathematical sophistication. Similarly, pedagogy courses need to provide teachers with tangible tools for using representation meaningfully in all phases of instruction. (Stylianou, 2010, p. 340)

Moreover, Stylianou (2010) highlights the works of Roth and McGinn (1998), Hall and Stevens (1995), and Ochs, Jacoby, and Gonzales (1994) in the different roles representation can have in problem-solving. Stylianou notes that representations can help organize information, reduce cognitive load, allow manipulation of given information, detect wrong approaches, ambiguous perspectives and information, allow sharing of strategies, and negation of different ideas.

Similarly, Singletary (2012) focuses on *mathematical connections made in practice*. She notes that there are different levels of mathematical connections and different kinds of mathematical connections made in practice. Thus, not only can representations be used for various uses in problem-solving, but the level to which they can be used to make connections in problem-solving can vary.

***Mathematical Language***

In this study, the word language is used to describe students' mathematical vocabulary usage, and not language in the sense of English, Spanish, French, or another language. As seen in the policy documents earlier, an emphasis is put on students being able to voice their reasonings mathematically and using formal mathematical vocabulary and terminology. However, getting to that stage is no easy task as "teachers have a significant role to play to mediate between the use of learners' ordinary English and the verbalization of symbolic language in an appropriate mathematical language" (Molefe, 2006, p. 510).

Research has shown that students' mathematical vocabulary is sustained and increased when connections are made to everyday language (Herbel-Eisenmann, 2002; Mullen, 2009). Researchers have classified these stages in various ways: Pirie (1998) used six classifications, Herbel-Eisenmann (2002) used three categories, and Molefe (2006) looked at a subset of Pirie's six in connection with Herbel-Eisenmann's three.

Molefe (2006) described the three categories to be *Ordinary Language*, *Mathematical Verbal Language*, and *Symbolic Language*. Molefe's *Ordinary Language* matches that of Herbel-Eisenmann's *Contextual Language*, and the combined *Mathematical Verbal* and *Symbolic Language* in Molefe's categorization matches Herbel-Eisenmann's *Official Mathematical Language*. Molefe's categorization, however, does not take into account what Herbel-Eisenmann (2002) classifies as *Bridging Languages*.

*Bridging languages* consist of *Classroom Generated Language* and *Transitional Mathematical Language* (Herbel-Eisenmann, 2002); each of which considers the idea of connecting students' everyday language with the mathematical terminology they are learning. Herbel-Eisenmann (2002) argues that in using multiple ways of referencing mathematical ideas, students learning was enhanced as more abstract ideas became accessible. This created a more natural flow to the conversation than just introducing and listing vocabulary words for the students to use and memorize (Herbel-Eisenmann, 2002). Thus, Herbel-Eisenmann used the following categories:

- Contextual Language-Language that is dependent on specific, re-occurring context or situations
- Classroom Generated Language-Language that is student- or teacher-generated. It pertains to the mathematical object, but is particular to the classroom in which it is generated
- Transitional Mathematical Language-Language that describes a location or process that is associated with a particular representation, which includes certain set phrases that are repeated often in the classrooms, but do not include a contextual reference

- Official Mathematical Language-Language that is part of the mathematical register and would be recognized by anyone in the mathematical community. (Herbel-Eisenmann, 2002, p. 102)

Therefore, the language types are useful when exposing students to mathematical vocabulary, as it is essential to provide meaning and practice across contexts, meanings, symbols, and diagrams (Riccomini, Smith, Hughes, & Fries, 2015). This view also aligns with the *lexicon perspective*, which "focuses on vocabulary acquisition (Dale & Cuevas, 1987; Rubenstein, 1996), emphasizing the importance of learning mathematical language and vocabulary for successful decoding and solving of mathematics tasks, such as word problems (Mestre, 1988; Spanos, Rhodes, Dale, & Crandall, 1988)" (Turner, McDuffie, Sugimoto, Aguirre, Bartell, Drake, … & Witters, 2019, p. 2). Turner and colleagues (2019) found this perspective to be the most common perspective amongst their early career teachers, which was of no surprise to them as it agreed with what literature had told them. Other, less common, perspectives include the *register perspective* and *situated-sociocultural perspective*. The first of which focuses on words multiple meanings in everyday life and mathematics, while the latter focuses on using and situating every day and mathematical meaning of words in prior perspectives while simultaneously using them to communicate and construct meaning (Turner et al., 2019).

Thus, Herbel-Eisenmann's types of language and the *lexicon perspective* (Turner et al., 2019) would be most evident in Riccomini and colleagues (2015) first of their three main purposes for teaching vocabulary in mathematics but can also be seen in the latter two purposes. They state that the first reason is to "provide initial instruction to promote the understanding and storage of word meaning in long-term memory" (Riccomini et al., 2015, p. 238). Their second goal takes place after the first, in that the purpose is to help students achieve mathematical fluency and retention of meanings. Lastly, building on the first two purposes, is to help students articulate their reasoning and justifications of mathematical concepts and relationships.

However, this research has classified, categorized, and remarked on what some teachers notice, but there is little research regarding mathematical language interventions (Riccomini et al., 2015) and much research still needed as to how to adapt this information for mathematics education courses (Turner et al., 2019). Moreover, only recently has there been a push for the use of precise and accurate mathematical language in the classroom. Therefore, most work on the subject is suggestive or highlights the benefits of supporting language but does not serve as a guide for the inclusion or supports of mathematical language in the classroom.

***Perseverance***

In combination with the brief literature review conducted by Bettinger and colleagues, research has connected student perseverance to many constructs: self-efficacy, motivation, mindset, locus of control or grit, or other beliefs (Bettinger, Ludvigsen, Rege, Solli, & Yeager, 2018). ; Dweck, 2006; Pajares & Miller, 1994Strategies for fostering perseverance seem to be of a similar nature. Dweck (2006) noted that perseverance is the product of a growth mindset. She states that students with a growth mindset see success as expanding their capabilities, thriving on challenges, and not giving up (Dweck, 2006). Similarly, Sun (2018) notes that students who have a growth mindset pursue goals to attain a deeper understanding. Bettinger and colleagues (2018), in agreement with many other researchers, note that “growth mindset interventions shape students' beliefs in their ability to learn and cause lasting improvements in school outcomes" (p. 2).

Dweck (2006) maintained that confidence is not always needed to persevere in a task. In order for teachers impart a growth mindset to their students, they must take care that their praises are of the child's learning process and not ability, that mistakes are not met with anxiety or concern for the child's ability but should not be glossed over either, and teaching should be focused on understanding and not memorization of facts, rules, or procedures (Dweck, 2006). In addition, teachers should also supplement textbook material with curricular tasks that incorporate opportunities for collaboration and sensitivities toward student autonomy (DiNapoli, 2016). Thus, "the aspects of classroom culture that seem to support student willingness to engage with challenging tasks are those related to the ways that the lessons are conducted and the expectations set for the students not only in terms of the mathematics but also the ways of learning it" (Sullivan, Aulert, Lehmann, Hislop, Shepherd, & Stubbs, 2013, p. 621).

Encompassed within the idea of *growth mindset*, Russo, Downton, Hughes, Livy, McCormick, Sullivan, and Bobis (2020) note that the increased study on the topic has informed and altered Australian teachers views and beliefs about struggle. Moreover, "[i]n the United States…creating opportunities for students to persist in problem solving is a tenet of effective teaching that is often described as creating the condition for *productive struggle*" (Sengupta-Irving & Agarwal, 2017, pp. 115-116)

*Productive struggle* ensues when "students expend effort in order to make sense of mathematics, to figure out something that is not immediately apparent" (Hiebert and Grouws, 2007, p. 387). Warshauer, Starkey, Herrera, and Smith (2019) found that preservice teachers (PTs) in a mathematics content course, were unfamiliar with the ideas of productive struggle and generally saw struggle as something negative. Additionally, "PTs placed the responsibility of productive struggle on the student, not the teacher, when learning mathematics…and had not considered it as a teacher-driven educational tool for learning mathematics (Hiebert and Wearne, 2003)" (Warshauer et al., 2019, p. 26). However, although the semester was not long enough to fully develop "robust mathematical interpretations" of productive struggle, most PTs were able to indicate at least one teaching strategies notated from Warshauer (2015):

> (1) ask questions to help students focus on their thinking and identify the source of their struggle, (2) encourage students to reflect on their work, (3) give time and support for students to manage their struggles, and (4) acknowledge that struggle is an important part to learning and doing mathematics (Warshauer, 2015; Warshauer et al., 2019, p. 25;\).

Furthermore, there is evidence that shows mixed results regarding teachers' comfort with pedagogies that lead to students engaging with struggle, especially low-performing students (Russo et al., 2020). Although beliefs often differ from what is incorporated into practice, Russo and colleagues (2020) found that most teachers in their study (n=93) held positive beliefs about the value of struggle, citing "benefits of struggle were the opportunities it provided students to persist through challenge, take risks, build autonomy, build confidence, foster self-efficacy, learn through mistakes, and acquire a growth mindset" (Russo et al., 2020, p. 6), and only nine teachers holding descriptive beliefs that contained neutral or negative ideas.

In an effort to illuminate teaching moves that could be made in the daily-classroom that help foster perseverance, Lewis and Özgün-Koca (2016) shared five categories of teacher moves to foster student perseverance in problems solving:

1) Selecting Mathematical tasks that require and support perseverance,
2) Talking about strategies for problem solving,
3) Demarcating phases in problem-solving process,
4) Naming feelings attendant to problem solving, and
5) Narrating internal processes.

Thus, research on these teacher moves is relatively new, and has not yet made its way to the teacher preparation work. Therefore, similar to other research about mathematical practices, Warshauer and colleagues recommend that teacher educators "introduce opportunities to connect PTs mathematical content knowledge to practices like understanding productive struggle in mathematics early in their teaching continuum" (Warshauer et al., 2019, p. 26).

***Summary of Teaching Practices***

This section is a brief overview of each of the five mathematical practices I proposed for this study. The research presented here provided the basis for the framework that was used to analyze the ePSTs' support of the practices. Moreover, research shows how these five practices are not only connected to each other but also foster student autonomous problem-solving.

**Student Autonomous Problem-Solving (APS)**

Although never explicitly mentioned, the mathematical practices from above, which were outlined by policy documents, also verify the importance of student autonomy. Individual research, however, made the connection between the two and established how in supporting the mathematical practices, one also supports student APS. Moreover, in noting this importance, one can also see how, like all practices, student autonomy is woven into the supports used for fostering these mathematical practices. Although policy documents do not mention student autonomy, it is worth noting that they consider *ambitious teaching* which "requires that teachers teach in response to what students do as they engage in problem solving performances, all while holding students accountable to learning goals that include procedural fluency, strategic competence, adaptive reasoning, and productive dispositions (Kilpatrick et al., 2001; Lampert, 2001; Newmann & Associates, 1996)" (Kazemi, Franke, & Lampert, 2009, p. 12). Thus, *ambitious teaching* can be seen as a way to support student APS and student agency.

In my study, I took a Self-Determination Theory (SDT) view of student autonomy. According to SDT, "autonomy is the psychological need to experience one's behavior as emanating from or endorsed by the self rather than being initiated by forces or events that feel alien or with which they do not identify" (Reeve, Deci, & Ryan, 2004, p. 34). Thus, student autonomy is dependent upon the support given by the teacher and the overall classroom culture. Based on the lens of SDT, research has shown that teachers who support student autonomy are more likely to listen and respond to student dialogue, ask for the opinions of students, allow students to use instructional materials, and consider students' emotional perspectives (Reeve et al., 2004; Reeve & Jang, 2006; Bozack, Vega, McCaslin, & Good, 2008).

According to Bozack and associates (2008) review of Reeve and colleagues (2004), fostering a sense of student autonomy in the classroom can be done in eight ways:

1) listening carefully,
2) creating opportunities for students to work in their own way,
3) creating opportunities for students to talk,
4) arranging learning materials and seating patterns so that students manipulate objects rather than passively watch and listen,
5) offering encouragement when students show effort and persistence,
6) giving hints and praising mastery and progress,
7) replying to student-generated questions in a contingent, satisfying way, and
8) acknowledging students' perspectives.

(Bozack et al., 2008, p. 2395)

Research shows that classrooms which support an autonomous-student environment produce students who are equipped with numerous opportunities for academic growth (Bozack et al., 2008), such as "greater perceived academic competence (Deci, Schwartz, Sheinman, & Ryan, 1981), a preference for optimal challenge (Shapira, 1976), better academic performance (Boggiano, Flink, Shields, Seelbach, & Barrett, 1993; DeCharms, 1976), increased achievement levels (DeCharms, 1976) and self-regulation, and positive coping behaviors (Turner, Meyer, Midgley, & Patrick, 2003)" (Bozack et al., 2008, p. 2391).

Although education supports the ideas and use of *ambitious teaching* and using students' reasoning, little to no work has been done regarding preservice teachers' views of student autonomy (Decker & Rimm-Kaufman, 2008; Eckhoff, 2011), which seems to be a primary function of *ambitious teaching*. Consequential results such as Eckhoff

(2011), noted that the early childhood PSTs voiced concern about their own ability to support students' creativity, which they had said was supported in part by fostering student autonomy, but do not provide details as to how student autonomy can be supported.

## Conceptual Framework

### *Personal Interest*

As a young girl, I struggled with understanding mathematics, and it wasn't until later in my schooling that I realized that the source of my troubles in mathematics was that my studies at that age were procedurally focused and answer driven. Frankly, I did not know what I was doing or why I was doing it; it was all just a bunch of disjointed rules that didn't make sense. It wasn't until fourth grade that I was able to explore the meaning behind the mathematics and make sense of it. I have also found that my colleagues have had similar elementary experiences, which led me to question how we can better prepare elementary teachers to help students, even if the teachers themselves are not super comfortable with the mathematics itself. However, in order to do this, a baseline understanding of what ePSTs take into practice is needed.

### *Topical Research*

As mentioned above, an ePSTs course work can be viewed into disjoint pieces. Conceptual content courses tend to be disjoint and separated from methods courses and the school-based practicums (Grossman et al., 2009b; Hollins, 2011). Thus, we can view this as showing a disconnect between the theory-driven courses and the practice-based work. Now, some programs are trying to account for this divide by bringing in more approximations and representations of practice into the content and methods classrooms

(Chazan & Herbst, 2012; Estapa et al., 2017; Philipp, 2007). However, these are typically static representation (animations or written work). However, those who have implemented noticing components into their courses (Philipp, 2007; Star & Strickland, 2007) found that the ePSTs' noticing can improve over time, but this still does not give the educational community insight into how the ePSTs would support these practices. Although these situations can and have proven to increase ePST noticing, it is unknown if this improves an ePST's implementation of any given practice. This implementation is typically left until the practice-based course, at which time the ePSTs are often overwhelmed by the complexities of the classroom to implement the practices they were taught to focus on fully (Doyle, 1977). Thus, my study focuses on ePSTs helping in a two-week summer math camp that allows for approximations of these practices in a more controlled mathematical environment and removes many of the complexities found in a traditional classroom setting. Additionally, the ePSTs are still in their theory-based courses and have not moved on to elementary classroom observations or student teaching, thus allowing them to implement these practices before entering into their practice-based courses.

Each ePST has been introduced to the mathematical practices of this study through their coursework and the Governing Precepts of MathKidz. I hypothesized that ePSTs personal conceptions consisted of a collection of components of the practices they had learned about through their various course work and development. Additionally, because of the unfamiliarity with enacting these practices, I further hypothesized that ePSTs supports of these practices misaligned or possibly even did the opposite of what the ePST defined for a given practice.

My intent was to observe ePSTs participating in MEC as a way to determine how the ePSTs supports of the mathematical practices and student APS aligned or misaligned with their conceptions. As noted above, conceptions inform implementation but do not always align. Additionally, conceptions are formed through multiple experiences. Figure 1 illustrates this framework.

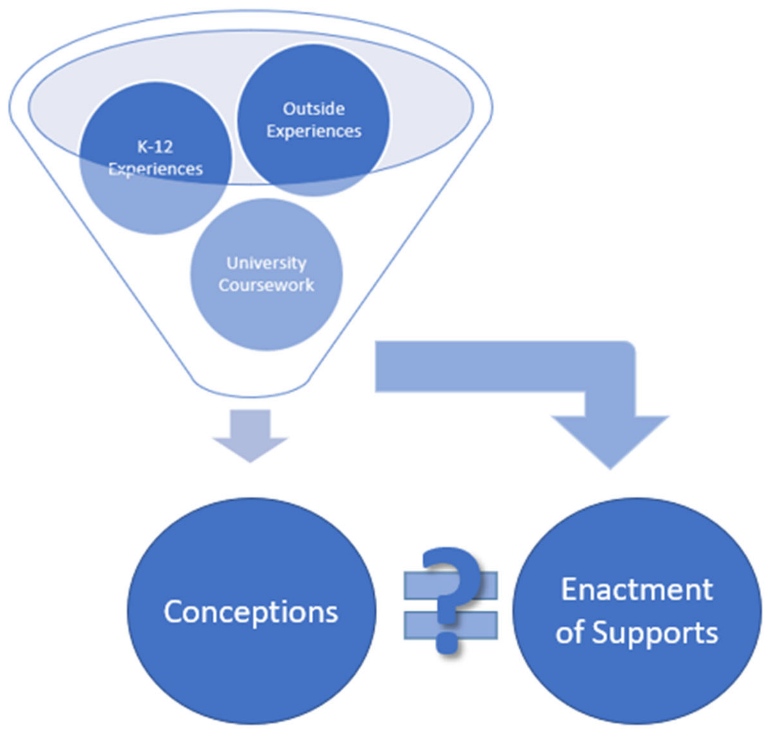


*Figure 1.* Conceptual framework

This conceptual framework provides a general idea of what this study explored. More specifically, it provides an image to support the research questions of this study.

1. What are the elementary preservice teachers' conceptions of
   a. Student *autonomous problem-solving* (APS), and
   b. the mathematical practices related to student autonomous problems-solving that are emphasized by the Governing Precepts of math camp?

ii. Fostering student *perseverance*

i. Pressing for student *justifications*

ii. Supporting students' development of *mathematical language*

iii. Allowing students to work with *mathematical troubles*

iv. Supporting students use of *visual representations*

2. What do the elementary preservice teachers notice about their fostering of student APS and their supports of these five practices during focused stimulated-recall reflection interviews?

# III. METHODOLOGY

Quality classroom experiences centered around professional practices are essential for teacher education (Ball & Forzani, 2009; Grossman et al., 2009a), as well as coursework structured to create a conceptual understanding which gives meaning to procedures (Stohlmann et al., 2014). One must also consider the complexity of the classroom when introducing new PSTs to such an experience (Doyle, 1977). Thus, being introduced to the kinds of pedagogies of practice mentioned above would be beneficial for PSTs to acquire and learn to use before entering into such a complex environment (Grossman et al., 2009a).

National policy documents such as *Standards for Preparing Teachers of Mathematics* (AMTE, 2017), *Principles to Action* (NCTM, 2014), *Principles and Standards for School Mathematics* (NCTM, 2000), and *Common Core State Standards for Mathematics* (CCSSM, 2010) have listed various key teaching practices that every teacher, including novices, should be familiar with before entering the classroom. These practices also align with the Governing Precept that MathKidz curriculum values.

Research has shown that PSTs do not focus on salient events, classroom practices, or student understanding, but they can be taught to focus on such practices and events (Jacobs, Lamb, & Philipp, 2010; Sherin & Han, 2004; Sherin & van Es, 2005; Star & Strickland, 2007). Therefore, in working in a summer camp that focuses on and supports these teaching practices, PSTs receive a unique opportunity to view and support these practices.

The practices that are focused here are fostering perseverance, supporting student autonomy, acknowledging multiple representations, working with student errors, working towards the use of precise mathematical language, and seeking student justifications. These practices can occur at various levels and can be supported in a variety of ways. Thus, the objective of this study is to understand how ePSTs support and value specific teaching practices while they interact with elementary students during a summer math camp and to seek the answers to the following research questions:

1. What are the elementary preservice teachers' conceptions of
    a. Student *autonomous problem-solving* (APS), and
    b. the mathematical practices related to student autonomous problems-solving that are emphasized by the Governing Precepts of math camp?
        iii. Fostering student *perseverance*
        i. Pressing for student *justifications*
        ii. Supporting students' development of *mathematical language*
        iii. Allowing students to work with *mathematical troubles*
        iv. Supporting students use of *visual representations*
2. What do the elementary preservice teachers notice about their fostering of student APS, regarding their supports of these five practices, during focused stimulated-recall reflection interviews?
3. What previously lived experience do the elementary preservice teachers call upon to explain their in-the-moment decisions?

In knowing and understanding how ePSTs view and support these mathematical teaching practices, MathKidz and other mathematics teaching opportunities for teacher preparation can use these findings to better serve and support the development and establishment of these and other essential teaching practices. In addition, these findings will lend insight for teacher educators to better understand how ePSTs are justifying their actions to support students. This insight can also be used to design content and methods courses that focus on the needs and exposure of PSTs to such practices earlier on in their teacher preparation.

**Pilot Study**

I conducted a pilot study at the summer camp one summer prior to my data collection. I recruited five ePSTs who were participating as Fellows in MEC to consent and join an approved pilot study regarding their camp experience. This study was conducted to uncover any connections to university coursework and any overall benefits of MEC as perceived by the ePSTs. I observed five and interviewed four ePSTs about their perceived connections and benefits. In conclusion, the pilot study produced evidence to support further investigation and led to the refined research questions of my current study.

The pilot study was not limited to a particular level or experience of the ePSTs. The participants in this pilot study consisted of all the ePSTs who were hired on as Fellows that summer. The participants were each in a different classroom and spread across different levels; there were three participants in level 1, one in level 2, and one in level 3 for a total of five ePSTs in my pilot study. In addition to the classroom differences, some of the participants also had prior experiences with MEC. Before the camp, one participant had worked for the camp once before, and one participant had already worked for the camp twice. Data gathered from this pilot study consisted of a 20-minute classroom observation, daily reflections (see Appendix A for prompts), and a single interview (see Appendix B for protocol) later that year. Of the five participants observed, only one participant, level 2, did not respond to scheduling a follow-up interview.

During the pilot study, I conducted 20-minute classroom observations of each participant to discover what each classroom looked like, as well as some of the responsibilities and conversations the ePSTs had in their setting. These observations were then discussed with the experienced teacher, Lead Teacher, of the corresponding classroom to assure that these observations were typical of the everyday camp classroom and to discover what other events frequently occurred when I was not present.

These observations showed that the ePSTs assisted in classroom management during the class discussion as well as individual and group work activities. The ePSTs were most involved during the group and individual work. They walked around the room from student to student or group to group and made sure everyone was on task and supported the students in their learning by asking and answering questions.

The daily reflections verified this finding and added more specifics on the type of questions ePSTs asked and how they responded. The reflections showed that the ePSTs observed and took part in multiple interactions involving students thinking. In line with the MEC Governing Precepts, several instances involved some level of student autonomy, perseverance, and justifications, as reported in the reflections. Other examples mentioned multiple representations, errors, and mathematical language, but not as frequently. The reflections provided evidence that these mathematical practices were witnessed, if not supported, by the ePSTs.

The amount and depth of the reflections on these practices were limited, which corresponds with what research says about PSTs noticing (Jacobs, Lamb, & Philipp, 2010; Star & Strickland, 2007). Written reflections by the ePSTs were extremely vague and difficult to categorize in terms of how practices were supported. However, in interviews, the ePSTs made it very clear that they noticed different techniques that they were either taught in their university courses or that they had never seen before and tried them out for themselves during camp. For instance, one participant mentioned how she had not yet experienced working with children in a type of classroom setting saying,

> I know how to teach math without children there, but getting the experience of working with children, and seeing how their responses affected how I respond to things and the different levels they can have…I haven't done any [observations] yet. I've done some things in high school, and I've done babysitting, but really having a classroom setting where we have a certain lesson that we are trying to teach and how to overcome the kids not understanding, but not as strict as a class; but still having just the chance to see what I can say to the kids and giving those things we get taught in class and like questions to ask or things to do, giving me a chance to actually work with them and put them into practice.

Another participant mentioned the perspective of her Lead Teacher in aiding the way she taught.

> Throughout the camp, throughout the days, it started getting a little bit easier talking to [the students], just seeing from another teacher's perspective on how she does it, I kind of learned from that and just applied it to how I was doing it, so it got easier.

Here the ePSTs watched how her Lead Teacher worked with the students and then used what she noticed to help adjust her own teaching.

These statements, combined with the ePSTs' vague descriptions of the different teaching practices, led me to believe that the ePSTs do in fact support these practices in the classroom and that video recordings and observations of these interactions would be a more productive way of capturing such instances. The pilot study, therefore, informed my study and design significantly.

**Design**

I took a qualitative approach to study the supports of mathematical practices that ePSTs used in a summer math camp for elementary and middle school students. Qualitative researchers study events, objects, or people in their natural settings, and attempt "to make sense of, or interpret, phenomena regarding the meanings people bring to them" (Denzin & Lincoln, 2011, p. 3). A qualitative approach allowed for in-depth explorations of ePSTs' interactions with students. Moreover, a case study design was used to provide a detailed look into each of the ePSTs' supports, views, and interactions (Yin, 2009).

The interactions took place in a two-week summer math camp affiliated with University X located in southern United States and will be referred to as MEC. MEC had been established for over 25 years, was built on a research-based curriculum, and is a nationally recognized summer camp for its promotion of mathematics for all students at a high level. Within MEC, this study focused on three particular ePSTs who worked for MEC as Fellows in the summer of 2019.

In order to saturate each ePSTs' case, multiple forms and sources of data were collected. This allowed for the design to evolve throughout data collection as needed in order to capture the participants' views and actions better. The final product was, therefore, written so the reader will have a sense of realism and will be able to engage in the story and believe the findings. In addition, this study was written and conducted with what Creswell (2013) considers to be *good characteristics* of a qualitative study: rigorous data collection, qualitative framing of assumptions and characteristics, use a qualitative inquiry approach, begin with a single focus or concept, extremely detailed, analysis using multiple levels of abstraction, use persuasive writing, explain how the researcher's background affected the study, and is conducted ethically. In the interest of keeping with good characteristics, it is important for me to be transparent with any personal experience with the camp or participants, as well as any underlying assumptions that affect the interpretive nature of this work.

**Assumptions**

I worked as a camp teacher for MEC for three years previous to the study. I had also taught elementary and middle school content courses to preservice teachers. That experience, in addition to my own elementary mathematics experience, led to my interest in support of mathematical practices used by ePSTs. This experience had the potential to bias my interpretations if not addressed appropriately. Thus, appropriate steps and actions (discussed in the data analysis section) were made to ensure that conclusions were not tainted by my opinions and that all results were reported truthfully from the data.

Due to the qualitative nature of this study, there are also the four main underlying assumptions of qualitative research: ontological, epistemological, axiological, and methodological (Creswell & Poth, 2018). These assumptions may have also influenced the study. Creswell and Poth (2018) defined the assumptions as follows; (1) ontological assumptions relate to the nature of reality, (2) epistemological assumptions, through proximity with the subjects, refers to what the researcher counts as knowledge, (3) axiological assumptions refers to the researcher's positionality within the field, and (4) methodological assumptions refer to the evolving research process.

This case study covers the use of mathematical practices from different ePSTs. Additionally, I was also in the classroom and observed the same events. Thus, triangulation of the events were formed from observations, interviews, daily reflections, and surveys. These supports of practices, as reported by the ePSTs, were shaped not only by the classroom and classroom teachers but also by the professional development that took place after the morning camp session each day. Thus, the professional development activities were reported when the ePSTs acknowledged the concepts in their reflections. Finally, as the study progressed, the interviews were semi-structured to reflect how each ePST interacted with the students and how the ePSTs justified their actions.

**Participants**

The participants for this study were purposefully selected from the ePSTs majoring in interdisciplinary studies from University X, who had completed at least one of the two content courses, and were participating as Fellows in camp. At University X, a PST has only two-degree tracks to become an elementary generalist: Interdisciplinary Studies (EC-6 ESL) or Interdisciplinary Studies (EC-6 Bilingual). Therefore, the participants are referred to as ePSTs. The recruitment of the participants occurred in two stages. When signing up to work for the camp, Fellows are invited to participate in research and asked to sign a consent form for research purposes for MEC. This resulted in six ePSTs selected to participate as Fellows. At the time of selection, one of the six ePSTs had two years of MEC experience, two had one year of experience, and three had no years of experience. From the six ePSTs, four were chosen and assigned to two sections of camp's Level 1.

These four were chosen in coordination with the MEC administration. One of each experience-level was chosen with the addition of a second new Fellow. Thus, one ePST with two years of prior experience, one ePST with one year of previous experience, and two ePSTs with no prior experience were chosen to serve as Fellows in the Level 1 classrooms. The choices of the experienced ePSTs were based on the previous camp research data and the ePSTs' willingness to respond to research participation requests. The two new Fellows were chosen based on their applications and hiring interviews.

During the second stage of recruitment, I verbally contacted the four ePSTs at the introductory meeting in April. A $20 visa gift card, to be delivered upon completion of the final interview, was offered as an incentive to partake in the additional interviews outside of the MEC research. For the purposes of full disclosure, the two ePSTs who had prior camp experience were a part of my pilot study, and the most experienced ePST had served as a Fellow in my camp classroom her first summer. The four ePSTs sat down with me after the first day of camp and signed written consent forms agreeing to participate in this study. However, due to missing data, one of the ePSTs was removed from the study within the first week. Thus, this study reports on three participants.

The participants will be known henceforth as Amy, Becky, and Linda (all pseudonyms). Amy, Becky, and Linda were examples of your typical elementary preservice teacher, who were excited to work with students and better their own practice. The ePSTs all said in their interviews or initial survey that they could not explain why the basic algorithms of addition and subtraction of integers worked before learning the models used in this setting or from their content course. The ePSTs also mentioned that they hoped to learn how to teach math in helpful and engaging ways for their future students from the current MEC experience. Amy had two years of prior MEC experience and had already completed a pre-K observation block at the time of the study. Linda had one year of previous MEC experience and had not completed any observation blocks yet. Becky had no prior MEC experience and had just finished her content courses. Table 2 summarizes the participants' backgrounds.

*Table 2*. Summary of participants background

| | Amy | Linda | Becky |
|---|---|---|---|
| Course Completion | Math Content 1<br>Math Content 2<br>Math Methods<br>Pre-K Observation Block | Math Content 1<br>Math Content 2 | Math Content 1<br>Math Content 2 |
| Experiences | Church Camps<br>Pre School Block<br>Babysitting<br>Church Nursery<br>MathKidz Fellow (2 years) | High school intern for kindergarten classroom<br>Volleyball coach<br>Babysitting<br>MathKidz Fellow (1 year) | Babysitting<br>High school intern for elementary school<br>Clothing store for pre-teen girls<br>Grader for Math Content 1 |
| Views toward teaching/math | "I have learned that math can be fun, and while teaching can be intimidating, and teaching math can be tough, it is very rewarding. I have enjoyed helping students enjoy math. I have also enjoyed finding a love for learning and teaching math." (Pre-Survey) | "There were math concepts such as the explanations for adding and subtracting integers. The reasons that the 'rules' worked were never taught to me. I was really excited to have that knowledge." (Pre-Survey)<br>"I hope to learn more tips, skills, and ways to teach math to my future students. I want to be able to help my students find and grow a passion for math." (Pre-Survey) | "Not that I'm not good at math, I actually really enjoy math and I like teaching it, it just takes me a long time to understand something. I need to look at something for a while." (Day 2)<br>"math is something that's so hard to teach, I'm so happy I decided to do this [camp]." (Day 2) |

*Table 2*. Continued

| | Amy | Linda | Becky |
|---|---|---|---|
| K-12 experience | "this [car model] was the first model that actually made that make sense to me…I knew, okay, well, I can just subtract, and then I would keep the sign of the bigger number. I never understood why that worked." (Day 4) | "It is a process, and it's super easy to just fall into, like, 'I follow a step-by-step thing, and just follow the steps, and then whatever.' I was super great at that. You give me steps, I'm perfect. But then if someone's like, 'Why do you do that stuff?' I was like, 'Couldn't tell you.'" (Day 5) | Talking about questioning why things worked "…and I don't really think my teachers did that to me when I was in school." (Day 4) "when I was going through school I had the subtraction sign to my negative and I don't remember what my teacher told me but I would just like make that into a giant positive, and so I never really understood why or anything like that." (Day 4) |

**Setting**

MEC is a summer camp hosted by MathKidz. MathKidz mission statement says, "[MathKidz] is a center for innovation in mathematics education at [University X]. Our mission is to research and develop model programs and self-sustaining learning communities that engage K-12 students from all backgrounds in doing mathematics at a high level". Its vision is "to be a nationally recognized leader for innovative and research-based model programs that significantly improve mathematics education". This study was at MEC, which is a half-day mathematics summer camp program for elementary and middle school students and takes place at the local high school in the same city as University X.

The camp employed a variety of different faculty positions from all levels of experience. Firstly, MEC hired seven elementary and middle school trained teachers who had completed the MEC professional development. They were knowledgeable in the Governing Precepts [Appendix E], which articulates the core ideas on which MathKidz is based and had taught at the camp for a number of years. These teachers are the primary instructors at each level and are referred to as Lead Teachers (LTs). Next, MEC employed eight graduate students (primarily Ph.D. students studying Mathematics Education) to team-teach the upper levels (Levels 4 and 5). The department had recognized these graduate students for their knowledge and understanding of mathematics, as well as their teaching skills in the college classroom.

MEC also served as a professional development program and worked with three Professional Development Teachers (PDTs) that summer. These teachers were practicing middle school teachers seeking professional development credits. There was one PDT in the Level 1 classroom. Lastly, there were undergraduate students hired on as Fellows (classroom aides). These undergraduates were engineering majors, mathematics majors, or PSTs of all levels. Thus, a typical classroom consisted of one LT or a pair of graduate students and two Fellows. Two classrooms also contained PDTs.

The camp portion, which involved the elementary and middle school students, took place from 8 am to 12 pm with a 30-minute break around 10 am. The student body was created through open enrollment and was primarily populated with students from the area, but also included students from other school districts and states. MEC served approximately 180-200 students that summer, which was partitioned into roughly two classes of 20 students per level, with 5 levels total; there were three classes of Level 3 this particular year. Students in the camp were placed into one of five different levels according to grade and reported exposure to mathematical topics.

The various levels cover the following topics:

Level 1: Integers & Algebraic Modelling (Grades 3-4)

Level 2: Algebra & Geometry with an exploration of fractions (Grades 4-5)

Level 3: Geometric relationships with a focus on graphing (Grades 5-6)

Level 4: Combinatorics with a focus on Counting (Grades 6-8)

Level 5: Logic, Number Theory, Algebra, Geometry, & Counting (Grades 7-8)

This study only focuses on the two Level 1 classrooms. Because ePSTs are generally familiar with integer manipulation, the mathematics in Level 1 should not cause a hindrance to the ePSTs' exploration of student thinking and teaching supports (Schack et al., 2013, p. 384).

In coordination with the MEC administration and to accommodate this study, the four ePSTs were assigned as Fellows to the two Level 1 classes. Each of the Level 1 classrooms contained one experienced Fellow, one novice Fellow, and a Lead Teacher. One of the two Level 1 classrooms had a PDT in it as well. Thus, classroom A had a Lead Teacher, Amy, Becky, and 19 children. The Lead Teacher in classroom A, Ms. Fray, was a practicing $4^{th}$ grade teacher with several years of teaching experience and seven years of experience teaching Level 1 by the start of the camp. Classroom B had a Lead Teacher, Linda, Tori, and 21 children. The Lead Teacher in classroom B, Mrs. Berry, is no longer a practicing teacher but did have multiple years of teaching experience at a local middle school and had taught Level 1 for eight years by the start of the camp.

During the morning camp classroom time (Implementation), the whole class discussions were led by the Lead Teacher. Group work and individual work frequently occurred, which was when the Fellows' role often came into play. The Fellows help facilitate group and individual work by answering questions, providing feedback, and assisting with other classroom needs and management. PDTs do not have a set role to play in the classroom. Most PDTs play a role similar to that of Fellows, but some simply choose to take a more observational stance. The PDT in classroom B took a role similar to that of the Fellows.

After each camp day concluded, the Fellows, PDTs, LTs, and the students ate lunch together as a class in the school cafeteria.  The students were either picked up by a parent or guardian or took the bus home.  At 12:30, the LT, PDTs, and Fellows returned to their classrooms for debriefing and reflection, which took approximately 45 minutes.

Written reflection prompts with space to write were in the classrooms for each person after lunch. After everyone had written their individual reflections (self-reflection), the LT led the group in a debriefing to discuss and analyze instances noticed in class that day and plans for the following day (shared reflection).

Finally, following the reflection and debriefing, the professional development seminar sessions (PD Seminar) were held for one hour. The LTs and PDTs were in one seminar, and the Fellows were in another. The seminars were facilitated by a university faculty and focused on a particular topic in the Governing Precept, typically the topic of the individual reflections for the day. Moreover, the seminar provided resources and presentations on the Governing Precept that would be the focus for the following day's camp class.

MEC was founded upon research-based practices, provided an approximation of practice, time to self-reflect, time to reflect and decompose practices with a community of practice, and a seminar decomposing and reflecting on the research-based practices. Figure 2 illustrates the camp structure.

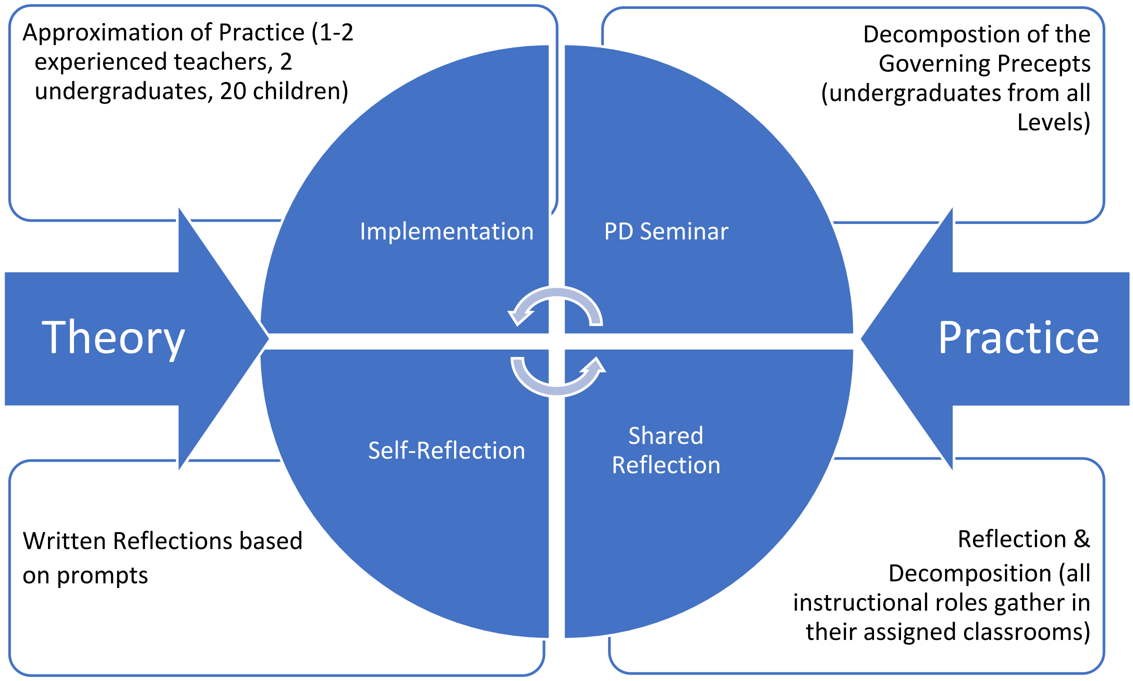


*Figure 2.* Illustration of the MEC structure

This allowed all the teachers to implement, reflect, learn, and assimilate knowledge regarding their practice of teaching. Thus, by participating in this unique experience, one can begin to uncover the conceptions of certain mathematical practices the ePSTs hold, how they support and reflect on them, what experiences they called upon in their conceptions or supports, and how their supports and reflections aligned with their conceptions.

**Data Collection**

Due to the qualitative nature of this study, dense descriptions of each case were obtained from multiple sources. The primary sources for this study were the surveys and survey reflections, interaction video recordings, stimulated recall interviews, and clinical interviews. Secondary sources of data included classroom observations, participants written daily reflections, videos of the afternoon classroom reflection debriefs, and videos of the afternoon PD sessions.

***Surveys and Survey Reflection***

I asked all the participants to complete an initial survey after the first day of camp regarding their views and definitions of the six topics. After each participant completed their clinical interview, I handed their surveys back and asked them if they would add or alter anything. These survey reflections were recorded and later transcribed.

***Interaction Video Recordings***

I alternated observation days between the two Level 1 classrooms to video group or one-on-one interactions between the ePSTs' and the students. The primary focus of these videos was on the supports used by the ePSTs' to have the students enact the mathematical practices that MEC values in their Governing Precepts. I made notes about the set-up of the interaction as well as other salient features of the dynamics and content of the conversation after each day. I also used microphones that attached to each participant during the filming of the interactions, so I could stand farther away with the camera to decrease the perception of intrusion felt by the students or ePST.

It should be noted that interactions and conversations regarding children were recorded only if parents and children opted into the MEC research, which was approved by the University IRB. Written parental consent was obtained for each child, as well as verbal assent from the child. All consent forms were signed and turned in, but some chose not to participate in the research; thus, only consented and assented conversations and interactions were collected and included in the study's data.

***Stimulated-Recall Interviews***

Based on my notes and observations from each day, I chose one to two video interactions for the stimulated-recall interview. The videos were selected based on audio clarity, completeness of recording, length and difficulty of the interaction, and visibility of supporting the mathematical practices. Each ePST was shown a clip of her own interaction and was asked to describe and discuss what occurred during the given interaction and why they chose to support the student(s) in such a way. I also asked participants to identify any teaching practices they saw within the specific video interactions, and if they called upon any previously lived experiences to help them in the situation.

Before the interaction video was played for the ePST, a basic set-up of when the interaction(s) took place and with whom was given to the ePST. As a guide, outlines of the interview questions were provided (See Appendix C). The ePSTs were allowed to watch the videos, or parts of the videos, as many times as they wanted.

I opted for stimulated-video recall to allow the ePST the opportunity to revisit the moment in its fullness instead of relying on their memory recall. I did this with the intent to make sure the descriptions of each interaction, justification, and explanation were accurate and true to each ePST and not lacking in detail. I also asked follow-up questions if I was unsure of the direction or explanation the ePST was giving. The participants also had binders with the camp's Governing Precepts and any of the professional development materials, which they sometimes referred to during the interviews.

Some interviews took place during the camp break time (approximately10:00 am to 10:45 am), while others took place after the seminar dismissed (approximately 2:30 pm to 3:15 pm). Table 3 provides an overview of the interview schedule.

*Table 3*. Stimulated-recall interview schedule

| Day | Classroom Recorded | Break Interview | Post Seminar Interview |
|---|---|---|---|
| 2 | A | | Becky |
| 3 | B | Amy | Linda |
| 4 | A | | Amy |
| 5 | B | Becky | ** |
| 6 | A | Linda | Becky |
| 7 | B | Amy | Linda |
| 8 | A | | Amy |
| 9 | B | Becky | * |
| 10 | | Linda | |

*Becky was not able to attend camp on the last day, so her clinical interview was conducted in place of Linda's one-on-one interview.
**Removed ePSTs interview

***Clinical Interviews***

I asked each participant to view videos from a professional development CD or outside research that corresponded to each topic under study. The purpose of these clinical interviews was to see how the participants' views of the topics aligned with their own supports and enactments. The videos and questions were presented in a PowerPoint presentation format to the participants, so the video could be embedded along with the questions. Unlike the stimulated-recall interviews, the questions for the clinical interview were not shown to the participants ahead of time. Rather, the questions in the clinical interview were created to gradually probe and focus the participants' attention on a specific topic. The first question for each video involved asking the ePST to state what they noticed about the video or interviewer. Each of the subsequent questions provided a little more focus on the teaching supports in the video. Thus, each question was presented and then answered immediately before I presented another question. This question-answer pairing allowed the participants to state what they noticed or found important without external influences. (See Appendix D)

I used five videos with four of them under three minutes in length and one video 7.5 minutes in length. Two videos focused on student autonomy were recommended by an expert and from an outside research project. The longest video came from a professional development CD, which encompassed the practices of perseverance, multiple representations, and student error. These practices were confirmed to be present in the video by an outside expert. Another video from an outside research project was used, which encompassed the practice of student justification. Similarly, this practice was confirmed to be present in the video by the same outside expert. Finally, after a discussion with one of my mentors, a video that showcased a student's reasoning with informal vocabulary was chosen from a professional development CD. (See Appendix D for video descriptions.)

Videos 1 and 2 showed a researcher working on a single problem with a student (each video had both a different researcher and student). Video 1 showed the researcher demonstrating a lesson and not supporting any student autonomy, while video 2 showed the researcher supporting the student autonomy. Video 1 was played first, then followed by questions. Video 2 was then shown and followed by the same questions. I asked the participants about how the teacher supported the student, and if they would do anything differently. Additionally, I ask what they noticed about the questions being used and if the student was benefiting from them. Lastly, after these questions were answered for both videos, participants were asked to compare and contrast the videos and explain whether they believed the students benefited equally or not.

Video 3 focused on a fourth-grade student's justification for why he thought $-5 + -1 = -6$. The student built on his prior knowledge of positive numbers, adding, and subtracting to reason through his thinking and justify his answer. Participants were asked to describe what they noticed about the video and the student's way of answering the question. Participants were asked to comment on the idea if no justification was shown. For instance, if the video only showed the student saying the $-5 + -1 = -6$, with no follow-up justification shown.

Video 4 showed a single student solving a three-digit addition problem with the assistance of two researchers. The student began by incorrectly denoting 1000 while adding using the partial sums method of addition. The student self-identifies this error while restating the problem and answer to the researchers. The student continued to persist through the correction of this error using multiple representations with aid from the researchers in the video. The participants were then asked what they noticed about the video, and what mathematical practices were being supported. Additional questions regarding the student's thinking, the researcher's questioning, and the student's final understanding were also asked.

Video 5 was of a student reasoning about whether the conjecture $a + b - b = a$ was a true statement. I only showed the first two minutes of this video since this was where the student used her informal definitions and descriptions to justify her reasoning. The content courses the ePSTs had already taken had covered number system. Thus, the ePSTs should have been familiar with terms such as whole numbers, integers, and rational numbers. The student in this video did not use this formal language but instead made up her own terms and definitions for these sets of numbers. This does not, however, affect the student's ability to reason through the given conjecture. Thus, this video was chosen to facilitate a discussion on the use of mathematical language. Participants were asked to describe what they noticed about the clip and if anything in the student's explanation stood out to them. Follow up questions asked about the student's vocabulary, and if it was okay that she did not use formal mathematical language.

***Secondary Data***

The secondary data collection of participants' written reflections and daily camp component videos were used for reference if a participant referred back to an event that day to triangulate the findings. Secondary data was collected for researcher reference and was not used for the analysis of an event, but as supporting evidence and insight into an unfamiliar event. Daily recordings that were not used for stimulated-recall interviews were also used as secondary data.

**Analytic Framework**

Based on the pilot study, I determined six themes that became the topics of this study. In attempting to obtain a snapshot of the ePSTs' knowledge regarding these topics at their stages of understanding, noticing, and implementation, I created an analytic framework to capture the various categories of each topic.

Figure 3 represents the unified analytic framework for the five mathematical practices and student APS that I applied to data collected for this study. The different categories of teacher moves have been designed according to a corpus of research on each particular topic. Additionally, the categories were intended to describe the ePSTs' understandings, which led to further development and refinement of the categories. This unified analytic framework has two components that are similar but serve slightly different purposes. The first part contains the Five Student Mathematical Practices and their respective categories. These practices and their categories denote what the ePSTs ask or notice of their students to support the students' mathematical practices (i.e., what the ePST asks the student to do). This is not to be confused with the larger core ideas of mathematical teaching practices but to represent categories of teacher moves to elicit or engage students in varying levels of a mathematical practice(s). However, these categories of teacher moves are all related and an essential part of those core teaching skills mentioned in NCTM's (2014*) eight mathematics teaching practices*, as seen in the section on policy documents in Chapter 2.

As the learning strands are interrelated (NCTM, 2014, p. 7), so are the mathematical practices, and the categories of teacher moves used to support the students. Thus, these categories are not mutually exclusive, nor are the practices independent of one another. For example, an ePST may ask a student to justify their thinking by explaining their work (*explain how*) while also asking about how their representation relates to the problem (*single representation, linking a representation to an idea*).

The second part of the framework contains seven ways to support student APS. These seven ways are things the ePST does to support the students or provide opportunities to engage in different ways (i.e., what the ePST does to help support the student). Like the categories of teacher's moves to support the students' mathematical practices, these seven supports can also be done in multiple ways.

<table>
<tr><th>Students' Mathematical Practices</th><th colspan="12">Categories of Teacher's Moves to Support the Students' Mathematical Practices</th></tr>
<tr><td>Justification</td><td colspan="4">No Explanation</td><td colspan="4">Explaining How</td><td colspan="4">Justifying Why</td></tr>
<tr><td>Visual Representations</td><td colspan="3">Single Visual Representation</td><td colspan="3">Multiple Visual Representations</td><td colspan="3">Linking Visual Representation to an Idea/Concept</td><td colspan="3">Connecting Multiple Visual Representations</td></tr>
<tr><td rowspan="2">Mathematical Troubles</td><td colspan="4">Teacher Correction</td><td colspan="4">Teacher Assisted in Student Correction</td><td colspan="4">Student Correction</td></tr>
<tr><td colspan="4">No Identification</td><td colspan="4">Student Identification</td><td colspan="4">Teacher Identification</td></tr>
<tr><td>Mathematical Language</td><td colspan="4">Contextual Language</td><td colspan="4">Bridging Language</td><td colspan="4">Formal Language</td></tr>
<tr><td>Perseverance</td><td colspan="4">Praises Unsuccessful Effort of Answer</td><td colspan="4">Praises Process</td><td colspan="4">Fosters Perseverance</td></tr>
<tr><td colspan="13">Seven Ways Teachers Can Foster Student Autonomous Problem-Solving</td></tr>
<tr><td colspan="4">Listen Carefully</td><td colspan="3">Create Opportunities to Work in their Own Way</td><td colspan="3">Create Opportunities for Student Talk (group or partner talk)</td><td colspan="3">Manipulate objects or Kinesthetically work instead of passively listen</td></tr>
<tr><td colspan="5">Offer Encouragement when students show Effort or Persistence OR Praising Mastery or Process</td><td colspan="4">Giving Hints and Replying to Questions</td><td colspan="4">Acknowledging Students Perspectives and Interests</td></tr>
</table>

*Figure 3.* Analytic framework

This framework was used to analyze the conceptualization of the ePSTs from their surveys and clinical interviews. Similarly, this framework was also applied to the daily interviews and interaction videos.

The mathematical practice of justification was based on Melhuish and colleagues (2015) concept that justifications can be incomplete or incorrect, but should focus on *why* a claim, procedure, or decision holds. An ePST can choose to accept a student's answer without any explanation, press for an explanation for what they did or how they did it, or press the student for an explanation that focuses on why their ideas work mathematically. This category did not distinguish between a generic press for justification such as, “Why?”, from a specific press for justification such as, “Why did you move right when adding?”.

The mathematical practice of using *Visual Representations* was based on Stylianou’s (2010) call to make PSTs more aware of multiple representations and their uses. I further defined my categories of *Visual Representations* by Singletary’s (2012) work with mathematical connections. Thus, this study focused on ePSTs’ decisions to use a single representation, notice or press for multiple representations, linking a representation to an idea, or make connections between multiple representations. The first two categories, *Single Representation* and *Multiple Representations*, relate to Singletary’s (2012) ideas of *No mathematical connection* or *Suggested mathematical connection*, in that the ePSTs noticed, pressed for, or reflected on the representations but did not suggest or suggest what the representation(s) were connected to. Furthermore, I used Singletary’s ideas of *Provided mathematical connection* and *Provided-and-explained mathematical connection* when defining the last two categories of *Visual Representations*. I described *Linking a Representation to an Idea* as connecting a Representation, or multiple representations, to a problem, concept, process or idea. Similarly, I defined *Connecting Representations* as providing a connection between representations that is more than answer focused.

The practice of mathematical troubles was based on definitions, trajectories, and ideas from Ingram’s (2012) dissertation work on whole-class interactions. However, first, I note that Rach and colleagues (2013) asserted that teachers tend to correct or notify students of their mistakes. Conversely, it is better for the student if they self-identify and correct their own mistakes (Ingram, 2012). In addition, in agreement with Ingram, I added that mistakes could also go unnoticed or corrected with the help of teachers.

A mathematical trouble is more than just a mistake or error but can also include other difficulties that occur in an interaction (Ingram, 2012). Similarly, "[r]epairs involve resolving a source of trouble to enable the interaction to continue successfully whilst correct only applies to the replacement of something 'incorrect' with the 'correct' form." (Ingram, 2012, p. 164) Thus, I focus on what happened after a mathematical trouble occurred, in that I looked at who identified the trouble, if at all, and who made the correction. I used the word correction to dictate who resolved or used the repairs to reconcile the larger trouble. Therefore, a teacher (the ePST) could have made the correction, a student could have made the correction, or the teacher may have repaired a smaller trouble which led to a student correction. I refer to the last correction option as *Teacher Assisted in Student Correction* because of the alignment it had with my participants' descriptions. It is also worth noting that Ingram had nine trajectories for a student trouble, but due to the individual discussion nature of this study, Ingram's was regarding whole-class conversations, I do not include the "other-initiation" trajectories. Additionally, I defined an identification to belong to the person who verbally or physically noted that something was incorrect, not to be confused with a teacher who probed a student to explain their work when an answer happened to be incorrect.

The practice of *Mathematical Language* stemmed from a combination based on Herbel-Eisenmann's (2002) framework for classroom dialog and Molefe's (2006) discussion of Pirie's (1998) ways of mathematical communication. My framework categorizes whether ePSTs supported students' use of *Contextual Language*, *Bridging*

*Language*, or *Formal Language*. Each of these categorizations were based on Herbel-Eisenmann's and Molefe's ideas.

*Contextual Language* was drawn directly from Herbel-Eisennmann and related to Molefe's *Ordinary Language*. This is "language that is dependent on specific, re-occurring contexts or situations" (Herbel-Eisenmann, 2002, p. 102) or "language that learners use in their everyday vocabulary, which is part of their informal talk" (Molefe, 2006, p. 507).

Herbel-Eisenmann breaks *Bridging Language* into two categories, but for this study, I left this as a single category and compounded the definitions. *Bridging Language* refers to "[l]anguage that is student- or teacher-generated. It pertains to the mathematical object, but is particular to the classroom in which it is generated" or "[l]anguage that describes a location or process that is associated with a particular representation" (Herbel-Eisenmann, 2002, p. 102).

Lastly, I used *Formal Language* to denote the language which corresponded to Herbel-Eisenmann's *Official Mathematical Language (OML)*. I chose the categorization of *Formal Language* because of Molefe's (2006) classification of Herbel-Eisenmann's *OML* as *formal*. Moreover, Molefe cites Primm to note that the mathematical registry is still developing. Still, words which belong to it must hold or express the inherent mathematical properties needed for its mathematical purpose. Thus, I define *Formal Language* as Herbel-Eisenmann established *OML*, in that "[l]anguage that is part of the mathematical register and would be recognized by anyone in the mathematical community" (Herbel-Eisenmann, 2002, p. 102). Additionally, the definition of *Formal Language* used here also included symbolic or notational language.

The purpose of the *Mathematical Language* categories was to capture when ePSTs support the students' use and understanding of the language they used. A student may have used *Contextual Language*, but the ePST may have chosen to press the student to use a different language such as *Bridging Language* or *Formal Language.* Additionally, the ePST must have drawn attention to the language for it to have been captured by these categories; simply using or allowing the words to be used was not enough.

The mathematical practice of *Perseverance* was based on both Dweck's (2006) growth mindset ideas and Lewis and Özgün-Koca's (2016) ways of fostering perseverance. Based on Dweck's approach, the ePST could have chosen two routes: (1) the route which can produce a fixed mindset and decreased perseverance by praising the student's unsuccessful effort or answer, or (2) they could have chosen the direction of a growth mindset and praised a productive process that yielded understanding.

The third category was adapted from Lewis and Özgün-Koca's (2016) shared teaching moves "that constitute teachers' efforts to foster student perseverance in problem solving" (Lewis & Özgün-Koca, 2016, p. 109). Adapting some of the teaching moves from Lewis and Özgün-Koca's whole class orchestration to a small group or individual conversations surrounding pre-determined problems, I established five moves similar to their five themes: (1) attending to students' emotional needs, (2) focusing the discussion on the strategy or different strategies, (3) changing the participation format of the conversation, (4) creating opportunities for students to reflect on their work, stuck points, or the language of the problem, and (5) creating an opportunity for the students to extend their knowledge beyond the problem.

Finally, *Fostering Student Autonomous Problem-Solving (APS)* was evaluated based on the eight ways of fostering student autonomy (Bozack et al., 2008; Reeve et al., 2004). However, Bozack and colleagues' eight ways were meant for all sizes of classroom conversations, including whole-class discussion. Thus, some ways of fostering student autonomy were supported infrequently or in similar styles to other ways of fostering student autonomy. Therefore, I arranged Bozack and colleagues' eight ways into seven ways to *Foster Student APS* for small group or individual conversations. My first four categorizations were the same as Bozack and colleagues' first four ways, and my final three categories were comprised of the rearranged and slightly modified final four ways from Bozack and colleagues.

Bozack and colleagues' last four ways of fostering student autonomy were: "(5) offering encouragement when students show effort and persistence, (6) giving hints and praising mastery and progress, (7) replying to student-generated questions in a contingent, satisfying way, and (8) acknowledging students' perspectives" (Bozack et al., 2008, p. 2395). I modified these to fit the situations presented in my study since most student questions were in regard to an answer or a mathematical trouble and not a new idea or conjecture. Additionally, the ePSTs would often answer a students' question with a follow-up question or a hint. Thus, I designated the final three ways of fostering student APS as (5) offering encouragement or praise when students show effort, persistence, or mastery of a concept or process, (6) giving hints or replying to student questions in helpful ways, and (7) acknowledging students' perspectives and interests. Although a slight addition was made to "(8) acknowledging students' perspectives" to create my seventh category, it still aligns with Bozack and colleagues' ideas since they view it as related to the question, “How does the teacher make information relevant to students?” (Bozack et al., 2008, p. 2395).

In summary, a combination of ideas, theories, existing frameworks, and additional information gained from my participants informed the categories of these six topics (Five Practices and Student APS) that were the focus of my study. Although some practices have mutually exclusive categories, not all the categories within a practice are mutually exclusive. For instance, under the practice of *Visual Representations*, the categories of *Single Representation* and *Linking a Representation to an Idea* can happen simultaneously. Moreover, one or more ways of fostering student APS can coincide. I used this framework to describe the ePSTs’ conceptions, supports, and reflections of the six topics.

Additionally, by supporting the five practices, the ePSTs also support APS. General connections made from the definitions seen in Chapter 2 include using representations as a way to actively engage students in manipulating objects during problem-solving, pressing students to develop their mathematical language, or provide justifications requires listening carefully to students. Additionally, by having the students provide justifications, the students are being allowed to work in their own ways. Fostering perseverance encompasses multiple ways to foster student APS, such as changing the participation format to include group work or attending to students’ emotions or encouraging them to reflect on problems. When working with mathematical troubles, helpful hints or responses to student questions are often needed. Lastly, using or pressing for representations, mathematical language, and working with mathematical troubles all lend themselves to fostering student APS through acknowledging students' perspectives.

**Data Analysis**

Data analysis happened in three phases. The first phase occurred during the collection to provide context for the interviews. After filming each day, I examined the data to find clear evidence of the ePSTs supporting at least one of the teaching practices from this study. These videos provided context, so questions could be asked to gauge if my interpretation of the situations were accurate.

The second phase of the data analysis took place after all the data had been collected from each participant. Participants' clinical interviews were transcribed and coded according to the analytic framework. The framework was then revised to better capture and distinguish any subtle differences or descriptions the ePSTs were providing. A fellow Ph.D. student assisted in revising the codes to check for definition clarity and reliable usage of each code in the clinical interviews. A second fellow Ph.D. student then conducted a reliability check using a clean and unidentifiable copy of the clinical interview data and newly structured analytic framework. The participants' surveys and survey interviews were then analyzed using the verified analytic framework to help answer research question one. Thus, providing insight into how the participants viewed these practices to explain why they did or did not recognize these practices in their own teaching.

All interaction videos were transcribed and coded according to the levels of support given for each practice, if they appeared. This coding was based on the levels described in the analytic framework of this study. This coding informed the results of the second research question by describing the extent the ePSTs supported these practices. Additionally, the levels of support mentioned by the ePSTs in the clinical interview helped to triangulate their views with their actions.

The daily interviews were transcribed and coded through an open coding scheme (Creswell & Poth, 2018) in addition to the levels of support from the analytic framework. These coding schemes informed why the ePSTs acted to support the students in the manner that they did, and to what level they saw themselves supporting these practices. Thus, this information completed the results for the first research question and provided the justifications needed to answer the third research question.

In the third phase, the codes from the interaction data were compared to the participants' categorical definitions of the practices found in the written survey, survey interview, and clinical interview. Similarly, the interview codes were compared with the participants' categorical definitions. Codes were compared across the participants to look for commonalities and differences that could be explained by the various levels of experience in this camp setting. These comparisons allowed for a more coherent and logical description of how and why the ePSTs supported the students through the interactions.

As a comparison across the cases was made in phase three, I wanted to ensure similarities in participants' anticipated degree path and coursework, so as to make sure all ePSTs had the same coursework history to call upon (although some were a little farther along than others). Additionally, every effort was made to ensure that a rich amount of interactions and interviews were captured from each participant. This, unfortunately, called for the elimination of one of the original four participants during the study.

Tori had several (3 days) absences during the 10-day camp for personal reasons, and the classroom observation data decreased significantly than the other three participants. I also became aware, partway through the data collection phase, that Tori was planning to undertake an alternative certification method and had a different major (Bachelors in Applied Arts in Education) from the other three participants. For these reasons, I chose to remove Tori from the study.

**Trustworthiness**

I ensure reliability for this study with a variety of techniques. Firstly, I verified my findings through triangulation across surveys, video analysis, reflections, clinical interviews, and stimulated recall interviews. This collection across multiple sources allowed for the participants to discuss, review, and reflect on their conceptions, ideas, and teaching supports in multiple different ways, which illuminated how their views aligned with the supports they provided to the students. Secondly, I was able to conduct a member check with two of the three participants about their conceptions and views in this study. I spoke individually with the participants to verify the accuracy of their conceptions I perceived from the time of the study. Next, a colleague provided an external audit of my coding framework using examples I gave of my codes. This audit provided more nuanced descriptions of how I was defining and applying my codes.

Lastly, I conducted an external reliability check for the coding of my clinical interviews and stimulated recall interviews. I selected a random 40% of the tasks from the clinical interviews to be checked, with at least one task belonging to each participant. Similarly, I selected a random 25% of the stimulated-recall interviews to be checked, or a random one interview out of the four for each participant. The external coder was provided with my audit-modified analytic framework and anonymized transcripts from the random selection and asked to code independently for the appearance of a coding category from my analytic framework. I based the reliability on the appearance, or lack of appearance, in a given task.

For the clinical interviews, the defined unit of analysis was the video task. Each check clinical interview task returned with greater than 80% agreement, with the remaining differences resolved through discussion. Similarly, the stimulated-recall interviews served as their own self-contained unit for analysis. Although video-tasked based, the participants would often intertwine their video reflections throughout the interview, making the tasks impossible to separate reliably. Thus, interviews were checked holistically for the presence of coding categories from my analytic framework. However, different from the clinical interviews, codes had the option of being present in two ways: the participant noticed or pressed for a given code in her interactions, or the participant would have liked to change their interaction to involve or modify a given code. The check for Case 1 returned 82% reliability, Case 2 returned 78% reliability, and Case 3 returned 76% reliability. In both Case 2 and Case 3, the external coder consistently missed instances of representation and perseverance, which caused the lower percentages of reliability. Nevertheless, after discussion, all differences in coding were resolved.

During resolution discussions, conversations also reaffirmed the coding definitions of the analytic framework to better account for our discussions and agreed-upon codes. I then reevaluated all remaining interviews for the presence of coding categories according to the finalized framework.

# IV. FINDINGS

The goal of this study is to see how elementary preservice teachers conceptualize, use, and reflect upon supporting young students' mathematical practices. Data to support this study was collected from three cases and analyzed using qualitative methods. This chapter will focus on answering the research questions for each of the three cases.

1. What are the elementary preservice teachers' conceptions of
    a. Student *autonomous problem-solving* (APS), and
    b. the mathematical practices related to student autonomous problems-solving that are emphasized by the Governing Precepts of math camp?
        i. Fostering student *perseverance*
        ii. Pressing for student *justifications*
        iii. Supporting students' development of *mathematical language*
        iv. Allowing students to work with *mathematical troubles*
        v. Supporting students use of *visual representations*
2. What do the elementary preservice teachers notice about their fostering of student APS, regarding their supports of these five practices, during focused stimulated-recall reflection interviews?

The first section will focus on Amy, the second section will focus on Linda, and the third section will focus on Becky. The last section will focus on a cross-case analysis of the thee ePSTs.

At the beginning of the camp, each participant was asked to write down their conceptions and rationales related to a) perseverance, b) student autonomous problem-solving, c) justifications, d) mathematical language, e) mathematical troubles, and f) visual representations. Specifically, to align with language matching the Governing Precepts I asked about a) Fostering persistence, b) Student Autonomy (Student-generated dialog and strategies), c) Justification of student thinking, d) Precise mathematical language, e) working with mistakes, and f) using multiple representations/strategies. These responses were revisited in an interview after the two-week camp concluded.

Over the two-week camp, the participants partook in the daily camp routine, reflection time, and professional development. Additionally, each participant was interviewed through stimulated recall videos every other day. These interviews were based-off of recordings taken that day or the morning of the previous day. These videos and reflections will help to answer research question two, regarding what practices were the ePSTs supporting and which of the practices did they focus on in their reflections.

## Case 1: Amy

Amy was the most experienced participant in my study. She was in between her pre-K and elementary observation block (exploration-focused field experience) when camp took place. This summer was Amy's third time participating in the camp and professional development setting. Each year the camp focuses its professional development around the governing precepts of MEC and the experiences that take place within the classrooms.

Amy's first year at MEC took place immediately following her first university mathematics content course for teaching. Her university instructor for the class was also one of her MEC lead teachers, with me as the other. The MEC class content involved early set theory, counting, combinatorics, and probability. So, although the material did not align with her university mathematical content courses mathematics topics, it did align with the ways to support children's mathematical thinking. The following summer, Amy worked as a level one fellow. Thus, Amy's second year at MEC allowed her to work with supporting the governing precepts with children in the same content area as this study, but with a different lead teacher. Thus, Amy had worked with the governing precepts of MEC and supported students using the governing precepts prior to this study.

### *Conceptions*

After the first day of camp and before the professional development portion began, I asked Amy to define student autonomous problem-solving (APS) and the five mathematical practices of this study that are related to APS: perseverance, justification, mathematical language, mathematical trouble, and visual representations. I also asked her what she valued about each of these practices. I did not differentiate between student

APS and the other five practices during the study but focused on the practices equally. Thus, Amy was given a list of six mathematical practice-related foci and asked to write about her conceptions and values of all six.

**Student APS.** When asked about student autonomy in the form of student-generated dialog or student-generated strategies, Amy stated the following:

> Observing students take charge of their learning. The value in this is that students that can work on strategies and stuff end up having a sense of ownership of their learning. (Amy, written survey)

Thus, Amy's views of student APS involve using and following the students' strategies and ideas (*create opportunities for students to work in their own way*) instead of using a teacher-directed strategy. Amy indicated that she valued the students' ideas (*listen carefully*) and wanted the students to use their ideas when solving problems to create a sense of ownership in their learning. When asked to reflect on this definition at the end of the two-week camp, Amy stated that "student autonomy is built like overtime working with the teacher, because you're trying to figure out like what interests them, like how to get interested in that subject" (Amy, survey interview). This statement suggests that Amy also conceptualized supporting APS as making the problems interesting and relatable to the students (*acknowledge student perspectives and interests*).

Moreover, before the survey interview, Amy partook in a clinical interview where she watched multiple videos (of other people). During the clinical interview, I asked Amy to comment on what practices she noticed and what she thought about the teachers' support of those practices. For example, when reflecting on a task that showed in one video a teacher-led strategy, followed by a video with a student-led strategy, Amy was able to identify the lack of Student APS in the first video before even viewing the second. Amy spontaneously noted that "she [the teacher] guided him [the student] really fast,…I don't think it was so much of him figuring out – it out on his own, but like being guided to the answer". Amy also added that she didn't believe the student fully understood how he came to the answer under the teacher's guidance and strategy. Whereas in the second video that allowed the student to use her thinking, Amy mentioned, "I think [the student benefited] more than the first one [student] because she [the second student] really is figuring a lot of this stuff out by herself." Therefore, not only did Amy conceptualize student autonomy as a way to allow students to take ownership of their learning, but also as beneficial to building understanding.

Additionally, Amy noticed that the manipulatives in the second video were more accessible to the student saying, "I liked – the first thing that I noticed was the blocks were in front of her to start" (Amy, clinical interview, Task 1 Video 2). Amy noticed that the availability and proximity of the manipulatives to the student served as a way to support the student's autonomy in problem-solving. From this statement, I include the idea of *arranging learning materials so students can manipulate objects* as a way Amy conceptualizes fostering student APS. Although she did not mention this in her survey and survey reflection, this was a teaching move spontaneously mentioned by Amy when I asked her what she noticed about the teachers' approach when supporting the student. Therefore, due to the spontaneity of her answer and her positive remarks towards it, I included it in Amy's conceptions of student APS.

**Perseverance.** When conceptualized perseverance, I asked Amy to think about persistence as this was the term used in the governing precepts. She wrote the following:

> Allowing students to have enough wait time. Asking guiding questions instead of giving direct answers. The value is creating a growth mindset which gives students endurance to work on hard problems longer. (Amy, written survey)

Thus, Amy's definition aligned with the ideas of *fostering perseverance* because it focused on providing time for the students to work on the problem, while focusing them on the process and strategies through questioning during the problem-solving process. Additionally, one can see that Amy attributed the value of perseverance to the amount of time spent working on a problem and establishing a growth mindset. At the end of camp, when I asked Amy to reflect on what she had written about perseverance, she said, "I think like if I would add something, something that we talked about in seminar was asking purposeful questions and so I think that's more important than just like guiding questions, … purposeful questions would be like asking questions for understanding". This addition, although clarifying what type of questions Amy would use to foster perseverance, still did not add or alter Amy's conception of the mathematical practice.

**Justifications.** Similarly, when asked about justification of students' thinking, Amy mentioned the following:

> It is really important as a teacher to get your students to explain their thinking. This is valuable because it is the best way to survey their thinking. (Amy, written survey)

Here, Amy focused on students explaining their thinking, but not distinguishing between explaining *how* they were thinking about something and *why* they were thinking about it. However, similar to her rationale for student APS, Amy found justifications to be valuable in the sense that it provided the teacher insights into their students' thinking.

When reflecting on this idea during the written end of camp reflection, Amy noted that just getting an answer did not help the student nor the teacher, because it didn't lend insight into the student's understanding. Amy continued to say, "they [the student] can get a right answer by just kind of guessing or just going through the process, but then (they) don't know how to explain the process, because they're really not understanding." This extra input from Amy during the survey reflection showed that Amy cared about students, both explaining the process of getting their answer and also explaining why it worked. Therefore, Amy's definition of justification aligned with both *explaining how* and *justifying why*.

Moreover, in multiple tasks during the clinical interview, Amy remarked that the teachers in the videos did not push the students for justifications, but just accepted the students' answers and moved on. Additionally, regarding a video when a student did justify his answer, I asked Amy what her thoughts were if the video were to end before the student justification, to which she responded, "I would have been upset (laughs). Because you know, that doesn't – just getting the answer doesn't show any understanding."

**Mathematical Language.** Amy stated the following when asked about her conceptions about the practice of using precise mathematical language.

> This comes from practicing the math language and modeling how the math language should be used. The value of this is having a better understanding of math and being able to articulate that understanding. (Amy, written survey)

Amy's survey conception show that she believed using mathematical language, such as proper textbook vocabulary, will allow for better student's understanding and communication of mathematical ideas. This definition emphasized the idea of using only *formal language*, which can be understood by the larger mathematical community outside the classroom. However, when reflecting on her survey, Amy mentioned that although teachers need to model formal language, it also depends on where the students' current understanding lies. Amy recalled a clinical video task as an example of how the student in the video was lacking the official vocabulary but was still able to define what she was talking about and communicate her thinking. Thus, Amy's conceptions about using mathematical language after the survey reflection also illustrate her understanding of the importance of using both *formal language* and *bridging language*.

**Mathematical Troubles.** Amy, again relating to students' understanding, noted the following conception when referring to student errors and mistakes. (Mistakes in the camp setting also include misunderstandings and misconceptions, which is why this study refers to them as mathematical troubles.)

> Mistakes are great to work with because if you can work with students and ask them questions which guides them to better understand these concepts. (Amy, written survey)

Here Amy mostly focuses on a rationale, but one can gather that Amy leans towards students using their ideas with some assistance from the teacher when working with mathematical troubles. Amy's conceptions about how to identify the mathematical trouble were unclear from her survey. In the survey reflection, Amy mentioned that a student can form a better understanding of the material and affords a deeper exploration of why a process works. Additionally, through a guided exploration of their thinking, Amy mentioned that she noticed that the students would be able to slow down and figure out where they made a mistake. When I prompted her to elaborate on this idea more, Amy said, “I want them [the students] to kind of see like hey, maybe that’s not right.” Thus, Amy not only conceptualizes using mathematical troubles as something the students should correct themselves (*student correction*) or should be assisted with in their correction (*teacher assisted in student correction*), but she also sees the benefit in the students finding the mathematical trouble themselves (*student identification*). Additionally, in the clinical interviews, Amy voiced her frustration when a wrong answer went unexplored or was corrected by the teacher, and happily pointed out when a student noticed their own error and self-corrected or was assisted in the correction by the teacher.

**Visual Representations.** Lastly, Amy was asked about her conceptions regarding the use of multiple visual representations.

> The benefit of this (as someone that really struggled with math), is that seeing the problem or concept represented in many ways helps to make those concepts concrete in your brain. (Amy, written survey)

Amy’s survey conception statement focused on using multiple representations (*multiple representations*) as beneficial for understanding but did not go beyond linking each individual representation (*single representation*) to a given concept or idea (*linking a representation with an idea/concept*). Similarly, in the survey reflection interview at the end of the two-week camp, Amy agreed with the beneficial nature of having a model or something tactile to work with and how it aids the student when trying to learn a concept and explain their thinking. Additionally, during the clinical interviews, Amy frequently and consistently pointed out when a visual representation was present and being used, and even made reference to the beneficial nature it had in aiding in the student’s understanding and use of linking a strategy or idea to the problem at hand. Amy did not mention the idea or nature of connecting multiple representations together or using a representation to aid in the understanding of a different representation in the survey, survey reflection, or the clinical interview. Rather, Amy would always link the representation back to the process, problem, or concept under study, and never mentioned using the models to understand the steps or procedures of a process.

**Summary of Amy's Conceptions.** Through the survey given at the beginning of camp, her survey reflection interview, and her clinical interview Amy conceptualized the five mathematical practices and student autonomy. Additionally, without being asked or prompted to do so, she connected some of the mathematical practices together but also to student APS. Figure 4 highlights Amy's conceptions of the five mathematical practices and student APS according to the analytic framework used in this study.

| **Students' Mathematical Practices** | **Categories of Teacher's Moves to Support the Students' Mathematical Practices** | | | |
|---|---|---|---|---|
| Justification | No Explanation | Explaining How | Justifying Why | |
| Visual Representations | Single Visual Representation | Multiple Visual Representations | Linking Visual Representation to an Idea/Concept | Connecting Multiple Visual Representations |
| Mathematical Troubles | Teacher Correction | Teacher Assisted in Student Correction | Student Correction | |
| | No Identification | Student Identification | Teacher Identification | |
| Mathematical Language | Contextual Language | Bridging Language | Formal Language | |
| Perseverance | Praises Unsuccessful Effort of Answer | Praises Process | Fosters Perseverance | |

| **Seven Ways Teachers Can Foster Student Autonomous Problem-Solving** | | | |
|---|---|---|---|
| Listen Carefully | Create Opportunities to Work in their Own Way | Create Opportunities for Student Talk (group or partner talk) | Manipulate objects or Kinesthetically work instead of passively listen |
| Offer Encouragement when students show Effort or Persistence OR Praising Mastery or Process | Giving Hints and Replying to Questions | Acknowledging Students Perspectives and Interests | |

*Figure 4.* Amy's conceptions

Moreover, we can form a picture of how Amy conceptualized the use of these practices and how they related to her conceptions of student APS. Figure 5 summarizes the connection between Amy's conceptions and fostering student APS.

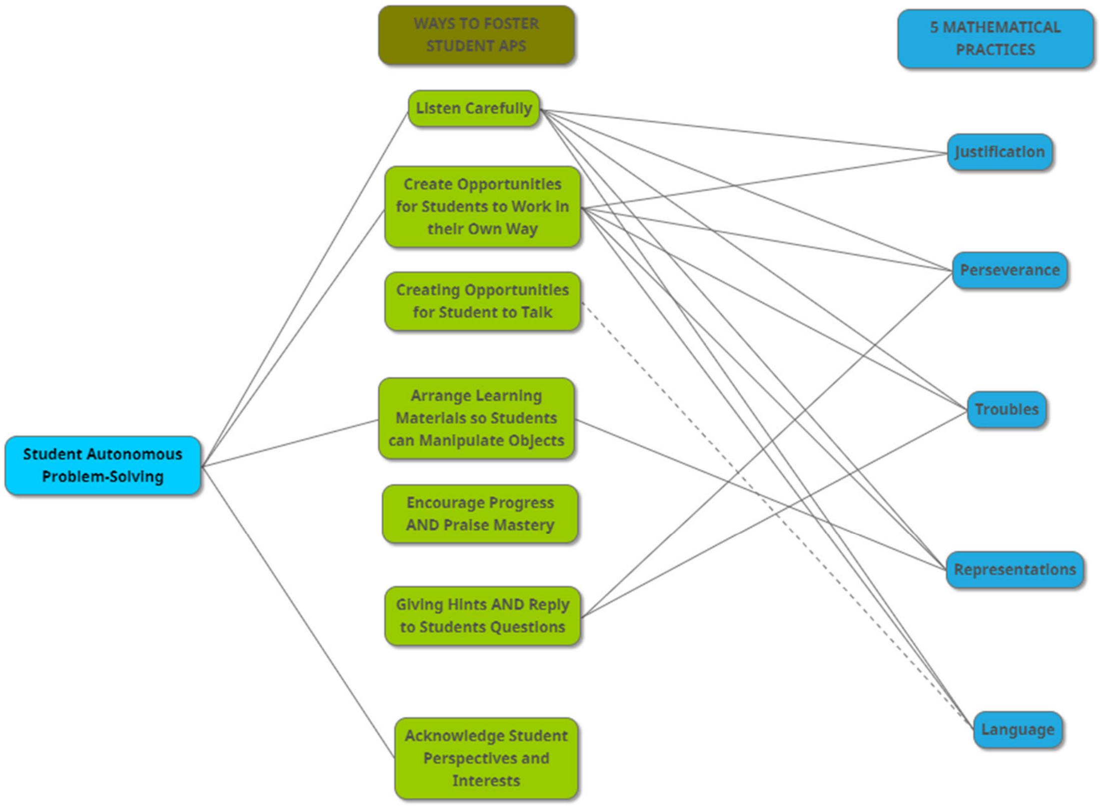


*Figure 5.* Amy's conceptions as related to student APS

Figure 5 illustrates Amy's conceptions of student APS as related to acknowledging student perspectives and interests, listening carefully, creating opportunities for students to work in their own way, and arranging learning materials so students can manipulate objects rather than passively listen. On the other side of Figure 5, one can see how Amy's conceptions of the five practices relate to these ways of fostering student APS or different ways that Amy had not considered initially. Additionally, in the survey reflection interview, Amy remarked how her conceptions were still the same and extended upon these ideas by adding information and clarification.

### *Implementations and Reflections*

Now that we know how Amy viewed and valued student APS, and the five practices, I turn to look at how Amy implemented and reflected upon those instances throughout camp to answer the second and third research question. In this next section, I focus on four stimulated recall interviews. I showed Amy one or two video clips of her interactions with students during camp from that day or the day before and had her reflect on the clips and the practices she supported in them.

**Day 2: First Reflection Interview.** On the second day of camp, I collected video recordings of individual and small group conversations involving Amy and the students. After reviewing the footage that day, I selected two interactions for Amy to reflect upon the following morning. These videos were the most audible (technical difficulties) and contained the most complete and mathematically interactive engagements between Amy and the students. Additionally, these videos illustrated Amy's clear focus on justification from the students. In the following paragraphs, I summarize each of the interactions from the second day of camp, followed by a synopsis of how Amy supported, or attempted to support, students in the five mathematical practices and student APS. After each description and synopsis, I turn to focus on Amy's reflection of the interaction and her relevant lived experiences.

***Interaction 1.*** On the second day of camp, Amy worked with one group consisting of Jaime, Sharon, Gina, and Gayle. They were trying to figure out the distance between pairs of integers on a number line:

a) 2 and 2

b) -2 and -2

c) -2 and 2.

Jaime and Gayle were working independently on part *c*, while Sharon and Gina were struggling with starting *a*. Jaime told Amy that he was confused, but thought the answer was 4; however, Gayle thought the answer was 2. When Jaime pressed Amy to evaluate his answer, she simply shrugged and said, "I don't know. You tell me. Why do you think it's four?" Thus, prompting Jaime to explain both *how* and *why* he believed the answer to be four. In response to this, Jaime provided a justification using the number line (*single representation*). Thus, this showed Amy creating an opportunity for Jaime to *work in his own way* and *manipulate the objects*. However, Amy still did not evaluate Jaime's answer but once again said "okay" and shrugged before turning to Sharon and Gina, who were indicating that they were stuck.

Still on part *a* and struggling, Amy demonstrated the problem, not adding any new information, but illustrating on the number line the idea of starting at two and moving to two (*linking a representation to an idea*). Similarly, Amy demonstrated part *b* for the girls before they arrived at part *c*, where Jaime had been waiting for them because he wanted to know if the answer was in fact 4. Before modeling part *c*, Amy pointed out that it could be "kind of tricky" because the girls were assuming the answer would again be zero. This act of appealing to emotions by acknowledging the difficulty or trickiness and changing the strategy to include the number line shows how Amy was *fostering perseverance* in problem-solving. It also shows how Amy was *listening carefully* to the students and acknowledging their thought of zero as an answer to part *c.* Moreover, because Amy brought forward *manipulatives,* Gina *actively engaged in manipulating objects* and began to point and count on the number line Amy had used to modeled part *c*.

Therefore, this act of modeling provided a *helpful hint* for the students that allowed them to finish their work on the problem.

In this interaction, one can see how Amy pressed Jaime to *explain how* his answer worked and to *justify why*. With Gina and Sharon, Amy's pressed for a *single representation* and *linking that representation to an idea*. Moreover, Amy *fostered perseverance* by attending to potential emotions due to the problems trickiness and bringing in the number line to change the strategy. Throughout the interaction, Amy fostered student APS by *listening carefully* to the students, *creating opportunities for the students to work in their own ways*, allowing and creating opportunities for the students to *engage in manipulating objects* instead of passively listening, and providing *helpful hints*. However, Amy's use of manipulatives with Gina and Sharon could be seen as a limitation for fostering APS since Amy brought it specifically into the conversation, which determined which strategy would be used.

***Amy's reflection: Interaction 1.*** At break time the next morning, Amy reflected on this interaction. During her reflection, Amy talked about how she had pressed the students to show her how and why the solutions made sense. Amy stated, "I am trying to show them, 'do the process, and right or wrong answer, you are able to show me what you got.'" (Amy, Day 2 Interview) Thus, this statement provided evidence that Amy reflected on her pressing for the students to *explain how*. Similarly, she reflected on pressing the students to *justify why*, stating, "We've definitely had to remind them …Just going through those processes and actually…trying to figure out like, 'hey, why is that 4 away, if it is -2 to 2?' or 'why is that from 0 to 5 and 0 to -5 are the same distance?'

trying to get those thoughts started and those questions to come out. Like the 'why' and not so much 'cool, it is that'." (Amy, Day 2 Interview)

Amy's reflection of wanting the students to justify their answers also provides evidence to suggest that Amy wants to *listen carefully* to the students' thinking. Also, the indication of "right or wrong answer" suggests that Amy *creates opportunities for students to work in their own way*.

Amy recalled her conversations about how answers were not the most important and it was better to work things out and focus on the process because the problems were going to "get tricky" and not worrying about the answer (Amy, Day 2 Interview). Thus, by reflecting on her conversations with students highlighting the difficulties of problems and the potential for frustration, and to instead focus on the strategy, Amy's reflections can be seen as *fostering perseverance*.

Similarly, since Day 2 was full of newer concepts, Amy said it was less about determining what steps were next and more about asking students where they were confused. Amy also noted that in these instances, she tried to illustrate the problem using the number line instead of telling the students where to start or what to do. This showed Amy reflecting on fostering student APS in multiple ways, such as *providing helpful hints* and *creating opportunities to manipulate* by bringing in manipulatives for further student use, but also *fostering perseverance* by having the students reflect on where and why they were stuck or confused about something and using the manipulatives to help visualize the problems (*single representation*).

Regarding this particular interaction, Amy attributed multiple supports to her lived experiences. In noting the importance of being able to reason and justify an answer in order to validate claims, Amy mentioned that this was something she was not used to when she was growing up. Amy said, “I think growing up, that is what I got used to. I memorized formulas, and I would still get the wrong answer sometimes.” (Amy, Day 2 Interview) Moreover, Amy reflected on her rephrasing of the question while illustrating the problem on the number line to Sharon and Gayle and asking them what they thought was next. Amy attributed this idea of illustrating the problem and asking the students where to start or what they thought should happen next to a previous Level 1 teacher.

Amy finished reflecting on this video by noting that all of these practices and ways of fostering student APS stem from working in the camp, participating in the professional development within the camp, and having taken the content course before her first year of camp with a camp teacher.

***Interaction 2.*** The second interaction I chose for Amy to reflect on was one that happened towards the end of the second day. Amy interacted with a group of boys (Pedro, Nigel, and Conner) who were having difficulty with the problem:

> In the year 540 BC, Pythagoras, a Greek philosopher and mathematician, formulated the Pythagorean Theorem, which is still used today. Archimedes, another famous Greek mathematician, worked on a variety of problems. In 240 BC, he formulated the area and volume of the sphere. Which formulation occurred earlier in history? (MEC Level 1)

The boys were all sitting on the floor, working on a poster together as a group. The boys had already read the problem, drawn a timeline, and established that they were trying to figure out which date happened earlier in history. Amy began by helping (*helpful hint*) the group by *bridging* the *contextual language* of BC to the concept of negatives. Amy then provided a *helpful hint* that pressed the group to use their *single representation* and *link the representation to an idea* by asking the group which side of the timeline was earlier and having Nigel put a tick mark labeled as 2018 on that end. This *helpful hint* and *linking* continued as Amy told the group, “That’s what year we are in. At the top…So we are in the present. Right there.”

The group was very unfocused, and time was beginning to run out for the day. Noticing this, Amy began asking questions to focus the group on the task and provide the students opportunities to respond to each other’s ideas (*group work* and *fostering perseverance*). Moreover, with every answer the boys provided, Amy probed them to *justify why* they chose their answer, which also shows how Amy was *listening carefully* and *allowing the students to work in their own ways*. However, as time was coming to an end, Amy began to focus the students to specific reasoning, “Okay, so that is further away. What did we talk about with the left and the right yesterday?”

In this interaction, we can see Amy struggle with the contextual language of the problem and choose to disregard BC and AD, and instead only focus on the numbers as implied negatives. However, this lack of labeling ended up confusing the group and left Amy struggling for words to help the students. Amy did manage to use the reference of the current year and their prior class knowledge to conclude with the group. During this interaction, Amy had difficulty with the mathematical language and struggled to provide helpful hints because of it but was able to use the visual representation as an aid in supporting the students.

***Amy’s reflection: Interaction 2.*** In her reflection, Amy noted that this was a particularly challenging question because the students did not understand the idea of BC and AD and that Amy had to explain those terms to them. Amy translated the context of BC for the student by using the *bridging language* of negatives (*providing hints*).

She also found it interesting that the students in that class would just accept her explanations without question, whereas in previous years, when she worked with some of the older kids, they would question her and want to know why. After accepting Amy's explanation that BC meant that the numbers would be negative, the group still struggled with what to do next. Amy continued reflecting on the series of reflective questions that she posed to make the group think about what the problem was asking and how they could answer it.

She further reflected on the boys discovering that they needed a number line, their confusion about where to place 240 after having placed zero, and her press for justification in their placement of 240. Amy reflected on her press for the students to *justify why* saying, "I was like, 'okay, why? Blah blah blah' and the other two were like, 'no! no! no! It goes over here.' I was like, 'okay, why' and so they explained that it would be closer to the zero" (Amy, Day 2 Interview). This reflection also provides evidence to suggest that Amy thought about her probe for justification as a way to *listen carefully* to the *students' working in their own way*.

After this, Amy reflected on Nigel's reasoning about placement of the 540 based on their placement of 240, and why she had the group place 2018 on the timeline (*single representation*). In this reflection of placement, Amy reflected on her press for the students to *link a representation to an idea*. She noted that "they were having a hard [time] trying to figure out which one [date] came earlier…That's why I had them put 2018 on the timeline…cause maybe that will be better reference and then see positive number on the other side" (Amy, Day 2 Interview). Thus, not only was Amy reflective of her wanting the students to link the context of dates to a number line but also in her utterance of a *helpful hint*. However, Amy did note that during that conversation, she was trying to compare the numbers in the problem to zero, but it wasn't working. Additionally, she was having a hard time with the wording of her questions because she was trying to ask which number came first without wording her question that way, even though she admitted to asking that exact question.

When asked if she would do anything differently, Amy said that she struggled with the wording of the problem and would have liked to have asked different questions that would have kept the language and better get to the point of the "negative dates happen further back". However, Amy thought she was asking questions that did not give the students the answer but focused on what they knew and how it applied and why. Additionally, Amy was able to ask reflective questions to make the students think about different strategies they could use. Therefore, Amy reflected on her contributions to *fostering perseverance* by "asking lead-in questions" (Amy, Day 2 Interview). She continued this thought by explaining that she was trying to see what the students knew about the problem and have them work as a team to make sense of the problem together (*creating opportunities for student talk: group work*).

***Amy's Day 2 Reflection Overview.*** Throughout the interview, Amy only mentioned one thing that she would change, and that was trying to keep the contextual language of the problem, which was something Amy had changed by trying to use bridging language from the class by referring to BC as negative years. This reflection showed how Amy was trying to provide the students with a hint as to which event happened earlier in history by using the current year as a way to link the concept of years to the number line representation. This also allowed Amy a way to provide a hint without asking the question directly, which she was trying to avoid because of the students' confusion about the context and language of the problem.

In her reflections, Amy mentioned how she pressed the students to link the number line and current year to the problem to help aid in their understanding, as well as reflect on where they were confused. However, the majority of her reflection focused on her noticing and presses for justifications, both explaining how and justifying why. Thus, showing that Amy was focused on creating helpful hints and questions, having students explain their work whilst she listened to their thinking and fostering perseverance to promote student APS. Table 4 summarizes the various practices Amy reflected on during the Day 2 reflection, evidence of said reflection, and lived experiences she called upon during those interactions.

*Table 4.* Selected evidence from Amy's day 2 interview

| Mathematical Practice | Category | Evidence from Reflection (Day 2 Interview) |
|---|---|---|
| Justification | Explain How | I am trying to show them, "do the process, and right or wrong answer, you are able to show me what you got". (Interaction 1) |
| | Justify Why | We've definitely had to remind them …Just going through those processes and actually…trying to figure out like, "hey, why is that 4 away, if it is -2 to 2?" or "why is that from 0 to 5 and 0 to -5 are the same distance?" trying to get those thoughts started and those questions to come out. Like the "why" and not so much "cool, it is that". (Interaction 1) |
| | | I was like, "okay, why? blah blah blah" and the other two were like, "no! no! no! It goes over here." I was like, "okay, why" and so they explained that it would be closer to the zero (Interaction 2) |
| Perseverance | Fostering Perseverance | We've definitely had to remind them that "hey, sometimes these are gonna get tricky , and doing it in your head isn't going to be the best option if you are trying to get the answer, because we are not worried about the answer. (Interaction 1) |
| | | Asking lead-in questions. Again, not giving them the answer. Really trying to see what they know… read the problem, … "what is something in this problem that we know… "okay, what is something else we know…" What is the question asking? What are we looking for? What do we even need to do?" That kind of took a team effort because they were like, "I don't know." (Interaction 2) |
| Mathematical Language | Contextual Language | That one was tough, because the question didn't explain BC and AD. I had to do a pre. (Interaction 2) |

*Table 4.* Continued

| Mathematical Practice | Category | Evidence from Reflection (Day 2 Interview) |
|---|---|---|
| Mathematical Language | Bridging Language | once I explained, “BC, these are going to be considered negative…they were like, “okay”. (Interaction 2) |
| Representations | Single Representation | girls were struggling with the 2, saying "we don't know this one". I was like, "we start at -2 and"...and I showed them...that's when I showed them on the number line. "How far away is it?" "It's 2." "Is it 2? Let's check." I was trying to visually show them why that was how far it was away, because that was tricky just thinking about it. (Interaction 1) |
| | Linking a Representation to an Idea | Then they were having a hard trying to figure out which one came earlier. I was trying to compare that to zero, and that wasn’t... …That’s why I had them put 2018 on the timeline. It wasn’t in the right place at all, but I was like, “hey, throw this on there” cause maybe that will be better reference and then see positive number on the other side. (Interaction 2) |
| Student APS | (1) listen carefully, (2) create opportunities for students to work in their own ways, (3) create opportunities for student talk: group work, (4) manipulate objects, (6) provide helpful hints and answer questions | |

**Day 4: Second Reflection Interview.** On the fourth day of camp, the number line and car model were the focus of the curriculum. I asked Amy to reflect on her thoughts about both the car model and other representations used to help build the students' understanding. Amy related that she never really made sense of negative and positive answers until learning these models. She also stated that using money and temperature were ways that seemed to help the students in the class.

When asked about the students who were struggling with negative signs and the car model, Amy brought up a previous experience and applying the idea of introducing and distinguishing language more clearly, for example, defining and using the word operation. Amy then reflected on a few of the interactions that took place that day using the car model or other pictorial representations.

***Interaction 3.*** Amy worked with Sharon and Pedro in solving the problem $-3 - 5$ using the number line. Amy noted that Pedro had the correct answer, but Sharon had written $-3 - 5 = 2$ on her paper. Amy began the conversation by asking Sharon to show her work (*explain how*) and Pedro to watch. While Sharon was explaining, Amy noticed that Sharon mistakenly moved the wrong direction and began to question Sharon with "why did you flip it around?", "Which one's positive?", "which way do we face if we're working with a positive number?", "Now, are we driving or are we going backward?" (*Teacher assisted in student correction*). This questioning not only served to help the student fix their mistake but also to focus the student on what the *bridging language* of driving and backward/reverse means in regard to the problem. However, only after Sharon concluded -8 did Amy begin to ask about Sharon's prior answer. Then, upon hearing Sharon say, "two", Amy asked, "So why is the answer not two?" This question from Amy functions both as a *teacher identification* of an incorrect answer and a press to *justify why* the answer was incorrect. Lastly, Amy proceeded to ask Pedro what answer he got for the question, which was correct, and if he could explain what he did (*explain how*).

This interaction illustrates Amy guiding the student to the correct answer through a series of questions, but also focuses on having the students explain their work. One can see that Amy was trying to get Sharon to recognize her mistake, but in a way that did not align with student APS or self-identification of mistakes.

***Amy's Reflection: Interaction 3.*** In regards to this interaction, Amy stated,

> I did want her to ... you know, she got two, and so I wanted her to kind of walk me through how she got the answer to see if she would get the same answer, but I was also, I did notice, I was asking her a lot of questions to maybe even try to get her to get the right answer and then see like, "Huh." So, when she was going through and she got the correct answer, I was asking her, "Oh, well then, how did you get two?" (Amy, Day 4 Interview)

This statement suggests that Amy reflected on trying to get Sharon to *explain how* she had come to her answer and notice her own mistake. However, Amy noticed that she was not allowing Sharon to *work in her own way* and was providing too many *unneeded hints* and guidance in pushing Sharon to find the error (*student identification*) and correct answer (*teacher assistance in student correction*).

By asking these guiding questions, Amy aided in the assisted correction, and inevitably helped Sharon to identify where she went wrong in the process. However, when Amy reflected on trying to have Sharon identify the error, Amy was unconscious of the fact that she had already identified that the original answer was incorrect (*teacher identification*) by her question of “So why is the answer not two?” (Amy, Interaction 3). Indifferent to who identified the error, Amy further reflected on how she pressed Sharon to explain why it was incorrect.

In reflecting about working with student errors, Amy remarked about her own experiences,

> Like, elementary and middle school, even high school. I would do a problem, and knew the formula or something, and I would mess up, but I wouldn’t realize it right away. I couldn’t look at an answer, like if this was my problem. I wouldn’t be able to see 10 and notice that that was wrong based off of the problem. It really wasn’t until I started doing car model stuff that that clicked. Okay, I can think of an answer, look back at the problem and realize I made a mistake, and go back and figure out where it is. And I really think that goes with having understanding of why we’re doing this, why that ties back into these problems, and just having the understanding of that. (Amy, Day 4 Interview)

Thus, Amy noted the importance of understanding why something works and how that can help students identify their mistakes. Similarly, Amy mentioned that she probably would have asked Sharon to talk through some of the problems instead of asking directed questions. Amy attributed this reflection to her current lead teacher, who said that the students will get the answer if you walk them through it because you tell them the next step with the question.

However, Amy made it a point not to just ask students to explain their thinking if they had mistakes. Pedro, on the other hand, had the correct answer, but Amy simply wanted him to *explain how* he got his answer and *justify why* his answer made sense and worked with the procedure, saying

> I wanted him to show me how he got it too and explain which way the car's facing and why, and why are we in reverse, again? Just to make sure he was understanding the same thing. Are you focusing on the operation right now, or are you focusing on if that number's positive or negative? (Amy, Day 4 Interview)

This also shows how Amy was reflecting on the *bridging language* of the class and making sure the students could all understand it. She continued this though further, mentioning that a lot of the students were uncomfortable justifying their answers, so she wanted to have them start explaining their correct answers too.

> I think what I'm trying to do is just make them feel comfortable justifying their answers regardless of if it's right or wrong. .... I'm not trying to tear them down, "Oh, you got the wrong answer", but just having them be comfortable saying like, "Oh, this is how I figured this out" and showing me. (Amy, Day 4 Interview)

Additionally, this statement shows how Amy was reflecting on fostering student APS by *encouraging the students* to continue to justify their answers regardless of correctness. Consequently, this also provide evidence to show that Amy reflected on *listening carefully* to the students and wanting the *students to work in their own way*.

In reflecting on this interaction, Amy related the idea of pressing students about their thinking and following their train of thought as something she learned in past camp experiences. Saying, "Just trying to get on the same level of what they're thinking instead of just jumping at, 'Hey, this is what I would do', and then that totally doesn't make sense to them".

***Interaction 4.*** In this interaction, we again see Amy struggling with the idea of allowing students to work in their own ways and explore their own thinking. Interaction 4 began similarly to interaction 3, when Amy noticed that Millie had ten written as the answer for the problem 7 – 3 = __. However, instead of asking Millie how she got her answer of ten, Amy asked Millie to rework the problem for her (*explain how*). Similar to how Amy used questions to focus Sharon, Amy guided Millie in the process and used funneling questions whenever Millie said an incorrect step (*teacher assisted in student correction*). For instance, when Millie moved forward three spaces from seven, Amy stopped her and asked, "Are we adding?", "Right here? What are we doing?", "So, if we are subtracting, do we go forward, or do we go backward?" Once again, these questions not only serve to guide the student, but to orient them to the *bridging language* being used in the classroom. However, although Amy guided Millie in the process, she *created opportunities to manipulate objects* instead of having Millie just passively listen.

Similar to the interaction with Sharon, Amy guided the students' strategies and led them to fix their misunderstanding or errors in the problem. This, however, did not align with student APS, which was something that Amy stated that she tries to foster.

***Amy's Reflection: Interaction 4.*** Amy mentioned that she observed that not only was this solution incorrect, but the next several solutions were solved incorrectly as well. In noticing that Millie had multiple incorrect answers, Amy wanted to know how Millie was thinking and pressed Millie to work the problems again and *explain how* she was doing them.

Amy noted that because Millie was a little behind, Amy had asked Millie to redo the problem instead of reflecting on the original answer. Remarking that, when Millie did get a different (correct answer), Amy used the multiple answers to make Millie question her original answer in hopes that Millie would realize the error (*student identification*).

> I was trying to see what she would do first, and then when she was kinda like, "Hmm, maybe that could be the right answer." I was like, "Let's see if you can do it again and get the same answer." And so, when she did, I was kinda like, "Hmmm. What do you think about that?" (Amy, Day 4 Interview)

However, Amy also noted that because Millie didn't reflect on the incorrect answer as she had done with Sharon, Amy wasn't sure if Millie understood why her original answer was incorrect. "I didn't know if she really did get why this was not ten, yet?" (Amy, Day 4 Interview) Thus, Amy thought about how she could have Millie identify (*student identification*) and correct the mistake (*student correction*).

> And so, I didn't know if she really did get why this was not ten, yet? And so, I think tomorrow if we're working on a problem, too, and she does kind of a similar mistake, I'm gonna ask her like, "Hey, how did you get your answer?" first, "And show me what you actually did." Instead of stopping her and asking her, "Hey, do we flip the car around?" And just seeing where she goes first. (Amy, Day 4 Interview)

In reflecting on interactions three and four simultaneously during the interview, Amy noted that she wanted to incorporate the idea of having the students reflect and use their original work for exploring their understandings (*explain how*) and incorrect answers (*student identification*) by focusing on the strategy and double-checking their work (*fostering perseverance*).

Additionally, Amy commented on how she liked how all the students in the interactions that day had persevered and continued to work with her and explained their thinking. However, Amy did not attribute this to anything she did, even though she did *foster perseverance* in terms of having the students reflect on their incorrect answers or rework the problems. Amy attributed the students' willingness to persevere and share their thinking as a product of their growth mindset, which was established by their academic schoolteachers. So, even though Amy was asking guiding questions to foster a growth mindset, which was how she defined perseverance, Amy did not believe her questioning contributed to the students persevering in their problem-solving and explanations.

Similarly, Amy noted that she wanted the students to get used to explaining both their correct and incorrect answers. She wanted her pressing for justifications to be a normal occurrence to the students and not a marker of an incorrect answer. She stated,

> So tomorrow, too, with some of those problems, I do wanna pick out, "Hey, how did you do this?" And it's actually right, and showing them I don't care, necessarily if the answer's right or wrong. I really do wanna know what you're thinking, how you got there because they might have a different way or something that worked or, you know. (Amy, Day 4 Interview)

Thus, these final thoughts show that Amy focused on pushing students to explain and justify their thinking, whether it was correct or not, but also to *encourage the students* in their efforts and to have them *persevere* in their thinking.

***Interaction 5.*** Before I even played or mentioned the final video to Amy, she recalled the exact interaction when reflecting on videos 3 and 4, so Amy was very much aware of the justification and perseverance theme that these reflection conversations were capturing. Meryl had made the same mistake as Millie, saying that $7 - 3 = 10$; however, Meryl became very frustrated when redoing the problem.

While Amy was asking those same guiding questions to Meryl, Meryl exclaimed, "I'm getting confused now!" Amy acknowledged her frustration with an "okay", but pressed Meryl to explain a different problem (*fostering perseverance*). However, Meryl started expressing her frustration by lightly hitting her toy car to her head. Amy responded to this action, "Don't hit yourself, you're doin' good! This is hard stuff, girlfriend!" Meryl seemed to take to this response and answered the question (*praising process* and *offering encouragement*). Amy then realized that Meryl was being confused by the different processes for the operation and the signs, so she wrote on Meryl's paper to distinguish between the two (*bridging language*) and took some time to make sure Meryl understood the difference (*teacher correction*). Amy ended their conversation with, "Okay. Give me a high five. That was correct. You're doing good." (*praise process* and *offering encouragement*).

This interaction illustrated Amy's struggle to foster perseverance but showed how she praised the student and encouraged her for her efforts. It also showed how Amy *assisted the student in the correction* that would not have been possible without the added knowledge and clarification Amy provided. Although Amy's questioning didn't align with her views of student APS, her actions to *foster perseverance*, providing certain *hints*, and *encouragement* do.

***Amy's Reflection: Interaction 5.*** Amy noted that Meryl began "shutting down," which made things difficult because Amy was still trying to engage Meryl in the problem by pressing her to *explain* her work. Amy stated,

> So, I was trying to point out the ones that she got wrong first, but she shut down really quickly. And so that was really tough. I was trying to find ways to still be positive and have kinda like a, "Okay. Well, show me." (Amy, Day 4 Interview)

This statement from Amy included evidence of Amy's reflection on a *teacher identification* of errors and how she was trying to foster student APS through *encouragement*.

However, once Meryl began sharing her thinking, Amy said that she noticed that Meryl was misunderstanding the terms and procedure that was written on the board. Amy reflected on this by saying,

> And then that's when I, kind of, came to that realization, "Okay. She's not understanding which one we're talking about, if it's the operation or the positive or negative, and so that's when I kind of was like, "Hmm. Let's slow down and let me write it on your paper, that way you can refer to …So I was like, "Well, maybe if I write it on her paper and she did seem to get it after she saw it written out. (Amy, Day 4 Reflection)

Thus, Amy reflected on making a correction by writing the meaning of the *bridging language* on Meryl's paper and making sure that Meryl then understood the procedure (*teacher correction*).

Amy also noted that she began pointing out the problems Meryl had gotten wrong because the overall objective was to notice a pattern from the problems, which Meryl couldn't have done if she had wrong answers. Moreover, Amy noted that she was trying to get Meryl not to be so answer-driven and to focus on the strategies and problem-solving process. Amy said she wanted to move Meryl away from a fixed mindset by asking her more questions about her thinking, regardless of the correctness of her solution. Amy continued this thought by saying that she thought that she might want to change the participation format with Meryl and have her listen and explain her thinking to her peers. Thus, Amy's reflection showed how she was currently *fostering perseverance* and how she would like to foster perseverance more in the future.

***Amy's Day 4 Reflection Overview.*** In conclusion, one can see in Amy's Day 4 reflection that she was aware of and reflective about her push for students to explain and justify their work. Amy consistently questioned the students to share their thinking and asked them to explain how they worked through a problem. Additionally, when errors or misunderstandings occur, Amy wanted the students to realize and self-identify the error , but she mentioned the need to guide the students less and let them work in their own way to realize their mistakes. Similarly, Amy noted that she wanted to start having all the students, not just the ones she felt were more mathematically secure in their understanding, explore and reflect on their wrong answers so that they could realize and correct their mathematical troubles.

Lastly, Amy was aware that some of the students had a fixed mindset and shared how she was actively trying to foster perseverance and create a growth mindset. However, Amy did not always reflect on her actions as ways to support student perseverance or praising a student's process. It was unclear if Amy did not consider these actions as important to the fostering of perseverance or if she considered these actions as more of classroom/behavioral management and emotional support. Amy reflected on her questioning to push students to continue working, which was how she defined perseverance but did not relate these moves during her reflection to ways that supported the students in their perseverance.

Through these instances, Amy reflected on ways to better support Student APS by focusing on better working with the students' ideas and errors. Additionally, she wanted to create opportunities for student talk when fostering perseverance. Although Amy noticed the students' perseverance, and helped foster it, she did not reflect on her actions as being helpful in these efforts. Table 5 below illustrates the practices and APS Amy reflected on and selected supporting evidence.

*Table 5. Selected evidence from Amy's day 4 interview*

| Mathematical Practice | Category | Evidence from Reflection (Day 4 Interview) |
|---|---|---|
| Justification | Explain How | I wanted him to show me how he got it too and explain which way the car's facing and why, and why are we in reverse, again? Just to make sure he was understanding the same thing. Are you focusing on the operation right now, or are you focusing on if that number's positive or negative? (Interaction 3) |
| | Justify Why | |
| Mathematical Language | Bridging Language | |
| Perseverance | (want*) Fostering Perseverance | And so I was just trying to get them maybe to like, "Hmmm, maybe we should double check some of these and work through not just getting the answer, but actually going through that process of moving, turning, whatever." (Multiple Interactions) |
| | | *I think if I would have given her time, maybe, to reevaluate, maybe even asked, "Hey Gayle, can you show Meryl how to do this problem? How you did it?" See how she would've reacted to that. That's something I would have done different. (Interaction 5) |
| Mathematical Troubles | (want) Student Identification | I did want her to ... you know, she got two, and so I wanted her to kind of walk me through how she got the answer to see if she would get the same answer, but I was also, I did notice, I was asking her a lot of questions to maybe even try to get her to get the right answer and then see like, "Huh." So, when she was going through and she got the correct answer, I was asking her, "Oh, well then, how did you get two?" (Interaction 3) |
| | Assisted Correction | |
| | (want) Student Correction | I think tomorrow if we're working on a problem, too, and she does kind of a similar mistake, I'm gonna ask her like, "Hey, how did you get your answer?" first, "And show me what you actually did." Instead of stopping her and asking her, "Hey, do we flip the car around?" And just seeing where she goes first. (Multiple Interactions) |

*Table 5. Continued*

<table>
<tr><td>Mathematical Practice</td><td>Category</td><td>Evidence from Reflection (Day 4 Interview)</td></tr>
<tr><td rowspan="2">Mathematical Troubles</td><td>Teacher Identification</td><td>So, I was trying to point out the ones that she got wrong first, but she shut down really quickly. And so that was really tough. I was trying to find ways to still be positive (Interaction 5)</td></tr>
<tr><td>Teacher Correction</td><td rowspan="2">I was like, "Well, maybe if I write it on her paper and she did seem to get it after she saw it written out like that, and so then this one, again, she got backwards and I was still trying to figure out, and I think that was actually when I wrote it down because she was using, yeah, the positive or negative as the direction that you drive or reverse instead of the operation. (Interaction 5)</td></tr>
<tr><td>Mathematical Language</td><td>Bridging Language</td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>Student APS</td><td colspan="2">(1) listen carefully, (2) create opportunities for students to work in their own ways, (3) create opportunities for student talk: group work, (4) manipulate objects, (5) encourage, (6) provide helpful hints and answer questions</td></tr>
</table>

**Day 6: Third Reflection Interview.** On the sixth day of camp, the students were introduced to a new model to help visualize adding and subtracting integers, this model was called the chip model. When prompted to give her thoughts about the model, Amy said that she really liked the model, but it did not make sense to her at first. However, making sense of the models and mathematics was now an important aspect for Amy, but said it was not something she considered before college. She said she would use formulas without understanding them, so she could never identify if answers were incorrect, but with the chip model you can make sense of the math. Amy really liked the idea of linking a representation to a mathematical understanding.

Amy had several conversations with students this day, but the foci of these videos were on the use of representations, fostering perseverance, and justifications. Although there were mathematical troubles in the videos, this was not at the forefront of the conversations.

***Interaction 6.*** Amy's view of representations was evident in her interaction with Meryl that day. As Amy approached Meryl, who was working with the number line, Amy suggested that Meryl try the problems with the chip model (*multiple representations* and *fostering perseverance*). Meryl responded that she was checking her answer.

***Amy's Reflection: Interaction 6.*** In reflecting on this quick interaction, Amy said that she thought this was a good way for Meryl to notice her own errors, if she had any. Amy wasn't sure if Meryl had the answer to the problem from the chip model already, but thought that even if she didn't, it would give Meryl an idea of what she should be looking for. Amy also mentioned that she pushed the students to use the chip model because they were quick to reach for the number line which they were comfortable with but didn't get at the idea of zero pairs. So, she encouraged the use of both models (*multiple representations*) as a way to self-check their answer and make connections from each model to the problem (*linking a representation to a concept*). In reflecting on her encouragement to use both models, Amy also *fostered perseverance* in problem-solving. Amy attributed the idea of using multiple representations for checking answers to previous camp experiences.

***Interaction 7.*** Amy reflected on the rest of the group conversation. Mario, Gayle, and Allie were trying to solve the problem $-5 - 3$. The students at the table had each gotten -8 as their answer independently, but Mrs. Fray (the lead teacher) was telling the class that this was incorrect (*teacher identification*). In seeing their confusion, Amy worked with the *table as a group* and demonstrated to them what they were doing by placing five red chips in her hand and then placing another two in her hand. She then asked the group, "did I just add some or did I subtract some?"

Since the group was still confused, Amy continued by holding out 5 red (negative) chips and asked the group, "How would you take away 3 negatives?" Gayle responded by grabbing three chips from Amy's hands and proclaiming that there were two left (*opportunities to engage in the manipulation of objects actively*). Amy then prompted the students to show her what they just did on their boards (*explain how*). However, Allie began drawing out the original process that led them to the answer of -8. Amy quickly addressed this and asked Allie, "Why are you adding?" (*justify why*). When Allie couldn't respond, Amy went through the same process as earlier. Still, this time she had Allie model the physical removal of the chips from her own hand (*opportunities to engage in manipulation of objects actively*).

This interaction illustrates how Amy was trying to use questioning and helpful hints but did not work out as planned and ended up guiding the students' problem-solving. However, Amy worked on *fostering perseverance* by changing the participation format and getting the students to work together and redo the problem using their boards after she had helped Gayle to demonstrate the problem.

***Amy's Reflection: Interaction 7.*** In the interview, Amy reflected on this interaction by acknowledging how difficult it was for her because she didn't fully understand the chip model for the purposes of teaching. She said that she wasn't trying to guide the students to the next step but ended up doing just that and kept repeating herself because she couldn't think of different wording. Amy believed rephrasing the questions would have been more helpful for the students, but she needed a better understanding of the model to accomplish this noting, "having a better understanding of the chips and the Chip Model in order to guide those questions better and have better wording and stuff like that." (Amy, Day 6 Interview)

Amy also mentioned that Mrs. Fray was the one who brought the error to everyone's attention (noticing *teacher identification*). Amy was trying to press the students to explore their answer and *explain* what they were doing and *why* it related to the problem, but she couldn't understand what the students misunderstanding was and therefore couldn't provide helpful hints or questions. Thus, Amy was having the students use the representation she set up to explore how things should work. This, as Amy stated, was still not making sense for the students and Amy kept asking why the students were adding instead of subtracting.

In thinking about this struggle and her difficulty with the Chip Model, Amy noted that she found it more challenging this year than past years because her previous lead teacher focused on whole class mastery, whereas Mrs. Fray concentrated on the majority of the class, so not all the students had mastered the concept before moving on. Amy realized this as part of the struggle she saw with the students but believed it to be more realistic of what she would have to work with in a traditional classroom. Amy did mention that she liked how the students persevered but did not mention how she fostered this perseverance other than trying to get all the students to *work together* as a group. Thus, Amy *fostered perseverance* by changing the participation format to a whole group conversation, which is also a way to foster student APS. However, it was unclear if Amy saw group work and changing the participation format as a way to foster perseverance.

Additionally, in reflecting on her struggle, Amy referenced how she was trying to understand what the students were doing. She was attempting to ask question so she could *listen* to how the students were thinking in their own ways but couldn't figure out their thinking and thus had *difficulties in providing helpful hints*. Thus, she ended up *limiting the students' own ways of thinking* by showing the students a strategy to consider. Amy noted,

> I was looking at what they were doing, and they kept adding so I was trying to figure out ways to ask, "Why are you adding?" Without just saying that. Cause I kept saying that, you know? (Amy, Day 6 Interview)

In her reflection regarding her limiting of the students' own ways of thinking, Amy demonstrated a strategy the students could use to connect the chips with the concept of subtraction (*linking a representation to an idea*), which allowed the students to *manipulate* and *explain*, but did not allow them much freedom in their choice of strategy. Amy reflected on this by saying, "show me how you take negative way. Cause I had five red chips in my hand. Can you take three away from this pile? And I'm like "okay, okay what do I have left?" (Amy, Day 6 Interview) She did note that she would have liked to be less "hands in" and would have liked to bring the number line back in to help with the problem. This again showed Amy's reflection and continued press for and wanting to include multiple representations for creating links between the representations and the mathematical problem.

***Amy's Day 6 Reflection Overview.*** We can see Amy acknowledging and supporting student APS through using multiple representations and bringing them into the conversations, utilizing student talk through group work, and fostering perseverance. Amy also tried to support student APS, with some difficulty, by providing helpful hints in the form of questions and replying to student questions. Because of the challenges she was facing, Amy noted how she is still working on her understanding of the chip model so she can better understand students' strategies and provide helpful hints to aid in their own thinking. Table 6 provides a summary of Amy's Day 6 reflections regarding her supports of the mathematical practices and student APS.

*Table 6.* Selected evidence from Amy's day 6 interview

| Mathematical Practice | Category | Evidence from Reflection (Day 6 Interview) |
|---|---|---|
| Visual Representation | Multiple Representations | And I think at first, we do want to encourage them like, "Hey, maybe this is kind of confusing so what if you check what you've got on the number line? Because you know how to do that." And so, when she said that, I don't actually know if she had the answer or not yet. But I was like, "Well maybe you should get the answer with the number line." She'll be able to make sense of it on the chips. And so that's why I was letting her do that first. Kind of see what you get, if she was checking in it, then maybe she would see her error if she had one. (Interaction 6) |
| | Linking a Representation to an Idea | |
| Mathematical Trouble | (want) Student Identification | |
| Perseverance | Foster Perseverance | |
| | | [Regarding persistence] Yeah, that's probably the biggest thing (Amy trails off) I'm trying to get that whole table involved instead of working with one student because I feel like that kind of started, but then I noticed it was a similar struggle with everyone. (Interaction 7) |
| Mathematical Troubles | Teacher Identification | [Mrs. Fray] at some point it was like, "Hey, like a lot of people are getting eight or negative eight or whatever they were getting." (Interaction 7) |
| | | I was looking at what they were doing, and they kept adding so I was trying to figure out ways to ask, "Why are you adding?" Without just saying that. Cause I kept saying that, you know? (Interaction 7) |
| Justification | Justify Why | |

*Table 6.* Continued

| Mathematical Practice | Category | Evidence from Reflection (Day 6 Interview) |
|---|---|---|
| Justification | Explain How | show me how you take negative way. Cause I had five red chips in my hand. Can you take three away from this pile? And I'm like "okay, okay what do I have left?" (Interaction 7) |
| Visual Representations | Linking a Representation to an Idea | |
| Student APS | (1) listen carefully, (2) create opportunities for students to work in their own ways, (3) create opportunities for student talk: group work, (4) manipulate objects, (6) provide helpful hints and answer questions | |

**Day 8: Fourth Reflection Interview.** On the eight day of camp, the Wednesday of the last week, the camp holds an open house. Thus, videos for Amy were limited to the second part of camp (a two-hour window). The videos chosen for Amy's reflections were based on the theme of student APS and communication through use of language and mathematical troubles.

The interactions to follow stems from a part of a group conversation. Amy interacted with three students who were just introduced to equations with unknown quantities. The introductory problem they were given was $5 + \_\_ = 7$. Even though these three students were seated together in a group, Amy had two different conversations going simultaneously: one with Cedric and the other with Joni and Gina. We will first look at Amy's conversation with Cedric.

***Interaction 8.*** Cedric recalled the day's conversation about "*x* could be any number" to jumpstart his reasoning, saying, "Well, I thought what five plus *x* equals, I decided well I was thinking of the addition to five. So, I was thinking of five plus two." However, Amy pressed Cedric to *explain how* this idea worked, so Cedric brought in the idea that *x* could be any number. Amy questioned his use of mathematical language (*bridging language*) with prompts to be more precise in his wording (*formal language*), "But can *x* be any number if 5 plus something is seven?", "if I had five, *x* can be any number? *x* can be 206 and that will equal 7?" Amy continued to press Cedric with similar questions until he concluded that it depended on the operation and numbers in the equation (*teacher assisted in student correction*).

Amy then turned her attention to the other students in the group, but as she was finishing with them, Cedric prompted Amy to evaluate his explanation. However, Amy responded, “do you think it was right?” to which Cedric said yes and justified his answer.

This instance showed Amy questioning the student and supporting student APS by pressing the student to justify his answer, which led to the student self-evaluating his work through justification. Additionally, we can see how Amy *listened carefully* and allowed the student to *work in his own way.*

***Amy’s Reflection: Interaction 8.*** Amy was amazed by Cedric’s willingness to keep thinking about his explanation and thought it was impressive that she did not have to press him for a final justification, but that he offered it willingly. She found Cedric’s autonomous problem-solving to be refreshing, saying that normally “there’s a lot of hesitation, there’s a lot of lack of ownership of ‘yeah, this is what I did’” (Amy, Day 8 Interview). However, albeit refreshing, Amy found his wording to be problematic at times and unclear and was pushing him to use more *formal language* and to disentangle his language. Amy was trying to press Cedric to move from his *unclear bridging language* of saying it could be any number and two at the same time, to use more *formal language* and saying that x was equal to a specific number in these cases.

Amy noted that she pressed for more formal language and disentangled understandings because she thought that his vocalization and vocabulary created a misunderstanding that impeded Cedric’s trajectory and was trying to *assist in this correction* by asking him the question about 206. She remarked,

> I think that’s where the confusion was happening. Because he was sure that x equaled 2 in that instance. But he kept saying, but *x* can be anything. So that’s

why I came up, I think I said 206. If it can be anything, can it be 206 in this problem? That's when he was, like no. (Amy, Day 8 Interview)

Amy also mentioned in her reflection that she considered incorporating Cedric into the conversation with Gina and Joni (*changing participation format* and *creating opportunities for stunt talk),* but decided against it since they had very different ideas and didn't want to distract from Cedric's line of thought (*creating opportunities for student to work in their own way*). She noted that she notices that Cedric becomes excited and engaged when he gets to work, explain, and justify using his own thinking, which causes him to persevere more (*acknowledging student interests*). This noticing and fostering of Student APS was something that Amy attributed to a previous year's camp experience where a student would become engaged in a problem only if it was interesting to him. Thus, because Cedric was primarily focused on the idea of a variable being any number or a specific number, and Joni and Gina were more focused on finding a generalizable process for solving the equations, Amy decided to keep the conversations separate.

***Interaction 9.*** Amy's conversation with Joni and Gina began with Amy pressing the girls to explain their thinking for *why* they thought the answer to $5 + x = 7$ was two, but yielded guess and check responses. Amy then asked if there was another way to think about the problem, to which Joni responded that $7 - 2 = 5$. Amy began to probe further, noting that it still didn't explain where the 2 came from. Amy continued to question the girls until one provided the strategy of $7 - 5$. At this response, Amy's demeanor changed to show excitement at the girls thinking (*praise*). Amy persisted in pressing the girls for a more generalized strategy by extending the problem to $7 + \_ = 5$ (*fostering perseverance*). In creating the extension, Amy created opportunities for the students to *work in their own ways* and *explain how* they were thinking. This can be seen when Amy presented them with the problem by saying, "Show me. Here, let me write that problem down. If I had $7 + x = 5$ [writing in on Joni's board], show me how you would figure that out."

The conversation that followed highlighted Amy *listening carefully* to the students explaining how the process would be the same, but after Amy's revoicing the students realized their mistake (*student identification*) saying, "wait…no" but unsure of the correct solution. Amy continued to listen to the girls' ideas and *fostered perseverance* by suggesting that the girls try different strategies, including the number line. Amy responded to the girls' confusion by saying, "Where is your… (Gina: number line?) maybe. Would a number line be helpful?" However, the girls did not think a number line would be helpful, but after Amy asked, "No? Why not?" and not getting much more than an "I don't know." And "I never tried a number line for this." the class was called back together.

***Amy's Reflection: Interaction 9.*** Amy reflected that she was trying to push the girls to *justify why* the answer was two, saying,

> that table got it really fast, so it's trying to get them to a place where they could explain why it was two, not just because, "Well, I knew that." You know, those addition facts. That's why I was kind of asking all of those questions… (Amy, Day 8 Interview)

Amy continued to mention that this press for justification led the students to develop a strategy that she *praised* them for, but also led her to create an extension of $7 + \_\_ = 5$ to challenge the students' strategy (*fostering perseverance*). This extension, Amy mentioned, caused the girls to keep thinking and eventually notice their incorrect reasoning. Amy also recounted her excitement when the *student identified* her own error. Amy related back to herself at that age saying, "'Cause I know at her age I did not have that number concept of, hmm that looks wrong. I was just like, "oh I plugged it in right. Or thought I did" (Amy, Day 8 Interview).

If given more time, Amy said she would have liked to have pressed the number line idea more (*single representation*) but wasn't sure if she would have brought in the chip model too. She reflected on this by saying,

> [prompted about the chip model] I'm just now thinking there's a number line but there's also the chip model. I suggested the number line, and we kind of talked about this a little bit. Because I know that they have it down. The chip model gets funky and it probably would've with that one because you would've had to do, you would've had to figure out a way to get negatives on the board. And I think I'm still confused on how to do an equation like that, show it like that. So, I wonder how they would've acted if they would've had the chips. (Amy, Day 8 Interview)

This reflection also lends evidence to how Amy reflected on her suggestion to use the number line as a *link to the idea*. However, when I asked about the chip model, Amy seemed hesitant about the idea.

These ideas, and the fact that Amy wanted to follow the students' ideas and strategies further, show that Amy wanted to *foster* the students' *perseverance* in APS by bringing in different strategies and using the *students' ways of thinking about mathematics*.

***Amy's Day 8 Reflections Overview.*** Thus, Amy supported student APS by *listening* to the students' justifications, allowing students to *work in their own ways*, and *acknowledging their perspectives and interests*. Amy attempted to *bring in different learning materials* but did not force them on the students. Additionally, she used questions that served as *helpful hints* to allow the students to explore and to identify and fix their errors. Table 7 summarizes the findings from Day 8.

*Table 7*. Selected evidence from Amy's day 8 interview

| Mathematical Practice | Category | Evidence from Reflection (Day 8 Interview) |
|---|---|---|
| Mathematical Language | Bridging Language | I was trying to understand what he was talking about and make sure he was getting the concept that *x* can be anything if it's just written on a paper. But in certain circumstances ... (Interaction 8) |
| | Formal Language | |
| Justification | Justify Why | that table got it really fast, so it's trying to get them to a place where they could explain why it was two, not just because, "Well, I knew that." You know, those addition facts. That's why I was kind of asking all of those questions… (Multiple Interactions) |
| Visual Representations | Single Representation | [prompted about the chip model] I'm just now thinking there's a number line but there's also the chip model. I suggested the number line, and we kind of talked about this a little bit. Because I know that they have it down. The chip model gets funky and it probably would've with that one because you would've had to do, you would've had to figure out a way to get negatives on the board. And I think I'm still confused on how to do an equation like that, show it like that. So, I wonder how they would've acted if they would've had the chips. (Interaction 9) |
| | Multiple Representation | |
| | Linking a Representation to an Idea | |
| Mathematical Troubles | Teacher Assisted in Student Correction | I think that's where the confusion was happening. Because he was sure that x equaled 2 in that instance. But he kept saying, but *x* can be anything. So that's why I came up, I think I said 206. If it can be anything, can it be 206 in this problem? That's when he was, like no. (Interaction 8) |

*Table 7*. Continued

<table>
<tr><td></td><td>Student Identification</td><td rowspan="2">I got excited when she came to the conclusion that you had to do 7 minus 5 in the other problem. So I was like, “yeah you do, dang”…when I flipped it and she started doing the same thing and then saw it, I think she probably did the problem in her head and was like, “wait a second”. But the she didn’t know where to go from there. But I thought it was super exciting that she could actually recognize, that doesn’t look right. (Interaction 9)</td></tr>
<tr><td rowspan="2">Perseverance</td><td>Praise Process</td></tr>
<tr><td>Fostering Perseverance</td><td>That's why I flipped the problem around to get them to see does this order matter? Trying to get them to see we have to get x by itself, but not necessarily having them realize that right now, but having them realize this order does matter. (Interaction 9)</td></tr>
<tr><td>Student APS</td><td colspan="2">(1) listen carefully, (2) create opportunities for students to work in their own ways, (3) create opportunities for student talk: group work, (4) manipulate objects, (6) provide helpful hints and answer questions, (7) acknowledge students perspectives and interests</td></tr>
</table>

### *Conclusion*

In conclusion, there is evidence that Amy attempted and reflected on fostering student APS in all seven ways from the analytic framework but had difficulty with some. For instance, Amy had difficulty facilitating student group talk or fostering student talk in the case of Day 6 and Day 8, Amy tended to pick which manipulatives to bring into the conversation instead of allowing the student the choice of manipulatives and providing helpful hints or responding to student questions for unfamiliar topics as in the case of Day 6. However, Amy acknowledges these difficulties and reflected on ways to better foster these aspects of supporting student APS. Amy succeeded in listening to the students' explanations, ideas, and justifications, and acknowledging the students' perspectives and interests. Amy struggled from time to time when allowing students to work in their own way, especially in the case of mathematical troubles although Amy made multiple attempts and reflected on her attempts to try to foster this. Lastly, Amy fostered perseverance (multiple times) and praised students' processes. However, Amy did not acknowledge her own efforts in the students' perseverance. Amy recognized when students persisted, as in the case of Day 4, but did not attribute any of her own supports to students' persistence even if it meets her definition.

Amy attributed her success and reflectional improvements to mostly previous camp experiences with teachers and students. Amy also mentioned her own K-12 schooling, from a negative perspective, several times when referring to her rationale for why justifications were so important. Other experiences Amy called upon were her Pre-K block, which allowed her to put into practice what she had learned, along with her University content coursework, saying

> I think mostly this camp, I feel like this camp has influenced how I act in other settings too. Like I did my preschool block this year. So, it was kind of the same thing, why are we doing this? You know? Just that questioning and stuff has been incorporated from this. From this (camp) knowledge and stuff. (Amy, Day 8 Interview)

Most of Amy's difficulty in fostering student APS lied in her interaction with the students. Amy recognized what was occurring through the students' explanations (for the most part), but struggled when trying to respond to the students, provide helpful hints, or foster student talk. However, Amy found the additional practice, advice regarding questioning for providing helpful questions that function as hints, and content regarding student struggles from her lived experiences (camp, coursework, etc.) to be most helpful in her interactions. Thus, Amy might benefit from having more experiences that are dynamic, interactive, reflective, and involving student thinking to aid in the situations with which Amy most struggle.

Table 8 summarizes Amy's conceptions, enactments, and reflections throughout MEC.

*Table 8*. Summary of Amy's results

| | Conceptions | Enacted Supports | ePSTs' Reflections |
|---|---|---|---|
| Justification | How | How | How |
| | Why | Why | Why |
| Perseverance | | Praise Process | Praise Process |
| | Fostering Perseverance | Fostering Perseverance | Fostering Perseverance |
| Mathematical Language | | Contextual Language | Contextual Language |
| | Bridging Language | Bridging Language | Bridging Language |
| | Formal Language | Formal Language | Formal Language |
| Visual Representation | Single Representation | Single Representation | Single Representation |
| | Multiple Representations | Multiple Representations | Multiple Representations |
| | Linking a Representation to an Idea | Linking a Representation to an Idea | Linking a Representation to an Idea |
| Mathematical Troubles | Student Identification | Student Identification | Student Identification |
| | | Teacher Identification | Teacher Identification |
| | Student Correction | | (want) Student Correction |
| | Teacher Assisted in Student Correction | Teacher Assisted in Student Correction | Teacher Assisted in Student Correction |
| | | Teacher Correction | Teacher Correction |

*Table 8.* Continued

| Conceptions | Enacted Supports | ePSTs' Reflections | Conceptions |
|---|---|---|---|
| Student APS | Listening Carefully | Listening Carefully | Listening Carefully |
| | Allowing students to work in their own ways | Allowing students to work in their own ways | Allowing students to work in their own ways |
| | | Student talk | Student talk |
| | Manipulating objects instead of passively listening | Manipulating objects instead of passively listening | Manipulating objects instead of passively listening |
| | | Praising & Encouraging | Praising & Encouraging |
| | | helpful hints & questions | helpful hints & questions |
| | Acknowledging students' perspectives and interests | Acknowledging students' perspectives and interests | Acknowledging students' perspectives and interests |

## Case 2: Linda

Linda was the second most experienced participant in my study. She had finished her content math courses but had not yet begun her pre-K observation block (exploration-focused field experience) when camp took place. This summer was Linda's second time participating in the camp and professional development setting. Each year the camp focuses its professional development around the governing precepts of camp and the experiences that take place within the classrooms. The previous summer, Linda worked at MEC for the first time. That experience allowed her to work with supporting the governing precepts with children in the same content area as this study, but with a different lead teacher. Thus, Linda had worked with the governing precepts of MEC and supported students using the governing precepts prior to this study.

### *Conceptions*

After the first day of camp and before the professional development portion began, I asked Linda to define student autonomous problem-solving (APS) and the five mathematical practices of this study that are related to APS: perseverance, justification, mathematical language, mathematical trouble, and visual representations. I also asked her what she valued about each of these practices. I did not differentiate between student APS and the other five practices during the study but focused on the practices equally. Thus, Linda was given a list of six mathematical practice-related foci and asked to write about her conceptions and values of all six.

**Student APS.** When asked about student autonomy in the form of student-generated dialog or student-generated strategies, Linda stated the following:

> (H)aving conversation led by students that allows them to figure out solutions for themselves or between their peers. Teachers foster this by subtle directioning [sic] and asking students questions. This is valuable because we said speaker is more likely to learn something than the listener. (Linda, Written Survey)

Thus, Linda's views of student APS involve using and following the students' strategies and ideas instead of using a teacher-directed strategy (*opportunities for students to work in their own way*). Linda indicated that she valued the students' ideas and wanted the students to use and discuss their ideas with peers (*create opportunities for student talk*). Therefore, Linda *listens* and creates opportunities for students to work in their own ways and to converse with their peers. Additionally, by "directioning [sic] and asking students questions," Linda aligns with other ways of fostering student autonomy, *giving hints and replying to questions*. When asked to reflect on this conception at the end of the two-week camp, Linda mentioned that she still agreed with her definition and added the importance of having the student explain their ideas, so a teacher is not assuming what the student is thinking.

Moreover, before the survey reflection interview, Linda partook in a clinical interview where she watched multiple videos (of other people) and commented on what practices she noticed and what she thought about the teachers' support of those practices. In a video designated to highlight a teacher-led strategy, followed by another video with a student-led strategy, Linda identified the guidance given in the first video before even viewing the second. Linda spontaneously noted that the teacher was

> directing but not telling the child how to do it... but not necessarily telling the kid like why… It was pretty much just like let's break it down until you've gotten

> there and then let you like count how it is. (Linda, Clinical Interview Task 1 Video 1)

Although Linda noted how teacher-directed the strategy was, she added that she believed this to be a good thing and beneficial for the student. However, Linda did also mention that the teacher in the first video demonstrated the problem twenty divided by four by having the student make a group of four, and then consecutively prompted the student to then make each remaining group of four. Linda noted that after demonstrating the first grouping, she thought the teacher should have stopped and waited to see if the student would continue in the problem-solving process. When reflecting on the second video that allowed the student to use her thinking, Linda stated, "the student [in the second video] was doing a lot of it more independently, and then the follow-up questions were coming, whereas the other teacher [in the first video] had to really like integrate their questions into the entire process." (Linda, Clinical Interview Task 1 Video 2) Linda stated, "I think I liked this one [the second video] better" (Linda, Clinical Interview Task 1 Video 2). When I asked Linda if she believed if the two students benefited equally, she was hesitant to answer, saying, "I think it's how that child responds [to the next problem]" (Linda, Clinical Interview Task 1 Video Comparison). However, Linda stated that if the child expected the teacher to guide them through the process again, then the guidance wasn't beneficial, but if the child tried to do the next problem independently, then it was equally beneficial.

**Perseverance.** When conceptualized perseverance, I asked Linda to think about persistence as this was the term used in the governing precepts. She wrote the following:

> I would see this as an environment where students feel comfortable not getting the right answer on the first attempt. Instead its viewing problems as a journey that takes multiple attempts and you don't give up. These [sic] is extremely valuable when learning topics to truly understand the material. (Linda, Written Survey)

Thus, Linda's definition aligned with the ideas of *fostering perseverance* because it focused on providing time for the students to work on the problem while focusing them on the process and strategies instead of the answer during the problem-solving process. Additionally, one can see that Linda attributed the value of perseverance to learning and understanding mathematical concepts. During the clinical interview, Linda noted that follow-up questions served to foster perseverance in that it made the student continue to think about a problem.

**Justification.** Similarly, when asked about justification of students' thinking, Linda mentioned the following:

> I could see this in two ways. Either a teacher justifying or a student justifying their own ideas. When it comes from a student they present there [sic] ideas and provide reasoning that uses proper mathematic vocabulary. (Linda, Written Survey)

Here, Linda recognizes different ways justification can happen in a conversation but choose to focus on students explaining their thinking using proper vocabulary (*formal language*). Her definition here shows her conceptualization of the students' presenting their work (*explain how*), and "provide reasoning" (*justify why*). Later, Linda added

> Mrs. [Berry] was really good about that, of being like I don't want to take your words, I don't want to tell people what you're thinking, because those are your

> own thoughts.  So that justification is – it's more impactful if a student is giving their own justification, so then me just assuming that oh, the way they said that, I'm assuming certain things that they might not have that knowledge of, like giving me their own justification can let me see okay, they're still missing maybe this vocabulary, or they're missing this concept, or they're on the right track with it, they're just not quite there yet , whereas if I just take what little information they give me or an answer, I can be like oh okay, they got it, whereas that's not true, which kind of goes through the student generated dialogue of like asking questions, giving – like and I tried to do so many questions so that it's really like they're giving me exactly what they're thinking rather than me making some type of assumption with what they think. (Linda, Survey Reflection Interview)

Linda focused on the students explaining their thinking, but not distinguishing between *explaining how* they are thinking about something and *why* they are thinking about it. Rather, her focus was on understanding what the students are doing and why they are thinking that way jointly.  Thus, Linda focused on students' justification as a way to assess the student's understanding and help aid in mathematical troubles.

**Mathematical Language.** Linda stated the following when asked about her conceptions about the practice of using precise mathematical language.

> This is extremely valuable when working with peers so that your ideas are properly understood. This can be defined as using developed vocabulary to express and articulate ideas. (Linda, Written Survey)

Linda expressed the importance of using proper vocabulary in several instances, including her definition of justification. Linda saw precise mathematical language (*formal language*) as a key component of communication. Thus, Linda focused much of her understanding of language towards *formal language* and recognized *bridging language* as well. In her survey reflection, Linda remarked, "Like you need to be precise in telling people like you can use greater than –To mean something, but like what is it actually meaning, that was like really important. And I think it helped the kids understand a lot more of what we were talking about." This statement shows that Linda recognized the importance of using words that are synonymous with the formal language and hold meaning for students. Additionally, the statement provides evidence to suggest that Linda views the use of synonymous words (*bridging language*) in a classroom as a way to explain and incorporate formal language into the conversation.

**Mathematical Troubles.** Linda noted the following conception when referring to student errors and mistakes. (Mistakes in the camp setting also include misunderstandings and misconceptions, which is why this study refers to them as mathematical troubles.)

> This goes back to persistence. It is the ability to take mistakes and see how they can be altered to lead you on the right path. (Linda, Written Survey)

Here Linda tied a brief idea of mistakes to her idea of persistence. She related the idea of spending time and not focusing on producing a quick answer to exploring an error to understand the correct way. Linda's definition does not lend insight into how she views the importance of who identifies the mistake or who corrects it, but during the clinical interview, Linda noticed that a student identified their own mistake stating, "she came to that herself, of…like that can't be right. I think that was good because that was her discovering it on her own." (Linda, Clinical Interview Task 1 Video 2) Thus, Linda's noticing and evaluation of the student recognizing her mistake (*student identification*) was something that Linda valued when working with mathematical troubles. Additionally, in the clinical interview, Linda also spoke about letting students fix errors on their own (*student correction*) instead of teachers doing it saying "let them fix that error on their own, rather than just being like oh, it's an easy fix, let me like show you really quick" (Linda, Clinical Interview Task 3). In the same clinical interview task, Linda also mentioned how sometimes students need teacher assistance in their corrections (*teacher assisted in student correction*).

**Visual Representations.** Lastly, I asked Linda about her conceptions regarding the use of multiple visual representations.

> I would define this as coming to a conclusion/answer and explaining their reasoning. This can include using precise language to differentiate different strategies to reach an answer. This is valuable because every student won't comprehend the same way. (Linda, Written Survey)

Linda again focused on using precise language in her definitions of this mathematical practice. Here she used it as a way to differentiate strategies in that students may use different wording or vocabulary in their explanations. Additionally, Linda mentioned the idea that using *multiple representations* is good because not all students understand the same way. In her survey reflection, Linda mentioned kids were eager to show the class their ideas and how "trying to find like different ways to relate that to things that they [the students] would know… it helps us think of better ways to teach". This statement illustrates Linda's idea of using multiple representations as not only beneficial for the students but as a teaching strategy too. Moreover, in the clinical interview, Linda also talks about how visualizing a problem (*single representation*) helps students to make sense of a concept. Thus, Linda also considered how representations are linked to mathematical structures and ideas and can aid in one's understanding of a concept (*linking a representation to an idea*).

**Summary of Linda's Conceptions.** Through the survey given at the beginning of camp, her survey reflection interview, and her clinical interview Linda conceptualized the five mathematical practices and student autonomy. Additionally, without being asked or prompted to do so, she connected some of the mathematical practices together but also to student APS. Figure 6 highlights Linda's conceptions of the five mathematical practices and student APS according to the analytic framework used in this study.

<table>
<tr><th>Students' Mathematical Practices</th><th colspan="12">Categories of Teacher's Moves to Support the Students' Mathematical Practices</th></tr>
<tr><td>Justification</td><td colspan="4">No Explanation</td><td colspan="4">Explaining How</td><td colspan="4">Justifying Why</td></tr>
<tr><td>Visual Representations</td><td colspan="3">Single Visual Representation</td><td colspan="3">Multiple Visual Representations</td><td colspan="3">Linking Visual Representation to an Idea/Concept</td><td colspan="3">Connecting Multiple Visual Representations</td></tr>
<tr><td rowspan="2">Mathematical Troubles</td><td colspan="4">Teacher Correction</td><td colspan="4">Teacher Assisted in Student Correction</td><td colspan="4">Student Correction</td></tr>
<tr><td colspan="4">No Identification</td><td colspan="4">Student Identification</td><td colspan="4">Teacher Identification</td></tr>
<tr><td>Mathematical Language</td><td colspan="4">Contextual Language</td><td colspan="4">Bridging Language</td><td colspan="4">Formal Language</td></tr>
<tr><td>Perseverance</td><td colspan="4">Praises Unsuccessful Effort of Answer</td><td colspan="4">Praises Process</td><td colspan="4">Fosters Perseverance</td></tr>
<tr><th colspan="13">Seven Ways Teachers Can Foster Student Autonomous Problem-Solving</th></tr>
<tr><td colspan="3">Listen Carefully</td><td colspan="3">Create Opportunities to Work in their Own Way</td><td colspan="4">Create Opportunities for Student Talk (group or partner talk)</td><td colspan="3">Manipulate objects or Kinesthetically work instead of passively listen</td></tr>
<tr><td colspan="4">Offer Encouragement when students show Effort or Persistence OR Praising Mastery or Process</td><td colspan="5">Giving Hints and Replying to Questions</td><td colspan="4">Acknowledging Students Perspectives and Interests</td></tr>
</table>

*Figure 6.* Linda's conceptions

Moreover, we can form a picture of how Linda conceptualized the use of these practices and how they related to her conceptions of student APS. Figure 7 summarizes the connection between Linda's conceptions and fostering student APS.

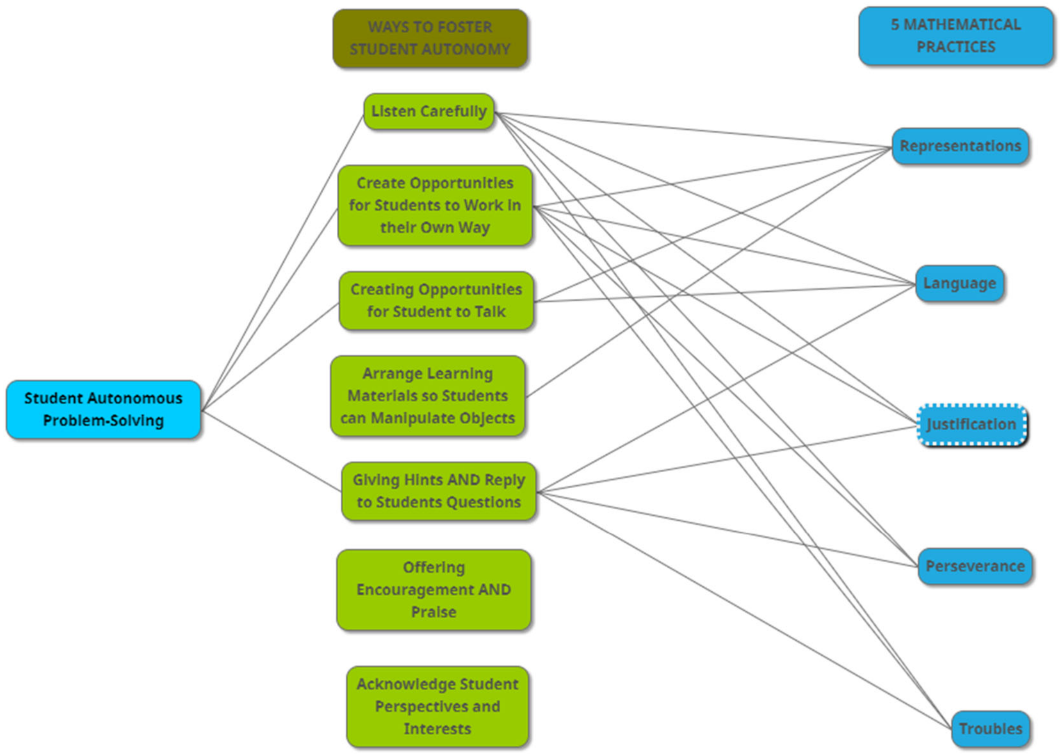


*Figure 7.* Linda's conceptions as related to student APS

Figure 7 illustrates how Linda's conceptions of student APS and the five practices relate to the ways of fostering student APS. On the left of figure 7, I link Linda's conceptualization of student APS to creating opportunities for students to talk, replying to student questions, giving hints, listening carefully, and creating opportunities for student to work in their own way. On the right of figure 7, I relate Linda's conceptions of the five practices to ways of fostering student autonomy. Thus, there are multiple links between how Linda views the five practices and how she supports student APS.

### *Implementations and Reflections*

The above are the findings for how Linda conceptualized student APS and the five mathematical practices throughout camp. I now report on how Linda implemented and reflected upon supporting the mathematical practices and student APS in a few selected instances throughout camp. These implementations and supports are based on four stimulated-recall interviews. I showed Linda one or two clips of her interactions with students during camp from either the day of the interview or the day before. Linda had much fewer clips than the other two participants in this study because her interactions tended to be significantly longer in time. After watching the clips, I had her reflect on them and the practices and supports she used in the videos.

**Day 3: First Reflection Interview.** On the third day of camp, I collected video recordings of individual and small group conversations involving Linda and the students. After reviewing the footage of that day, I selected a single interaction for Linda to reflect upon. This video contained the most complete and mathematically interactive engagement between Linda and a student. I also chose this video because of Linda's determination for the student to recognize their misunderstanding. In the following paragraphs, I summarize the interaction from the third day of camp, followed by a synopsis of how Linda supported or attempted to support the student in the five mathematical practices and student APS. After the description and synopsis, I turn to focus on Linda's reflection of the interaction and her relevant lived experiences.

***Interaction 1.*** On the third day of camp, Linda worked with a single student, Amiah. Amiah was working on the following problem:

> It is -7° C when Penelope visits Anchorage, Alaska. It is -9° C when Victoria visits Poughkeepsie, New York. Which temperature is colder? Explain.

Amiah thought that -7 was colder (I purposefully exclude reference to degrees here as it was not part of the conversation), which prompted Linda to ask Amiah to *explain how* she got her answer. Amiah noted that -7 was greater than -9, but Linda then asked if Amiah could draw a number line (*press for single representation*). However, Amiah drew 0 on the far-left side of the page, which led Linda to ask if that left room for negatives (*helpful hints*). This question was followed by instructions from Linda to draw a second number line with room for the negatives and to place the numbers from the problem on the number line.

When Linda asked which temperature was colder, and Amiah said $-7$, Linda prompted Amiah to *justify why,* by saying, “Why you think that one's colder?” Amiah’s response then prompted Linda to ask, “is it getting colder the further away I get from zero, or is it gonna get colder the closer I get to zero?” This conversation revealed that Amiah thought that temperatures became colder as they approached zero from either direction. Although, when prompted by Linda to think about temperature as a *contextual language* for less than or greater than, Amiah was able to determine that colder was synonymous for less than. At this realization, Linda requested that they move to the front of the classroom so that Amiah could work out the problem kinesthetically on the life-sized number line (*opportunities to manipulate actively*).

A similar conversation took place, and after returning to the same answer of $-7$, Linda asks Amiah, “Do you remember our symbols?” However, Amiah looked at where the symbols were on the board with a puzzled look (see figure 8 for symbol recreation).

$>$

"is greater than"

$<$

"is less than"

$=$

"is equal to"

*Figure 8.* Recreation of the symbols written on Ms. Berry's board during day 3

After Amiah defined the symbols incorrectly, Linda quickly redefined the symbols (*teacher correction*). However, after this brief interlude and returning to their spots on the number line, Amiah is still stating that $-7$ is colder. Linda takes another approach and begins to ask Amiah about seasons and appeals to Amiah's own experiences with summer and winter temperatures (*acknowledging student perspectives*). Linda illustrates this by having Amiah stand on three to represent the 30-degree winters, while she stood on 9, representing the 90-degree summers. Continuing with this thought process, Linda starts moving them down the number line one place at a time, repeating the question of who is colder. This *hint* and *manipulation of the representation* seemed to be successful until Amiah is on $-4$, and Linda is on $1$, stating that she is now hotter.

After questioning, "Which direction do I have to go to get to you," Linda realized that Amiah was still not understanding the symbols and proceeded to draw a small number line on the board with $-4$ and 1 that utilized the less than symbol as the leftmost arrow of the number line. This *hint* seems to have helped as Amiah is now on board that $-4$ is less than 1, but before Amiah could respond about their new positions after taking another step, a student standing near $-9$ on the classroom number line shouts, "I'm colder!" Linda proceeds to engage the student in their conversation (*creating* *opportunities for student talk*) by asking the student, “Why are you colder?” (*justify why*). However, the process with Amiah eventually resumes, and so does the confusion, until finally, Linda makes a breakthrough by asking about temperatures below zero. Linda then *praises* Amiah’s breakthrough and revision of her answer by saying, “Nice. Okay, so let’s go look at our paper again.”

In this interaction, one can see how Linda *listened carefully* and pressed Amiah to justify her answer and evaluate the correctness of her solution instead of Linda saying whether the answer was correct or not. Additionally, Linda *fosters perseverance* in this conversation by having the students focus on the strategies, bringing in the other student, and uses multiple forms of the number line. Lastly, this interaction also showed Linda’s push for the students to use *contextual* and *official language* by understanding what greater than and less than mean but also using them in the context of temperature. Thus, this interaction showed a variety of ways Linda engaged in fostering student APS.

***Linda’s reflection: Interaction 1.*** That afternoon, Linda reflected on this interaction. During her reflection, Linda noted the struggle of placing zero on the number line. Linda remarked that she noticed that there wasn’t going to be room for the negative numbers needed in the problem and brought this to the students' attention (*teacher identification*). *She assisted the student in the correction* by suggesting a secondary number line but did not say where the zero would go and left that decision up to the student. Thus, Linda also supported the *student to work in their own way* by leaving the placement of the zero up to the student.

Linda also mentioned how she realized that it might be more beneficial to use the large number line in the front of the classroom, which would support a continuous movement and fully engage the student in the movement. This also shows Linda reflecting on her supporting APS by creating opportunities for students to *manipulate objects actively*. She continued to talk about how the action of continuously moving left towards and into the negatives would show that as you move left, you get colder, saying, she was hoping the student would realize "the negatives are going to work in somewhat the same direction. If we continue to move, it's going to…You're still going to be getting colder." (Linda, Day 3 Interview) Thus, noting the use and understanding of *contextual language* in the problem, and reflecting on what Linda thought to be *helpful hints*.

However, when Linda reflected on this idea, she noted that she was hoping this would also allow the student to realize her misunderstanding (*student identification)* and correct herself (*student correction)*.

> I was hoping she would see, almost ignoring the numbers…our distance from each other hasn't changed, we're still moving in one direction. You should stay colder…I was hoping that she would recognize over the negatives it still worked that way. (Linda, Day 3 Interview)

Linda reflected on Amiah's confusion about the symbols and how she ended up drawing a number line using the symbol to explain the directionality. Amiah's confusion became obvious to Linda after Linda tried to link the concept of moving and placement on the number line to the value, and how one can think of the symbols, greater than and less than, as the arrows on the number line (*linking a representation to an idea*).

Thus, Linda reflected on the additional information that she added to assist in Amiah's correction. This *assisted correction*, the focus of the interaction, took longer than Linda had thought. Linda commented on how impressed she was with Amiah's willingness to stick with it and continue on to do another problem noting, “then she was like, ‘I want to do it on my own with no teacher.’ And I was like, ‘Solid. I think that’s a great plan, give that a shot and let’s see.’” (Linda, Day 3 Interview) Thus, this provides evidence that Linda reflected on *encouraging the student to show effort and persistence*.

Additionally, Linda *praised* Amiah after the interaction (this was not included in the stimulated-recall video since it was in another video), but Linda did not reflect on it. However, Linda reflected on how she tried to use the interruptions in the room to her advantage. When the other student kept popping in, Linda said she wanted to include their ideas in the conversation to aid Amiah. Thus, Linda reflected on the ways she *fostered perseverance* and student APS by *creating opportunities for student talk* by saying, “[w]hen [the other student] came in and most kind of like, “Oh, she’s colder.” I was like, Okay, Why?...maybe something that [the other student] would say would resonate with [Amiah]” (Linda, Day 3 Interview). This reflection also provides evidence as to how Linda reflected on perseverance as being connected to her prompting the student to *justify why*.

Finally, I had Linda reflect on what she would change about the interaction. She mentioned that she would probably change how she started the conversation and would have checked for an understanding of the words and concepts used in the problem saying,

> I started to recognize once we had started of, like, okay there's not the greatest grasp of what negative means. So maybe taking time to ask questions of like ... Some questions of, "So what is a negative number? What does a negative number mean?" (Linda, Day 3 Interview)

This quote shows that Linda reflected on how formal language can be confusing and cause trouble for students if they are not comfortable with the definitions of formal vocabulary words. Additionally, Linda reflected on how she led the conversation and would have liked Amiah to have driven the conversation more.

> I feel like I did a lot of leading of it, too. I would have liked to been like, "Okay, you tell me where to go on the number line." Like, "What do you think we should do next? Why are you thinking this?" Getting her to really speak what she was thinking out loud too. I think that would have been better. (Linda, Day 3 Interview)

This also shows that Linda would have liked to have *listened* more carefully to the student, allowed the *student to work more in their own way*, and *provide better open-ended hints*.

Regarding this particular interaction, Linda contemplated a few lived experiences that she found relevant. Linda attributed the use of multiple number line representations to previous camp experiences, noting that the students did better when they were actively working out the problem. Additionally, she related her ability to work with interruptions and distractions to her time spent playing volleyball. She says that she can ignore things that are going on within herself and can concentrate on what is important.

***Linda's Day 3 Reflection Overview.***

Thus, Linda reflected on all the reasons why I chose this video but also focused on fostering student APS by listening to students, allowing them to work in their own ways, communicate with each other, actively manipulate objects, providing helpful hints, and responding to questions. Linda also reflected on how impressed she was by Amiah but did not reflect on her praise towards Amiah. Table 9 summarizes Linda’s Day 3 reflections and select evidence from her interview.

*Table 9*. Selected evidence from Linda's day 3 interview

| Mathematical Practice | Category | Evidence from Reflection (Day 3 Interview) |
|---|---|---|
| Mathematical Troubles | Teacher Identification | But now that we're introducing negatives it's [students are saying], "Okay, but I always put the zero on the left." Well, if you do now where is the room for the negatives? You don't have any more paper. |
| | Teacher Assist in Student Correction | Trying to get [them] to see if we can shift zero or whatever we need to a certain place so that you have room for the actual values that you're going to need. |
| | (want) Student Identification | I was hoping she would see, almost ignoring the numbers…our distance from each other hasn't changed, we're still moving in one direction. You should stay colder…I was hoping that she would recognize over the negatives it still worked that way |
| | (want) Student Correction | |
| Visual Representations | Single Representation | So, I was like maybe if I talk it over with the positives, that'll somehow lead us into realizing that the negatives are going to work in somewhat the same direction. If we continue to move it's going to ... You're still going to be getting colder. So, I was hoping that, that would work. |
| | Linking a Representation to an Idea | |

*Table 9*. Continued

| Mathematical Practice | Category | Evidence from Reflection (Day 3 Interview) |
|---|---|---|
| Justification | (want) Explain How | I feel like I did a lot of leading of it, too. I would have liked to been like, "Okay, you tell me where to go on the number line." Like, "What do you think we should do next? Why are you thinking this?" Getting her to really speak what she was thinking out loud too. I think that would have been better. |
| | Justify Why | When [the other student] came in and most kind of like, “Oh, she’s colder.” I was like, Okay, Why?...maybe something that [the other student] would say would resonate with [Amiah] |
| Perseverance | Fostering Perseverance | |
| Mathematical Language | Contextual Language | okay, what does negative really mean in relation to temperature? In relation to other things. So, I think that was the battle. |
| | (want) Formal Language | I started to recognize once we had started of, like, okay there's not the greatest grasp of what negative means. So maybe taking time to ask questions of like ... Some questions of, "So what is a negative number? What does a negative number mean?" |
| Student APS | (1) listen carefully, (2) create opportunities for students to work in their own ways, (3) create opportunities for student talk: group work, (4) manipulate objects, (5) offer encouragement and praise, (6) provide helpful hints and answer questions | |

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**Day 5: Second Reflection Interview.** On the fifth day of camp, the number line and car model were the focus of the curriculum. I asked Linda to reflect on her thoughts about both the car model and other representations used to help build the students' understanding. Linda related that she never really made sense of addition and subtraction of integers until learning the car model. When asked about other models used to help aid in the understanding so far, Linda could not think of any.

***Interaction 2.*** The second interaction took place on the fifth day of camp and involved Linda working with a student, Shea, on the problem $3 - 5$. Shea had written two as her answer, so Linda prompted Shea, "Can you show that one to me?" (*explain how*). Shea quickly began by moving her car to zero. Linda then began questioning Shea, "which direction do you face?", "how many do you drive?", "do you need to change directions?", to each of which Shea responded correctly. Linda *praised* each response, "Right. Awesome.", "Cool.", "Awesome." Then Linda began including "Why?" questions to press Shea to *justify* her reasoning.

However, a different type of *praise* took place when Shea responded to Linda's question, "What in this problem tells you you are going to be in reverse?" When Shea responded, "the operator", Linda excitedly repeated Shea and said, "I like that word." Despite this brief exchange of excitement over the vocabulary (*formal language*), the conversation continued as before and ended with an answer of negative three (Note, $3 - 5$ was the problem). Linda recognized that the error occurred because of a change in reference point on the car (the cars were about 1 unit in size, so if a student was originally using the back wheels as a reference and then switched to their front wheels during the problem their answers were a unit off), and prompted Shea to recount and "make sure that we have the back wheels" are in the correct spot (*teacher correction*).

This interaction was chosen because of Linda's excitement and noticing of the student's vocabulary. Additionally, Linda identified a mistake, pressed the student for justification, and praised the student throughout the process.

***Linda's Reflection: Interaction 2.*** While talking about how she liked the car model, Linda noted that she believed the students benefit from having the model and being able to manipulate something and that this model is more beneficial than a number line on its own.

> the car model worked a lot more, or just trying to count on the number line and not ... The car helps you understand changing directions, so I think all of them, that was the thing that made it click, rather than just trying to look at a number line and figure it out. (Linda, Day 5 Interview)

Thus, this reflective utterance suggests that Linda believed the car model (*single representation*) served as a way to *link the representation to the idea* of adding and subtracting and creating *opportunities for students to actively manipulate objects* when exploring questions was important.

However, Linda continued to describe the difficulty when students changed their reference point on their cars, which sometimes leads to *teacher identification*. For instance, Linda reflected on a particular instance in the interaction, "So that one, the reason I had that student recount, is because they didn't switch. So, I had the back wheels on two, rather than on three. So that's where they counted." (Linda, Day 5 Interview) Linda noted that when driving forward (adding), the students typically used their front tire, but when they were in reverse (subtracting), they normally used their back wheels. Linda noted that the student in the video did not change her reference wheel before reversing and ended up with an error of one unit.

Linda reflected on pointing out the mistake by saying,

> I would've like to try and see if she could have figured that out by herself…I would've liked if I had stopped for a second and been like, "Okay, can she understand why the marker is important? Where her marker was?" (Linda, Day 5 Interview)

This suggests that Linda wanted to foster student APS by allowing students to *work in their own way*.

Linda also reflected on how Shea worked through the process and was responding to her press for justification. Linda noticed that she prompted Shea multiple times throughout the video to *explain* what she was doing and *why* saying

> I just want them to explain it to me. So, for that one, the questioning was, "Okay, if you're gonna change direction, why? Or if you're not changing direction, okay, why?"
>
> "Well, because five is still positive, so I don't need to change direction."
>
> Okay, are you driving in ... Are you forwards, are you going backwards? How do you know?" So, it was a lot of just making sure they know the process but understanding why the process is the way that it is.

Thus, Linda wanted to *listen* to the students and pushed the students to explain their thinking. In addition to students' explanations of their thinking and validation justifications, Linda also emphasized how important it is for students to build and use proper vocabulary. She continued to describe her excitement (*praise*), which came out as praise of *formal language* during her reflection.

> it's mostly important that they know them, and they're understanding what that means, but I think it shows they're understanding when they start using that in their vocabulary… that's why I got excited when she said operator, because it was one of the first students I had, when I was asking a question, say operator. (Linda, Day 5 Interview)

She noted that the students normally use the word "minus", which Linda considers to be an informal language for subtraction or operator (*bridging language*).

> Usually, it's just minus.
>
> Interviewer: Minus? Plus or minus?
>
> Linda: Yeah. So, they'll just stick to those, which is good. It can still make sense. But if they are gonna say, instead of saying negative two, because some of them do prefer minus two, just making sure that like, "Okay, so why is that minus different than a subtraction minus?"…That's a big important, of like, "If you wanna use that wordage, that's okay," but understanding there's a difference between the two. (Linda, Day 5 Interview)

However, she reflects that it's a good thing to use if they can still make sense of the problem.

Linda noted that the students would use words they were more comfortable or familiar with, but it was important for the adults in the classroom to use the official vocabulary (*formal language*) and model it for the students so they would get comfortable hearing it, understand it, and eventually use it.

In recognizing the importance of students using a language they are comfortable with, but modeling the language that is valued, Linda fostered student APS by having the students to *work in their own way*, *listening carefully* to their ideas, and providing *hints* for the students.

Lastly, Linda attributed much of her teaching to working in the camp the previous year. Linda remarked how working at camp last summer helped her realize the importance of vocabulary and helped her find ways to incorporate and model it. Additionally, she noted that she now has ideas of potentially difficult concepts and roadblocks for students and different strategies to guide and question students.

***Linda's Day 5 Reflection Overview.*** In conclusion, one can see in Linda's Day 5 reflection that she is aware of and reflective about her push for students to explain, justify, and use formal language. Linda consistently questioned the students to share their thinking and asked them to explain and justify why. Additionally, Linda recognized that she identified the error for the student but wanted the students to realize and self-identify the error.

During this interview, Linda reflected on ways to better support student APS by focusing on listening to students, allowing them to work in their own ways, manipulate objects, and provide helpful hints and questions. Linda also reflected on praising the student for using a vocabulary word; thus, this also shows how Linda encourages students when they show progress. Table 10 summarizes the mathematical practices and student APS she reflected on and select evidence to support my findings.

*Table 10*. Selected evidence from Linda's day 5 interview

<table>
<tr><th>Mathematical Practice</th><th>Category</th><th>Evidence from Reflection (Day 5 Interview)</th></tr>
<tr><td rowspan="2">Mathematical Language</td><td>Bridging Language</td><td>Usually, it's just minus.<br>Interviewer: Minus? Plus or minus?<br>Linda: Yeah. So, they'll just stick to those, which is good. It can still make sense. But if they are gonna say, instead of saying negative two, because some of them do prefer minus two, just making sure that like, "Okay, so why is that minus different than a subtraction minus?"…That's a big important, of like, "If you wanna use that wordage, that's okay," but understanding there's a difference between the two.</td></tr>
<tr><td>(notice) Formal Language</td><td rowspan="2">it's mostly important that they know them, and they're understanding what that means, but I think it shows they're understanding when they start using that in their vocabulary… that's why I got excited when she said operator, because it was one of the first students I had, when I was asking a question, say operator.</td></tr>
<tr><td>Perseverance</td><td>Praise Process</td></tr>
<tr><td rowspan="2">Visual Representations</td><td>Single Representation</td><td rowspan="2">the car model worked a lot more, or just trying to count on the number line and not ... The car helps you understand changing directions, so I think all of them, that was the thing that made it click, rather than just trying to look at a number line and figure it out.</td></tr>
<tr><td>Linking to a Representation to an Idea</td></tr>
</table>

*Table 10.* Continued

| Mathematical Practice | Category | Evidence from Reflection (Day 5 Interview) |
|---|---|---|
| Justification | Explain How | I just want them to explain it to me. So, for that one, the questioning was, "Okay, if you're gonna change direction, why? Or if you're not changing direction, okay, why?" |
| | Justify Why | "Well, because five is still positive, so I don't need to change direction."<br><br>Okay, are you driving in ... Are you forwards, are you going backwards? How do you know?"<br><br>So, it was a lot of just making sure they know the process but understanding why the process is the way that it is. |
| Mathematical Troubles | Teacher Identification | So that one, the reason I had that student recount, is because they didn't switch. So, I had the back wheels on two, rather than on three. So that's where they counted. |
| | (want) Student Identification | I would've like to try and see if she could have figured that out by herself…I would've liked if I had stopped for a second and been like, "Okay, can she understand why the marker is important? Where her marker was?" |
| | (want) Student Correction | |
| Student APS | (1) listen carefully, (2) create opportunities for students to work in their own ways, (4) manipulate objects, (5) praise and encouragement, (6) provide helpful hints and answer questions | |

**Day 7: Third Reflection Interview.** On the seventh day of camp, the students were working with variables. Linda had several conversations with students this day, but I only had her reflect on two videos. These videos both dealt with students' confusion regarding variables. Both videos focused on justification and language.

***Interaction 3.*** Jesse was confused about how to start the problems seen if figure 9.

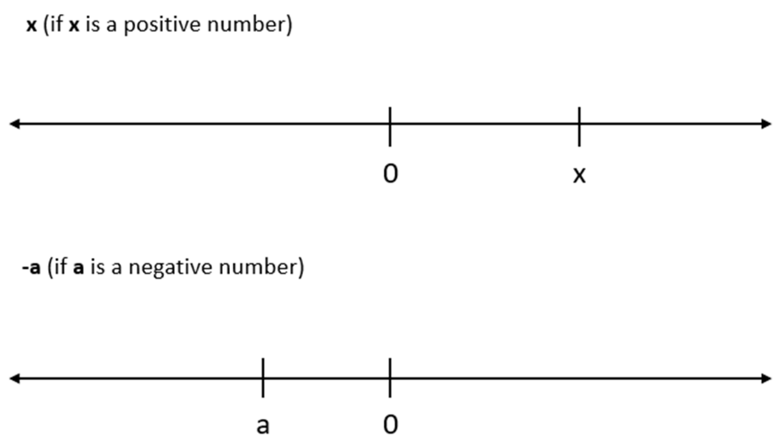


*Figure 9.* Recreation of Jesse's problem

After Linda finished explaining the problem to Jesse and asked, "Where is negative $x$ going to be?", he immediately responded, "Can I have a ruler?" Linda was very excited by this, and even said, "I think that's a great idea." (*praise* and *opportunity to work in own way*). Jesse measured the distance between 0 and $x$ and transferred the ruler to the left side of 0 and made a tick mark with the label $x$. Linda then pressed Jesse to think about his label, "is there anything missing you think from the problem?" However, Jesse didn't think so, so Linda (*teacher*) *identified* the mistake in his notation (*formal language*), saying, "You don't think so, okay. So, you have $x$ [pointing to the $x$ on the positive side] and $x$ [pointing to $x$ on the negative side]. Can they be the same number?" Then, following a brief conversation, Jesse marked the left most $x$ negative (*teacher assisted in student correction*), to which Linda agreed and *praised* by saying, "Okay, perfect!"

For the next problem, Linda, again, explained the question to Jesse, gesturing to each piece of the number line as she spoke. Similar to the previous problem, Jesse took his ruler and measured the distance, transferred it to the opposite side, and created a label. After a brief hesitation, he also labeled it as negative this time (*student identification* and *correction*). Linda had started asking a question during the hesitation but quickly changed her question partway through, "Okay, awesome. Looks good. Do you get why that's there?" This turn of talked served both as *praise* and a press to *justify why* Jesse though that $x$ needed the negative sign. This was followed up by questions about Jesse's use of his ruler (*justify why*) by asking, "So why are you using your ruler? Could you just like free ball it, if you wanted?" and "Yeah. But what does the ruler helping you do?" (*explain how*). Jesse surprised Linda by responding to this last question by saying, "finding the exact location", to which she *praised* his use of *mathematical language* by saying, "The exact location. I like that."

I chose this interaction because of the focus on mathematical language and Linda's press for justification. In addition to these foci, there was also a focus on the number line representation used in the problems.

***Linda's Reflection: Interaction 3.*** In reflecting on this interaction, Linda said that she was excited (*praise*) that Jesse wanted to use his ruler to make his representation accurate (*single representation*), saying "when he was like can I use my ruler? I was like of course you can use your ruler; I would love you to use your ruler. Because we did make such a big discussion of distance does matter, and the spacing needs to be exact."

(Linda, Day 7 Interview) Moreover, Linda talked about how she wanted Jesse to identify that he needed the negative symbol after some assistance from her.

> So, he used his ruler to put negative X on the negative side without the symbol, which we had discussions of is that negative X, what can that symbol also stand for? It stands for opposite of, so opposite of X. Then just discussing with him to get him to recognize we need that symbol so that they don't look the same, you can't have X and X on opposite sides of the number line. (Linda, Day 7 Interview)

In this utterance, Linda remarks about the questions that served as *hints* for Jesse to *link the representation to the concept* of opposites so that he could identify (*student identification*) and correct his error (*teacher assisted in student correction*) in his *formal language*.

Linda continued to reflect on language, and how no student from the previous year had used his language of "exact location". She noted that it was a great way to explain his thinking and that although they had talked about absolute value, not many of the students were comfortable using it in their explanations yet.

> Yeah, we've been talking about that [absolute value] a lot…So, I think they're still a little reluctant to use it. I think they all know what it means. So, if I had asked him what's the absolute value for ... can you tell me what that means? I think he would have known and a lot of them know, they're just not quite ready to use it quite when they're ... their explanations. They're not using that quite yet. So ... They have the idea. They have the concept, and so when I was asking him distance, I was ... you know ... I would have loved if he had said well the absolute

value in thinking any time I say distance that should immediately pop absolute value in their head, which it might of. He's just not articulating it yet. (Linda, Day 7 Interview)

Thus, Linda reflects on how the students are not yet using formal language but are using *bridging language* until then. Moreover, this provides evidence that Linda is reflecting on allowing *students to work in their own ways* and her *listening carefully* to the students.

Linda also thought about how she had switched her question at the end to pressed Jesse to *explain* his reasoning, focusing on *how* he measured but also *why*. She noted that she had started asking a question about the opposite sign, but he had fixed it on his own (*student identification and correction*), and she had to change her question. She did say that she would have preferred to have given him choices in her question and would have liked to give the student a more open-end question that allowed more freedom in his answer.

During her reflection, Linda also reflected on another student she worked with because Jesse's problem and struggle with variables were similar to that of Jagad. Jagad, instead of labeling $-a$ decided to use a different variable altogether. Linda spoke about Jagad's confusion and how she identified the problem but helped him correct the variables.

But he was using where they had negative A and then Y.

And I was like, where's that? Why are you using that? Which he didn't really give me an explanation. I'm like okay well let's look at the problem. What symbols did we use above? X and negative X, are those the same letter? He's like yeah. Is it a different symbol and he got that and I was like, so what does it need to look like

> on this problem with the A's, which had the other symbol ... which he eventually got that (Linda, Day 7 Interview).

In this reflection, Linda noticed that she pressed Jagad to *justify why*, but accepted *no justification* from him. Additionally, she noted about her identification (*teacher identification*) of the trouble and how she assisted him in the correction (*teacher assisted in student correction*).

During her reflection on this interaction, Linda focused much attention on formal language and justifying reasoning. In focusing on this, Linda also showed how she listened carefully to the students, wanted them to work in their own way, be active and manipulate while solving problems, provided helpful hints and questions, and praised the students as a positive reinforcement for their justification and proper vocabulary (*formal language*) usages. Additionally, Linda said she drew on her experience with the camp PD, and how they had just spoken about

> different reasons for questioning and what they could be, and one we brought up today was really cool. Like, questioning to see where they're coming from to…just different ways of using questioning, which I think is something I try to do with all of the students (Linda, Day 7 Interview).

Thus, Linda also reflected on her past experiences with respect to questioning and acknowledging the *students' perspectives* and prior knowledge.

***Interaction 4.*** Interaction 4 took place at the very end of class that day. Upon hearing this, Linda immediately knew which interaction I had chosen, saying, "that one was rough", because the student did not originally want to continue working and would only shrug at her questions. Ms. Berry had written $\mathrm{M} + {}^{-}\mathrm{N} = \mathrm{M} - \mathrm{N}$ on the board for the class to think about. Fred was leaning back in his desk, his whiteboard in his lap, with $6 + 3^{-}$ written on it. When Linda began to ask Fred questions, he was unresponsive. Linda, trying to *foster perseverance*, then asked, “what do you think six plus negative three is equal to?” to which Fred responded, “three”. When Linda prompted Fred to show her (*explain how*) on the number line (press for *single representation*), he again was unresponsive. Still wanting Fred to explore his answer, Linda asks if he could direct her in what he wanted to do. Fred proceeded to direct Linda and answered her prompts to *justify why* he was directing her in such a way; this conversation ends when they arrive at an answer of three.

However, Linda then addressed the fact that Fred put the negative sign behind the three (press for *formal language*), by asking, “So you put the negative sign in front…or behind it, does it matter if we have it in front or behind?” Unsure of himself, Fred responds, “No.” However, Linda presses further but becomes wary of Fred’s engagement in the problem and agrees to use Fred’s notation (*bridging language*), saying, “So, we can keep that there cause I get that.”

Moving on, Linda asked, “do you think that we could write the problem in a different way?” (*fostering perseverance*). Fred wrote $6 - 3^{-}$ on his board, which Linda asked Fred to read and then model with his car (*explain how* and *create opportunities to engage in manipulation of objects*). After modeling his work, Fred arrived at an answer of nine, to which Linda asked, “okay. So, let's look at this. Do these two questions equal each other?”, “No. So what's another problem that you could think of that would give you the same answer as this one [pointing to $6 + 3^{-} = 3$]? Do you want to think about it and tell me tomorrow?" This ending seemed to illustrate that Linda wanted Fred to both (*student*) *identify* and *correct* this problem.

I chose this video because of Linda's press for the student to persevere in problem-solving, her press for the student to justify his thinking, and her press to use correct notation. Also, because the problem was left open, this could affect how Linda viewed the perseverance or correction of notation.

***Linda's Reflection: Interaction 4.*** In the interview, Linda reflected on this interaction by acknowledging how difficult it was for her because Fred was "checked out" for the day and "ready for lunch". However, Linda mentioned that she was still able to press him to talk about the problem and liked how Fred had slowly become more engaged in the problem as they continued working on it (*fostering perseverance*), and eventually came to the realization that his two expressions were not equivalent (*student identification*). Additionally, Linda remarked how she wished there would have been more time, so she could have had Fred work with the other students (*opportunities for student talk*).

Moreover, Linda also talked about Fred using the visual representation of the number line (*single representation*), noting that this was something they were trying to do with the entire class. Linda had pressed Fred to use the number line at the beginning and slowly involved him in the movement along the number line (*engage in manipulatives*), all the while pressing him to make connections and *explain* the process (*linking a representation with an idea*). She also noticed that Fred, along with other students, would try to manipulate the process on the number line to match the answer that they expected to get, which required Linda to constantly probe Fred and the other students to explain their thinking as they worked and to justify their steps.

Linda also reflected on her pressing for formal notation (*formal language*) and how it was okay that Fred used his notation in their conversation because they both understood what he was writing (*bridging language*), but it is something he needed to fix before he explained to the class or if someone else were to look at his work. Linda additionally thought about how she had pressed Fred to *justify why* the symbols are important but should have "been really intentional, the way [she] was with Jesse" (Linda, Day 7 Interview) in the identification of the notational error (*teacher identification*).

Throughout the reflection, Linda focused primarily on her press for formal language and how she is going to continue her conversation with Fred to fix his notation, and her probing the students to understand and explain both how and why the processes work and relate this to the number line and notations. Linda showed how she supported student APS by listening to students, allowing the students to actively manipulate instead of passively listen, and provided helpful hints and questions to help the students make connections. Additionally, Linda reflected on how she allowed students to work in their own ways, and what she could do to better this practice. Furthermore, Linda indicated that she would have liked to foster student talk between Fred and his peers.

***Linda's Day 7 Reflection Overview.*** We can see Linda acknowledging and supporting student APS by allowing students to manipulate the representations, fostering perseverance, praising the students, listening carefully to what the students were saying, and providing questions and hints for the students. Linda also reflected on supporting student APS by considering how to ask better open-ended questions that allow more freedom in student answers and ways to involve students in student talk. Table 11 summarizes the selected evidence that supports the findings here.

*Table 11.* Selected evidence from Linda's day 7 interview

| Mathematical Practice | Category | Evidence from Reflection (Day 7 Interview) |
|---|---|---|
| Perseverance | (want*) Fostering Perseverance | *if I'd been like okay you think about this. Go work with somebody else for like five minutes or other students and then have that time to come back (Interaction 4) |
| | | I liked that as time went on, he was more willing, like by the time the second part came on, he was doing the number line himself. (Interaction 4) |
| | Praise Process | So, he, Jesse is one that, he likes to use a ruler. He likes it to be very exact, which I really like. So, when he was like can I use my ruler? I was like of course you can use your ruler; I would love you to use your ruler. Because we did make such a big discussion of distance does matter, and the spacing needs to be exact.<br><br>So, he used his ruler to put negative X on the negative side without the symbol, which we had discussions of is that negative X, what can that symbol also stand for? It stands for opposite of, so opposite of X.<br><br>Then just discussing with him to get him to recognize we need that symbol so that they don't look the same, you can't have X and X on opposite sides of the number line. (Interaction 3) |
| Representation | Linking a Representation to an Idea | |
| | Single Representations | |
| Mathematical Trouble | (want) Student Identification | |
| | Teacher assisted in student correction | |
| Mathematical Language | Formal Language | He could be like well I'm still meaning negative three. If he can explain himself, that's great. That's okay, and then we can get into that conversation, but if he wasn't there and it was just written and somebody came and was looking at it, then I would almost think like that's really important. (Interaction 4) |
| | Bridging Language | |

*Table 11.* Continued

| Mathematical Practice | Category | Evidence from Reflection (Day 7 Interview) |
|---|---|---|
| Mathematical Trouble | (notice) Student Identification | He just ... he kind of paused for a second, and then he added it. (Interaction 3) |
| | (notice) Student Correction | |
| | Teacher Identification | But he was using where they had negative A and then Y.<br>And I was like, where's that? Why are you using that? Which he didn't really give me an explanation. I'm like okay well let's look at the problem. What symbols did we use above? X and negative X, are those the same letter? He's like yeah. Is it a different symbol and he got that and I was like, so what does it need to look like on this problem with the A's, which had the other symbol ... which he eventually got that (Interaction 3) |
| Justification | No Justification | |
| | Justify Why | I think a lot of them, especially because we do use a lot of visual stuff 'cuz questioning and expecting them to give me their reasoning. So, when I was asking ... like specifically of why do you use the ruler? The precise location was great, but I wanted him to be able to say that, just not being like oh 'cuz now my line is straight, or you know ... various things you could tell me about why you use your ruler. (Interaction 3) |
| | Explain How | |
| Student APS | (1) listen carefully, (2) create opportunities for students to work in their own ways, (3) create opportunities for student talk: group work, (4) manipulate objects, (5) praise and encouragement, (6) provide helpful hints and answer questions, (7) acknowledging students perspectives | |

**Day 9: Fourth Reflection Interview.** On the ninth day of camp, the students were working with the chip model; this was only the second day that they worked with the model. That day the students had been introduced to the idea of zero pairs and were working with subtraction problems. The interaction chosen for Day 9's reflection was split into two parts because the student left the classroom for a class picture. This interaction was chosen because of the in-depth conversation about the model, struggle with zero pairs, the use of multiple representations, and the overall perseverance in problem-solving.

***Interaction 5*** Linda was working with a student, Eilis, on the problem $2 - {}^{-}3$ using the chip model (*single representation*). Linda had asked Eilis what she was thinking about (*listening carefully*), to which she responded, "I take away all of them". Acknowledging and repeating her idea, Linda also asked if Eilis could show her the two from the problem (press to *linking the representation to the problem*). She followed this up by asking Eilis questions about the problem and operation, and what she thought could be done, "What can we do to get some negatives on our board, without changing the fact that this equals two?" However, this last question also served as a *helpful hint* that zero pairs were needed.

From this point, the conversation turned to Eilis's discovering how to represent the value of positive two with the inclusion of three negative chips. In Eilis's first attempt to understand this idea, she added three red chips with her two yellow. Linda then began to probe Eilis about the new value of her board, and what she could do not to change the value of two. Eilis then asks, "Can you take away these two?" referencing the two positive chips on her board and noting that the answer would then be negative three. At this point, Linda prompted Eilis to use the number line (*multiple representations*). After a few seconds of thinking quietly and working on the number line, Eilis stated that the answer is negative one. However, the conversation continued with the thought that negative one was going to be the final answer (*no correction*).

Linda began questioning Eilis using the chips (*single representation*) about the value of two when adding in a positive chip—then noting, “One. Okay, I can't do that, cause I have to keep my value two. So, what can you add to keep this [point to the two yellow chips] two, but get negatives on your board?” (*teacher identification*). This is followed by another attempt and similar conversation until Linda says, “So, what if...I need these two [2 yellow chips] to stay, and you need these three negatives [pointing to the 3 red chips on the side]. What can you add with these negatives to keep it two.” However, before she could respond, Eilis was called out into the hallway, but not before the student next to Eilis, Chelsi, who had been following along with their conversation, mentioned the idea of zero pairs.

When Eilis came back to her desk, she created three zero pairs plus the two yellow chips. Eilis stated, “So I can take it away”. Eilis and Linda finish out the problem, but Linda continued the conversation asking, “So, do you want to double-check ourselves on our number line to see if we can get five?” (*multiple representations*). Eilis was using her finger but was confusing herself, so Linda wrote the problem on Eilis’s whiteboard and asked that she use her car. After this, Eilis was able to answer all of Linda’s guiding questions for the problem and ended at the answer of positive five.

This instance shows Linda questioning the student and supporting student APS by pressing the student to check their work by manipulating multiple representations, allowing the student to work in their own way in exploring the chip model, listening carefully to the students' ideas, and providing helpful hints based on the student's ideas.

***Linda's Reflection: Interaction 5.*** Linda first began her reflection by talking about how this was one of the first subtraction problems the students had done using the chip model (*single representation*). The goal for these first few problems was for the students to recognize the need for zero pairs (*linking a representation to an idea*).

> Yeah, so we were asking them to show a subtraction problem with their chips, which I think that was one of the first ones we had done with subtraction, so it's about them recognizing the fact that they have to add in zero pairs so that they can have negative chips on their board. (Linda, Day 9 Interview)

However, she also mentioned that she wanted Eilis to use the number line in the hopes that she could work backward with the chips (*multiple representations*).

Linda continued her reflection about why she pressed Eilis to use multiple representations saying, "in case she wasn't trusting the fact that the chips worked, I was like 'let's check on the car model too' to make sure that they can see multiple ways of how they got that answer, and that answer is the correct answer." (Linda, Day 9 Reflection) Linda also attributed this idea to one of the camp's professional development seminars. In one of the seminars, the PD group watched a video clip of a girl struggling to solve the subtraction problem of $70 - 23$ but was using multiple representations to solve the problem. In using the multiple representations, the student in the video became aware that one of her answers was wrong.

Linda remarked that she did not correct (*no correction*) Eilis when she found an incorrect answer on her number line and said, "I might have confused her a little bit by not correcting that it wasn’t negative one”. However, Linda also stated that she hoped Eilis would question her answer if she used multiple models. Moreover, from her previous camp experience, Linda was aware that addition with the chip model was normally okay for the students, but “subtraction with the chip models like a whole other beast”. Also, Linda noticed that the problem was written far away from Eilis and was causing her some difficulty in the car model, which Linda knew Eilis was normally fairly comfortable with. Linda stated, “that’s why I was like ‘Let’s write it [the problem] on the board, so you’re clearly seeing’.” (Linda, Day 9 Interview) Additionally, Linda remarked that she wanted Eilis to use the actual car instead of her fingers to model the number line because she wanted to clearly see all the steps Eilis was performing and that just using fingers can get confusing.

When I asked Linda what other lived experiences she drew on for this interaction, she mentioned that outside of class, they talked a lot about questioning, and “making sure your questioning doesn’t imply that they’ve done something wrong.” (Linda, Day 9 Interview) This thought of “imply that they’ve done something wrong” also was visible in Linda’s thought regarding *student identification and correction* of this error.

> So I was trying to keep my tone, especially even cause I noticed I was wrong so I didn't want to go up and be like, "Hey, even I was wrong, that's not the right answer." I really wanted it to be of even though I knew it was wrong and we needed to address that and fix it, finding a way to have her figure out that it was wrong, so that's why I was like "Let's do the car model.", but I let her kind of direct the car model cause I wanted her to get five, and be like "Oh, okay ... it's five.". So, we did something wrong eventually instead of me being leading of like

"Hey, it's actually not negative one." (Linda, Day 9 Interview)

This statement also suggests that Linda was reflecting on having students *work in their own ways*, *actively manipulate objects*, and providing *helpful hints and questions*.

Lastly, I asked Linda to reflect on her supports of the mathematical and teaching practices. Linda mentioned the idea of working with multiple models and having the students work through those problems to check their own work, but also spoke of *fostering perseverance* by "trying different things" (Linda, Day 9 Interview) and just watching the students work through the process (*listening carefully*) and reminding them that the value of the number needed to stay the same when adding in negative chips to create the zero pairs (*hints*).

Linda reflected on her use of multiple representations, students' checking their work, and fostering perseverance. Additionally, in the end, she mentioned that in the case of Eilis, the break served a purpose and allowed Eilis a little more time to think about the problem. However, relating to the previous interview about Fred, Linda said that she didn't like leaving that one open because it was too long of a time between working on the problem. Linda also stated that it depended on the student but preferred the shorter breaks.

***Linda's Day 9 Reflection Overview.*** In this interaction, Linda caught on to all the major reasons I had selected the video. Additionally, one can see how she actively engaged the students in the problems, listened carefully to their ideas, and tried to provide helpful hints and questions. However, Linda mostly reflected on the questioning and helpful hints aspect here, which requires listening carefully, but did not reflect on how she was trying to listen carefully as much as before since it is normally related to probing the students for their justifications which was not present here. Table 12 highlights the findings and evidence for Linda's Day 9 reflections.

*Table 12*. Selected evidence from Linda's day 9 interview

| Mathematical Practice | Category | Evidence from Reflection (Day 9 Interview) |
|---|---|---|
| Mathematical Troubles | (want) Student Identification<br>(want) Student Correction | So I was trying to keep my tone, especially even cause I noticed I was wrong so I didn't want to go up and be like "Hey, even I was wrong, that's not the right answer." I really wanted it to be of even though I knew it was wrong and we needed to address that and fix it, finding a way to have her figure out that it was wrong, so that's why I was like "Let's do the car model.", but I let her kind of direct the car model cause I wanted her to get five, and be like "Oh, okay ... it's five.". So, we did something wrong eventually instead of me being leading of like "Hey, it's actually not negative one." |
| | No Correction | I could tell that she wasn't understanding the fact that you had to keep those two, your value had to be two before you took anything away, and so when I had her do it on the number line and she got negative one, I don't know why I didn't even do the math in my head and I was like "Oh yeah, it's negative one, she knows how to do the car model." |
| Perseverance | Fostering | And it's also persistence because they'll put the negative three, and I'm like "Well you can't just have those you gotta keep it two.", and then they'll kind of sit there and try and do it again, and I'm like "Wait, it's gotta stay two." So, it's a lot of persistence of "I don't understand how to keep it two, but I'm going to keep trying different things." |

*Table 12*. Continued

| Mathematical Practice | Category | Evidence from Reflection (Day 9 Interview) |
|---|---|---|
| Visual Representations | Multiple Representations | So having her do it on her number line I was hoping she would kind of see "Okay, I know what my answer is now, now how can I get that represented on the board?" |
| | Linking a Representation to an Idea | |
| | Single Representation | Yeah, so we were asking them to show a subtraction problem with their chips, which I think that was one of the first ones we had done with subtraction, so it's about them recognizing the fact that they have to add in zero pairs so that they can have negative chips on their board |
| Student APS | (1) listen carefully, (2) create opportunities for students to work in their own ways, (4) manipulate objects, (6) provide helpful hints and answer questions | |

### *Conclusion*

In conclusion, there is evidence that Linda reflected on all seven ways to foster student APS from the analytic framework but has difficulty with some. For instance, Linda noticed she was leading the student and would point out student errors occasionally, which limited the students' ability to work in their own way. Additionally, made certain manipulatives accessible or brought them into the conversation. She acknowledged these difficulties and reflected on ways to better foster these aspects of supporting student APS.

However, Linda succeeded in praising students, listening to the students' explanations, ideas, and justifications, as well as allowing students to manipulate objects rather than passively listen. Although all of Linda's video primarily focused on a single student, Linda was able to bring in student talk from the students around that student and reflected on how she would have liked more time for the students to converse with their peers. Similarly, Linda also reflected on asking questions that were of interest to the students, would pique their curiosity, or bring out the students' background knowledge. This reflection showed how Linda would also support acknowledging students' perspectives and interests.

Linda attributed her success and reflectional improvements to mostly previous camp experiences with teachers and students. Linda also mentioned her volleyball experience, in that it helped minimize the distractions and chaos of the classroom. Other experiences Linda called upon were her University content coursework and her own experiences as a student, regarding learning styles and justifying reasoning. Table 13 summarizes Linda's conceptions, enactments, and reflections throughout MEC.

*Table 13*. Summary of Linda's results

| | Conceptions | Enacted Supports | ePSTs' Reflections |
|---|---|---|---|
| Justification | | None | None |
| | How | How | How |
| | Why | Why | Why |
| Perseverance | | Praise Process | Praise Process |
| | Fostering Perseverance | Fostering Perseverance | Fostering Perseverance |
| Mathematical Language | | Contextual Language | Contextual Language |
| | Bridging Language | Bridging Language | Bridging Language |
| | Formal Language | Formal Language | Formal Language |
| Visual Representation | Single Representation | Single Representation | Single Representation |
| | Multiple Representations | Multiple Representations | Multiple Representations |
| | Linking a Representation to an Idea | Linking a Representation to an Idea | Linking a Representation to an Idea |

*Table 13*. Continued

| | Conceptions | Enacted Supports | ePSTs' Reflections |
|---|---|---|---|
| Mathematical Troubles | | None | None |
| | Student Identification | | (want/notice) Student Identification |
| | | Teacher Identification | Teacher Identification |
| | Student Correction | | (want/notice) Student Correction |
| | Teacher Assisted in Student Correction | Teacher Assisted in Student Correction | Teacher Assisted in Student Correction |
| | | Teacher Correction | Teacher Correction |
| Student APS | Listening Carefully | Listening Carefully | Listening Carefully |
| | Allowing students to work in their own ways | Allowing students to work in their own ways | Allowing students to work in their own ways |
| | Student talk | Student talk | Student talk |
| | | Manipulating objects instead of passively listening | Manipulating objects instead of passively listening |
| | | Praising & Encouraging | Praising & Encouraging |
| | helpful hints & questions | helpful hints & questions | helpful hints & questions |
| | | Acknowledging students' perspectives and interests | Acknowledging students' perspectives and interests |

### Case 3: Becky

Becky was the newest participant in my study to the camp experience. She had finished her content math courses when camp took place. Becky had heard about the camp from her content course instructor, who also was the professional development director for the camp. Thus, although Becky had no experience with the camp, she was not unfamiliar with some of the practices found within the camp's governing precepts.

#### *Conceptions*

After the first day of camp and before the professional development portion began, I asked Becky to define student autonomous problem-solving (APS) and the five mathematical practices of this study that are related to autonomous problem-solving. I also asked her what she valued about each of these practices. I did not differentiate between student APS and the other five practices during the study but focused on the practices equally. Thus, Becky was given a list of six mathematical practice-related foci and asked to write about her conceptions and values of all six.

**Student APS.** When asked about student autonomy in the form of student-generated dialog or student-generated strategies, Becky stated the following:

> Student autonomy is where the student is encourage to read/explore/explain their thought process and reasonings about math or any other subject. This is very valuable for student's to practice so they have the opportunity to explore what they think before being told what something is/what is correct. (Becky, Written Survey)

Thus, Becky's views of student APS involved allowing the *students to work in their own ways* and explaining their thinking (*carefully listening*). Additionally, when reflecting during the clinical interview, Becky noticed that the students were able to *manipulate the objects* and that the interviewers were providing *helpful hints*. Moreover, Becky was able to distinguish how the questions or hints provided by the interviewers affected the students' ability to work in their own way. In other words, some questions allowed the student more flexibility in their solution strategy, whereas other questions provided a strategy for the student to follow. Becky also noticed a smaller detail in one of the task videos, saying, "and then he [the interviewer] asked something that she [the student] liked, so I guess like making it more like fun and like catered to her [the student]" (Becky, Clinical Interview). In reflecting on this detail, Becky showed that she also *acknowledged students' perspectives and interests*.

**Perseverance.** When conceptualized perseverance, I asked Becky to think about persistence as this was the term used in the governing precepts. She wrote the following:

> Persistence to me is defined as the continuing to push through something with determination. So I believe fostering persistence would be able to grow/develop the ability to push through negative behavior, confusion, and other obstacles that teachers may face in the classroom and turn it into positivity so the student can overall learn. (Becky, Written Survey)

Becky's definition discussed how teachers persist in the classroom to help their students learn, but also spoke about helping the students persist. Becky added that you shouldn't just give up on students and that it is important for the students to persevere; however, she also stated that she isn't sure how to "make them be persistent…if they don't want to" (Becky, Survey Reflection Interview). Thus, Becky talked about pushing through negative behavior and confusion to *foster perseverance* but was unclear about how she would do this.

In the clinical interview, Becky noted that the interviewers were persistent in asking the student questions, which encouraged the student to persevere in problem-solving. Becky said that by asking questions, the interviewers encouraged the student to keep thinking about the problem. Therefore, although unclear in her definition, Becky alluded to the importance of questioning related to fostering perseverance in students.

**Justification.** Similarly, when asked about justification of students' thinking, Becky mentioned the following:

> When there is justification of student thinking I think it could be both for the teacher to justify the students answer by asking why they answered the problem they did, or how they got to their answer. (Becky, Written Survey)

In addition to this definition, Becky reflected that it could also mean "talking it out and justifying how they [the student] know it going through the process" (Becky, Survey Reflection Interview). Therefore, Becky focused on students *explaining how* the process and *why* it worked.

**Mathematical Language.** Becky stated the following when asked about her conceptions about the practice of using precise mathematical language.

> Precise mathematical language is using vocabulary that is accurate to the math process so there is only 1 form of language in math. When we start using nonprecise math language misinterpretation can occur and cause confusion. (Becky, Written Survey)

Becky expressed the importance of using proper vocabulary (*formal language*) in her definition, but in the clinical interview, she commented that while proper vocabulary is developing, *bridging language* is sufficient. Becky saw precise mathematical language as a key component of communication. Thus, Becky focused much of her understanding of language towards *formal language* but did recognize *bridging language* too.

**Mathematical Troubles.** Becky noted the following definition and values when referring to student errors and mistakes. (Mistakes in the camp setting also include misunderstandings and misconceptions, which is why this study refers to them as mathematical troubles.)

> Working with mistakes is exploring the mistakes student make to help them learn from it. When the student is able to explore their mistake and identify where they went wrong and how to correct it, they are able to further understand how to get the correct answer and correct procedure for next time. (Becky, Written Survey)

In her reflection, Becky mentioned how important, in general, it was to explore mistakes. She continued by saying, “if they make a mistake, like don’t automatically cut them off and be like no, this is wrong, have them finish through it, even talk it out and then go from there and work on that mistake like together” (Becky, Survey Reflection Interview). Additionally, Becky noted that the benefits of working with mistakes goes beyond just having the students correct mistakes but can also address larger misconceptions.

Therefore, Becky’s definition and reflection focused much attention on the *student identifying and correcting* their own mistake, but also the need for the *teacher to assist in a student correction*. Moreover, Becky made it a point to note that she tried, as a teacher, not to identify the mistakes herself.

**Visual Representations.** Lastly, I asked Becky about her conceptions regarding the use of multiple visual representations.

> Using multiple representations/strategies, the student is finding/learning other ways to solve problems/find answers. There is typically not just 1 way to find an answer in math, and some students do better when they are able to with some strategies than others. When offering/using multiple representations/ strategies students can find what works best for them and their understandings. (Becky, Written Survey)

Becky mentioned that using *multiple representations* is good because not all students understand the same way and can find a way that makes sense for them and helps aid in their understanding. Additionally, in the clinical interview, Becky remarked about the benefits of having students work through their thinking using manipulatives to talk through their justifications. Therefore, Becky aligned with the use of *single or multiple representations* for students and *linking those representations to an idea or concept* to aid in the students' understanding.

**Summary of Becky's Conceptions.** Through the survey given at the beginning of camp, her survey reflection interview, and her clinical interview Becky conceptualized the five mathematical practices and student autonomy. Additionally, without being asked or prompted to do so, she connected some of the mathematical practices together but also to student APS. Figure 10 highlights Becky's conceptions of the five mathematical practices and student APS according to the analytic framework used in this study.

| **Students' Mathematical Practices** | **Categories of Teacher's Moves to Support the Students' Mathematical Practices** | | | |
|---|---|---|---|---|
| Justification | No Explanation | Explaining How | Justifying Why | |
| Visual Representations | Single Visual Representation | Multiple Visual Representations | Linking Visual Representation to an Idea/Concept | Connecting Multiple Visual Representations |
| Mathematical Troubles | Teacher Correction | Teacher Assisted in Student Correction | Student Correction | |
| | No Identification | Student Identification | Teacher Identification | |
| Mathematical Language | Contextual Language | Bridging Language | Formal Language | |
| Perseverance | Praises Unsuccessful Effort of Answer | Praises Process | Fosters Perseverance | |

| **Seven Ways Teachers Can Foster Student Autonomous Problem-Solving** | | | |
|---|---|---|---|
| Listen Carefully | Create Opportunities to Work in their Own Way | Create Opportunities for Student Talk (group or partner talk) | Manipulate objects or Kinesthetically work instead of passively listen |
| Offer Encouragement when students show Effort or Persistence OR Praising Mastery or Process | Giving Hints and Replying to Questions | Acknowledging Students Perspectives and Interests | |

*Figure 10.* Becky's conceptions

Moreover, we can form a picture of how Becky conceptualized the use of these practices and how they related to her views and values of student APS, as seen in Figure 11.

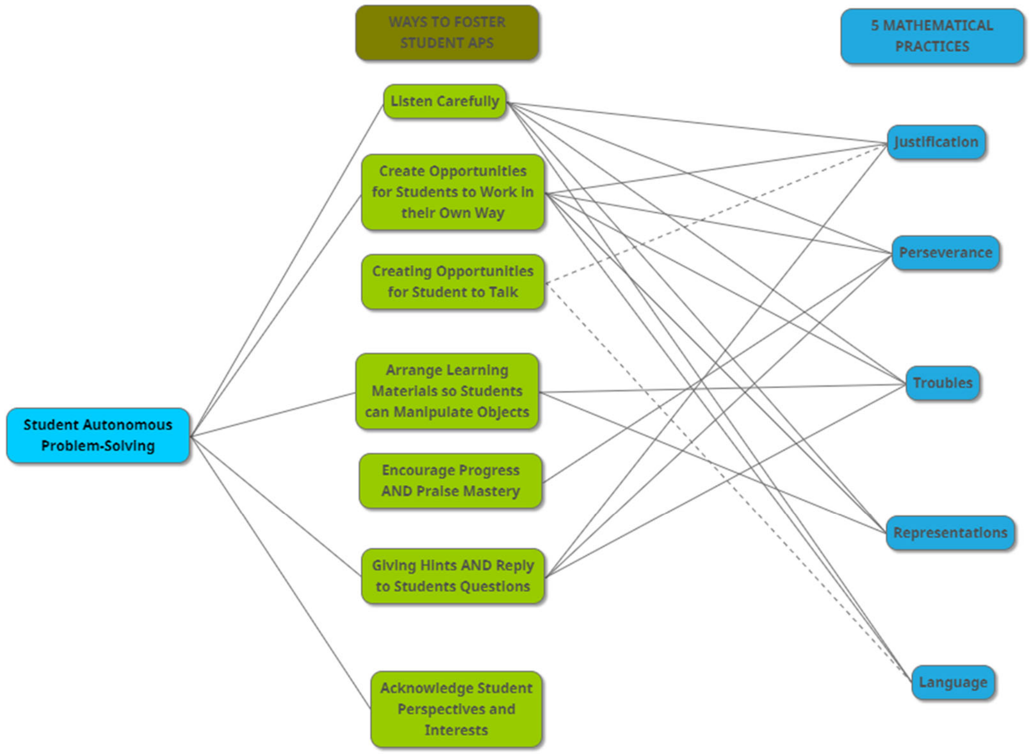


*Figure 11.* Becky's conceptions as related to student APS

Figure 11 illustrates how Becky's conceptions of student APS and the five practices relate to the ways of fostering student autonomy. On the left of figure 11, I linked Becky's conceptualization of student APS to arranging learning materials, giving hints, listening carefully and creating opportunities for students to work in their own way, and acknowledging students' perspectives and interests. On the right of figure 11, I connected Becky's conceptions of the five practices to the different ways of fostering student autonomy. The connection between language and creating opportunities for students to talk is dotted because it was unclear if the communication mentioned in Becky's conception of precise language included whole class or peer-to-peer communication. Similarly, there is a dotted line connecting Justification and student talk. Thus, there are multiple links between how Becky viewed the five practices and how she viewed student

APS. Additionally, as seen in figure 11, Becky thought of supports for student APS in ways outside of her own definition of fostering student APS.

***Implementations and Reflections***

The above are findings for how Becky viewed and valued student APS and the five practices throughout camp. I now report on how Becky implemented and reflected upon those instances throughout camp based on four stimulated recall interviews. I showed Becky one to three clips of her interactions with students during camp from either the day of the interview or the day before. After watching each clip, I had her reflect on them and the practices and supports she used in the videos.

**Day 2: First Reflection Interview.** On the second day of camp, I collected video recordings of individual and small group conversations involving Becky and the students. After reviewing the footage of that day, I selected two interactions for Becky to reflect upon. These videos contained the most complete and mathematically interactive engagement between Becky and the students. I also chose these videos because of the freedom given to the students. The focus of these videos revolved around Becky's support of different representations and perseverance.

In the following paragraphs, I summarize the interactions from the second day of camp, followed by a synopsis of how Becky supported or attempted to support the student in the five mathematical practices and student APS. After each description and summary, I turn to focus on Becky's reflection of the interaction and her relevant lived experiences.

***Interaction 1.*** Becky approached a group of students (Cedric, Meryl, and Gayle) working on a poster for the problem:

If Cave Travis is located 325 feet below the surface and Cave Crockett is located 413 feet below the surface, which cave is farther from the surface of the ground?

The students were confused about what the problem was asking. Becky asked Cedric to read the problem aloud. After checking what the students remembered from class, Becky summarized the class's conversation, and then asked, "what does that make you think that the surface of the ground is, or sea level? What is that starting point?" (press to understand the *contextual language* of the problem). After Meryl and Gayle both stated that the question was asking, which was furthest from zero, and they thought the answer was Cave Crockett, Becky asked for them to *explain*, "What made you guys think that?"

Next, Becky prompted them to draw a number line to show their thinking (press for *single representation* and *linking a representation to an idea*). The students placed zero on the number line and acknowledged that 400 also must fit on the number line. Becky then asked, "Do we have to go by ones, or by something that's bigger?" (*helpful hint*). This also *created an opportunity for student talk* as all three started discussing if they should go by tens, fives, or twos.

However, Cedric was unsure of the drawing and questioned the direction of the number line (*student identification*). He noticed that Meryl had drawn the points on what she was calling the positive side of the number line. Becky, then understanding what Cedric was asking, questioned Meryl, "now is this positive or a negative?" Meryl responded, "Positive. I flipped it around now." Becky probed Meryl for more information, but Meryl became confused and threw her hands in the air, "please help me." Becky attended to Meryl's emotional response with *encouragement* saying, "Well no, you were on the right track. You had it right. It's okay. We just got a little bit confused. You even told me...” (*fostering perseverance*) and continuing to reference another problem from the book that used the word underground instead of a negative sign (*contextual language*).

This interaction showed Becky’s press for the students to justify their reasoning and draw a representation of their thinking. It also showed how Becky addressed Meryl’s frustration and helped foster perseverance in problem-solving by acknowledging emotions and praising progress. Additionally, it showed how Becky used another problem in the book as a reference to address Meryl’s confusion.

***Becky’s reflection: Interaction 1.*** Becky reflected on the interaction with the students and noted how she was trying to remind them of what Ms. Fray had said earlier in the day (*provide helpful hints*). However, the students ignored Becky’s ideas because they were working so fast on their own. Additionally, she echoed Meryl’s confusion and her press for Meryl to *justify why* the numbers were negative.

> I asked her why she went below and she said, “Because there’s the word ‘below’,” or something like that, in the word problem, and that’s how she knew that it would be below the ground. Also, it was a cave, and she was like, “Caves are below the ground.” That’s how she knew that it’d be below. (Becky, Day 2 Interview)

In this reflection and further reflection, Becky reflected on how she was also pressing Meryl’s understanding of the *contextual language* of “below” to mean negative.

Moreover, Becky spoke of how she wanted to validate Meryl’s contributions (*praise*) and thinking when Meryl became upset. Becky mentioned how she wanted to provide some validation for Meryl without just saying that she was right, although acknowledging that she did do that in the interaction. She wanted to recognize Meryl's emotions and contributions so that Meryl would still participate in the conversation (*fostering perseverance*).

When I asked Becky what experiences, if any, she drew on for the idea of validating a student, she mentioned how she had babysat her entire life and spent some time working in a clothing store, and how those experiences plus her personality made her want to help people and cater to their needs. In a more general reflection, Becky also spoke about her content course informing her on the importance of justifying reasoning, and how she was already finding the camp experience to align with what she learned and provide exemplar teaching of those course ideas.

Lastly, I asked Becky to reflect on the governing precepts concerning the video interaction she just watched. Becky mentioned how the students were working together (*student talk*), trying to figure out how to start and how she had pressed them for justifications regarding the explanation of *how* and *why* they were reasonings in that manner (*listening carefully*).

> they didn't know how to start the problem. They were going back and forth of how they thought that they should, and I feel like that was good for them to do. …Then, I was trying to ask some questions and ask to see why they thought that, if it said below or whatever it said in the problem, why she associated it with being underground. I was trying to use communication to see where she was thinking of. Even though she got it right, I still wanted to see where she was coming from. (Becky, Day 2 Interview)

This statement also sheds light on how Becky was *creating opportunities for students to work in their own ways*. Similarly, Becky also spoke about questioning the students to make sure they understood the *contextual language* of the problem.

Throughout this reflection, Becky showed how she reflected on and fostered student APS. Becky talked about how she listened to the students' ideas, allowed them to work in their own ways, talk with their peers, praised and encouraged the students, and provided hints and questions to aid the students.

***Interaction 2.*** The second interaction was a short, but contained a quick conversation between Becky and two girls (Annie and Millie) who were finishing their poster for the problem:

> August is atop a hill that is 128 feet above sea level. Hugo is in a cave that is 512 feet below sea level. Who is farther away from sea level?

The girls had drawn a rough sketch of their poster idea in the workbook along with their answer. The image contained a person standing on top of a hill and a cave at the bottom. The person at the top of the hill was labeled 128, and a cave at the bottom of the poster was labeled as 512. The girls had drawn the entrance of the cave at sea level with a bridge that crossed over a body of water to another landmass. The girls had written their answer of 512 at the top of the poster and were coloring when Becky approached them; the two-minute warning had already sounded. Becky was only able to ask a few questions to probe the students about sea-level (*contextual language* and *linking the representation to an idea*) before the activity concluded (press for an *explanation of how*).

I chose this video interaction for Becky to reflect on because it used the ideas of a number line without one drawn. It also did not use negatives but remained in the context of the problem.

***Becky's reflection: Interaction 2.*** When Becky reflected on this interaction, she spoke about how she had wanted to ask the girls more questions because she didn't fully understand their drawing (*single representation*). She didn't think the girls completely understood how their picture represented their problem either (*linking a representation to an idea*), saying

> their picture didn't necessarily really represent it, so I feel like ... and maybe they just didn't make the picture to represent it, they just wanted to draw a picture…and not associating that the picture should've been representative of that. I was trying to help them figure out "Where is your sea level?" That's something that you need to know. That's your starting point, essentially, for this problem. (Becky, Day 2 Interview)

Becky also reflected on how she should have asked the girls to *explain how* their picture answered the question and how they knew where to start without sea-level.

Thus, Becky showed that she wanted the girls to explain how they got their answer and what their process was, so she could *listen* to how they were thinking about it. Later in her reflection, Becky also mentioned that because she didn't know much about the girls thinking, she didn't know if they understood that sea level meant zero (*contextual language*), and would have like to ask them to *justify why* they knew 512 was greater.

> I feel like I should've asked them "Well, then how did you know 512 was greater?" At sea level ... that's where it's starting from. I feel like I should've probed them more on that. "Well, then how did you know that this would be above? How did you know this would be below? You don't have your sea level anywhere, how did you know?"…I definitely should've asked that because, then, I feel like they would've maybe explained it and, then, if they weren't grasping the sea level, because I'm sure ... literally no one was associating the fact except for a select few after they had thought about it ... was associating that the sea level, the ground, anything like that was zero. (Becky, Day 2 Interview)

However, towards the end of her reflection, Becky started to wonder if the girls intended their picture to be like a line with the two points saying, "I don't know if this is a line, but they did put the 128 on top and the 512 on bottom." Regardless, Becky still found the girls' image to be unclear. When I asked Becky if she would have chosen this image when selecting work for a class discussion, she said that she probably wouldn't have at that moment but thinking about it now she might have.

However, Becky reflected that she though a class discussion around their incomplete work would make the girls explain how they got their answer, but also realize their work was missing information (*student identification*) and would have corrected it (*student correction*). This would have also allowed the girls to *link the representation to the idea* of sea-level. Moreover, in Becky's reflection of how this would have benefited a class discussion, she speaks about a whole-class conversation, looking at multiple pieces of work, and being reflective of strategies which are all ways to *foster perseverance*.

When reflecting on how she would present this image to the class, Becky also noted that she thought this image was missing something and was not a good representation of the solution. However, she said that she found presenting things with errors or missing information beneficial to the whole class and would hope that someone in the class would have noticed the missing information or would have asked a question. This reflection shows how Becky would support student APS by creating opportunities for *students to work in their own way* and *opportunities for students to talk*. Additionally, it also included having to *listen carefully* and *provide helpful hints, questions, or responses* to facilitate the hypothetical class discussion.

***Becky's Day 2 Reflection Overview.*** In reflecting on the two interactions, Becky focused on supporting the mathematical practices of visual representations, justification, perseverance, mathematical language, and hypothetical mathematical troubles. Moreover, in supporting and reflecting on these supports, Becky reflected on multiple ways to foster student APS. Table 14 summarizes these supports and provides selected evidence from the Day 2 interview.

*Table 14.* Selected evidence from Becky's day 2 interview

| Mathematical Practice | Category | Evidence from Reflection (Day 2 Interview) |
|---|---|---|
| Justification | Explain How<br>Justify Why | they didn't know how to start the problem. They were going back and forth of how they thought that they should, and I feel like that was good for them to do. …Then, I was trying to ask some questions and ask to see why they thought that, if it said below or whatever it said in the problem, why she associated it with being underground. I was trying to use communication to see where she was thinking of. Even though she got it right, I still wanted to see where she was coming from. (Interaction 1) |
| Mathematical Language | Contextual Language | I wanted to try to explain to her "Oh, well, just because there's not a negative sign in the problem doesn't mean that it's not negative. It says 'Below,' so therefore, it is negative." I don't think I got my words out. (Interaction 1) |
| Perseverance | Praise Process<br>Fostering Perseverance | I felt like I needed to give her some validation and that she was doing something right, she was in the right direction, she just kind of got confused on something or tripped up, but now she realizes that it is actually negative, to help her not get so discouraged and still want to participate because she had turned her body away and gave her pencil away. I just wanted to make her feel better, so that's why I had said that. (Interaction 1) |
| Mathematical Troubles | Teacher Correction | I was like, "Wait, you even said it yourself, it's a cave, so shouldn't it be below the ground?" |
| | | Mainly about the forwards and backwards…<br>It was like I just have to tell her. |

*Table 14*. Continued

| Mathematical Practice | Category | Evidence from Reflection (Day 2 Interview) |
| --- | --- | --- |
| Mathematical Troubles | (want) Student Identification | [hypothetical class discussion] they probably didn’t think that there was anything wrong. Sometimes, having the class ... not in a mean way ... be like, “So, let’s-”… “Let’s look at this picture. Does anyone have anything that they notice about this picture?” Something like that. Then, them being like, “Well ...” with the problem, so that they can see ... “Well ... “I honestly don’t even know if anyone would’ve picked it up. …I feel like if we would’ve opened it to the class, or if I was the teacher, I could’ve opened it up to the class, and just see if anyone noticed anything because, that way ... or ask them to present their picture to the class with their problem and how they came to their solution, or how they came to this, might help ... (Interaction 2) |
| | (want) Student Correction | |
| Representations | Single Representation | their picture didn’t necessarily really represent it, so I feel like ... and maybe they just didn’t make the picture to represent it, they just wanted to draw a picture…and not associating that the picture should’ve been representative of that. I was trying to help them figure out “Where is your sea level?” That’s something that you need to know. (Interaction 2) |
| | Linking a Representation to an Idea | |
| Student APS | (1) listen carefully, (2) create opportunities for students to work in their own ways, (3) create opportunities for student talk: group work, (5) praise and encouragement (6) provide helpful hints and answer questions | |

**Day 4: Second Reflection Interview.** On the fifth day of camp, the number line and car model were the focus of the curriculum. I asked Becky to reflect on her thoughts about both the car model and other representations used to help build the students' understanding. Becky related that she never really made sense of addition and subtraction of integers until learning the car model. When asked about other models used to help aid in the understanding so far, Becky could not think of any.

On the fourth day of camp, the students had been working primarily with addition problems using the number line. Becky was familiar with the car model from her University content course, but, as she explained, the model was still new to her. The video conversation chosen for Becky to reflect on revolved around mathematical troubles, particularly the troubles caused my mathematical language. Only one video was used because it focused on two different problems within the video and was the first sign of Becky changing her questioning strategy.

***Interaction 3.*** Becky was working with Darcy on two problems $^{-}4 - 3$ and $^{-}3 - 5$. The conversation began with Becky asking questions according to the order of the problem but changed the order of her questions after an error had occurred. She continued to use the second questioning method (which is how the curriculum teaches the car model) on the next question.

Becky immediately asked Darcy about her first thought of starting at negative four by saying, "How did you get there though?" (*explain how*). Becky continued to ask about the remaining parts of the problem until Darcy wanted to drive forwards in the negative direction (correct answer requires reversing while facing the positive direction).

Becky continued to probe until she realized that Darcy didn't understand what it meant to go backward (*bridging language*).

After two more attempts at starting the problem and Darcy moving forwards, Becky decided to count with Darcy to negative four. Still, instead of asking about the direction or the operation next, Becky asked about the sign of the second number. (This is how the curriculum suggests as the next step, but Becky may have been unaware of it.) Darcy identified that it was a positive number and that her car would then be facing right; however, Darcy remained confused about the operation, which prompted Becky to say, "When you move backward, you literally move back. And when you move forwards, you're moving forward. You are moving to the front. So, when we are moving backward, we are going to be moving back this way (pointing to the negatives)." (*teacher correction*).

After the conclusion of the problem, Becky chose the problem $^{-}3 - 5$ for Darcy and her to work together (*fostering perseverance*). She began by asking Darcy about the first number, if it was positive or negative, which way her car would face, and how many spaces she needed to move. However, instead of focusing on the operation, Becky decided to ask about the second number, the justification for turning the car around (*justify why*), and the meaning of its associated sign (*bridging language*), *praising* Darcy with a "Good Job" when she responded correctly to her questions.

The conversation continued with a similar structure of questions and correct answers: Are we adding or subtracting, do we move forward or backward, what is it going to look like to move backward? Darcy was able to correctly answer all of Becky's questions and demonstrate the process on her number line. Becky *praised* Darcy every step of the way by saying “good” or “good job”.

This interaction was chosen because of the focus on the change in question structure, the error, representation, and persistence shown throughout this video. Becky supported the student by praising the student, choosing an appropriate second task to extend Darcy’s thinking on the topic, pressing the student to explain her reasoning, and allowing the *student to manipulate a representation* while trying to *link the representation to the process* and ideas behind adding integers.

***Becky’s Reflection: Interaction 3.*** I interviewed Becky the morning of the fifth day of camp, and I began by asking what her thoughts were about the car model. Becky spoke about her introduction to the model in her University content work, and that it helped her to understand why subtracting a negative would have the same result as adding a positive. She noted that she believed this also helped the students understand the reasoning, more so than some silly random rule. She also liked the *physical manipulation* involved with the model saying, “So I really like the car model, I think it makes sense and I think it helps them visualize and they get to manipulate it so I really like it.” (Becky, Day 4 Interview)

Similarly, I asked Becky if there were any other ways, they were using in class aside from the car model. She noted that they were focused on the car model currently but would also do a life-size model in front of the classroom where the students could use their bodies to move around instead of toy cars. Additionally, Becky had looked ahead in the book and mentioned that the chip model would be coming up and thought that it could be beneficial in helping the students visualize these problems in different ways.

After watching the video, Becky remarked that one of the main issues that she had with Darcy, was the fact that Darcy did not understand what it meant to go backward or forward (*bridging language*). Becky reflected on how the car model used a specific language that could cause a problem if students were unfamiliar with it. However, she found it difficult to explain the language since it was already a descriptive bridging language that was being used but tried to use *helpful hints* and broke things down into smaller pieces for the students to digest.

Moreover, because Becky had difficulty in helping the students to understand the language, she also had difficulty in helping the students identify their mistakes (*student identification*) without pointing it out herself, which she did sometimes (*teacher identification*), saying "Mainly about the forwards and backwards…It was like I just have to tell her." (Becky, Day 4 Interview) and

> I was, definitely trying to not give her the answers if that made sense, and so I think that's why I kept trying to ask her questions, so that she could think about it and answer them herself, and be like, oh, you know. So, I definitely wanted her to like focus, oh I just realized that doesn't make sense. Yeah, I was definitely trying to help her struggles by asking her those questions for her to, well I guess now like thinking, I was literally asking her like a straightforward answer for her to get to where she was going. (Becky, Day 4 Interview)

Additionally, Becky continued to reflect on her questioning strategy while she was trying to help the students fix their mistakes and understand the process (*teacher assisted in student correction*). At this point in her reflection, Becky was already aware of how many questions she had been asking, remarking "now like thinking back, I feel like I might be asking too many questions" (Becky, Day 4 Interview) and that she considered just correcting the students (*teacher correction*) instead of asking so many questions. She continued to say how she would have liked for the students to have explained their thinking (*explain how*) and *why* they were thinking that way before her asking so many questions (wanted to *listen more carefully* and allow the students to *work in their own ways*). Specifically, justifying why they were using the model (*single representation*) in such a way, and if they were *linking the representation to the idea* of the operation and sign of the numbers.

> I wanna see why they turned it, so that they can associate, oh I turned it because we went from a negative to a positive, or whatever the case may be, so I kind of want to know their understanding and I think it also kind of helps guide them when they're moving too fast or they can't come up with these things on their own, trying to think of another instance of when I asked questions…Maybe if I was like okay, or if I asked them maybe to show me exactly what they did on the number line and after every move that they made, explain why they did this, I think that might help, I don't know if I did that. (Becky, Day 4 Interview)

Throughout the reflection interview, Becky mentioned ideas of *fostering perseverance*. Becky spoke about ways to slow down the students to focus more on the process and strategy instead of rushing through the problem and extending the task to include similar problems.

Becky attributed much of her support ideas to the PD seminar, Ms. Fray, and her University content course. She reflected on how these experiences made her more aware of questioning and the importance of having students explain their work and justifications. Additionally, Becky remarked when she was in her content course, she found this odd at first because none of her other teachers had done this, but now understands the importance of it.

***Becky's Day 4 Reflection Overview.*** In conclusion, one can see in Becky's Day 4 reflection that she was aware of and reflective about her push for students to explain, justify, and identify and play a part in correcting their own mistakes. Thus, Becky fostered student APS, or wanted to better foster student APS, by listening carefully to the students and allowing the students to manipulate instead of passively listen. Becky also reflected on her questioning and how she could better provide helpful hints in a more autonomous way, which would allow students to work in their own ways more often. Table 15 shows a summary of the mathematical practices and student APS that Becky reflected on.

*Table 15.* Selected evidence from Becky's day 4 interview

<table>
<tr><th>Mathematical Practice</th><th>Category</th><th>Evidence from Reflection (Day 4 Interview)</th></tr>
<tr><td rowspan="2">Justification</td><td>Explain How</td><td rowspan="4">I wanna see why they turned it, so that they can associate, oh I turned it because we went from a negative to a positive, or whatever the case may be, so I kind of want to know their understanding and I think it also kind of helps guide them when they're moving too fast or they can't come up with these things on their own, trying to think of another instance of when I asked questions…Maybe if I was like okay, or if I asked them maybe to show me exactly what they did on the number line and after every move that they made, explain why they did this, I think that might help, I don't know if I did that.</td></tr>
<tr><td>Justify Why</td></tr>
<tr><td rowspan="2">Visual Representation</td><td>Single Representation</td></tr>
<tr><td>Linking a Representation to an Idea</td></tr>
<tr><td rowspan="3">Mathematical Troubles</td><td>Teacher Assisted in Student Correction</td><td rowspan="2">I was, definitely trying to not give her the answers if that made sense, and so I think that's why I kept trying to ask her questions, so that she could think about it and answer them herself, and be like, oh, you know. So, I definitely wanted her to like focus, oh I just realized that doesn't make sense. Yeah, I was definitely trying to help her struggles by asking her those questions for her to, well I guess now like thinking, I was literally asking her like a straightforward answer for her to get to where she was going.</td></tr>
<tr><td>(want) Student Identification</td></tr>
<tr><td>Teacher Identification</td><td>They would tell me, when you subtract for instance, you move backwards but then I would ask them and they would say something different, and so she was just like well you just told me, so then I was just like trying to play with that without saying, without directly telling them, this is what you do.</td></tr>
</table>

*Table 15*. Continued

| Mathematical Practice | Category | Evidence from Reflection (Day 4 Interview) |
|---|---|---|
| Mathematical Troubles | (want) Teacher Correction | I should have just been like oh you're moving instead of asking her another question because I think I may have confused her; cause I think that she was like wait am I? But, yeah, sometimes to, maybe I'm taking it too much, I ask them questions to gain their thought process because I don't wanna give them the answer but I feel like if I give them something instead of asking another question that might also help too because if I ask too many questions I think that they're like, I don't know. I don't know what you're wanting, type thing. I think I ran into that today. |
| Mathematical Language | Bridging Language | I was like does she not understand like, what forwards and backwards means because that's like a big problem cause that's how, or that's how they're showing if they're adding or subtracting. If she doesn't even know what forwards or backwards means, then she can't show if she's adding or subtracting. |
| Perseverance | Fostering Perseverance | I had her do another problem, like that and I wanted to see if she could do it, again. Before she continued, because those were the hardest ones |
| Student APS | (1) listen carefully, (2) create opportunities for students to work in their own ways, (4) manipulate objects, (6) provide helpful hints and answer questions | |

**Day 6: Third Reflection Interview.** On the sixth day of camp, the students were working with the chip model and zero pairs for the first time. I asked Becky to reflect on three instances from that day: two with Conner and one with Meryl. The focus of all of these conversations revolved around mathematical Language and justifications. I had chosen these videos because of the time spent on the problem, clarity, and the students' unfamiliarity. Many of the other videos from this day involved students who would explain their work or thinking in detail and did not require much support from Becky.

The following subsections address each conversation I had Becky focus on, with a reflection of that conversation directly after. The last subsection contains Becky's reflection over further prompts regarding her interactions with Conner. Because the reflection was over both interactions and took place after all of the interactions had been reflected on, it is presented at the end.

***Interaction 4.*** Conner had completed all the problems on his page using his number line, but Becky asked him to show her using the chips (*multiple representations*). This particular conversation revolved around the problem $-6+1$, which Conner had already found to equal negative five. He had lined up six red chips and one yellow chip but counted all the chips together and stated that his answer was seven. Becky then pointed to the six red chips and asked Conner, "What are these?", "What does red mean?" and similarly with the yellow chip (*bridging language* and *linking a representation to an idea*). Becky also moved the single yellow chip to align vertically with the last red chip and asked Conner to recount. However, Conner was still just counting all the chips together to get an answer of seven, but after Becky reoriented him that the chips were different by saying, "We have negative 6, and we're adding one positive" (*helpful hint*) Conner answered negative five. Becky, affirming his answer but pressing him to *explain how* he got that asked, "Ahhh. Oh, how'd you get that?"

I chose this video because the primary focus was on understanding and working with Conner's misunderstanding and correction of his error. Through working with the error, Becky also focused on the visual representation of the chip model and pressed Conner for explanations.

***Becky's Reflection: Interaction 4.*** Becky first commented on how Conner had come up with the correct answer using the car model but changed his answers when he started using the chips (*multiple representations*). She noticed that he wasn't associating the different colors of the chips (*single representation*) with the correct parities (*linking a representation with an idea*) but would total the number of chips he had in front of him. To try and help Conner notice his mistake (*student identification*), Becky said she tried to make the zero pair in the problem more evident by aligning the pair vertically (*teacher assisted in student correction*).

> I think it helps them whenever they have them on top of each other so that they can see those zero pairs. Like oh, okay these I can cancel out and then I'll have this much left over. I don't know of course that's how I think of it and I think that, but he wasn't thinking about it that way or I think a lot of them weren't thinking about it that way but it helps and that's how we're trying to set it up. By putting the zero pairs on top and it's establishing that those zero pairs are zero.
> I think he had them side by side and so that's when he was counting up the 7. Then, so I was like, and I shouldn't have done this, but moving the one on top to

> help him visualize it that way so then he found 6 cause he just counted them ... well first he counted them and then he goes “7”. (Becky, Day 6 Interview)

Thus, Becky reflected on supporting Conner by *listening carefully* to his ideas, allowing him to *manipulate the materials*, and tried to provide *helpful hints*. However, Becky commented on how she didn’t like that she moved Conner’s chips as a way to offer a hint.

Becky also reflected on how she found this interaction to be challenging because Conner did not want to use the chip model. Becky noted that she had difficulty with Conner and some of the other students who were quiet or struggled because they didn’t like to talk or explain their thinking. Therefore, she noted that because of this she often has to ask these students to *explain how* or to *justify why*. This sentiment illustrates how Becky wanted to hear what the students were thinking, so she could help foster APS by *listening carefully* to the student and *creating opportunities for students to work in their own way*.

When I asked Becky what she thought about Conner’s answer and explanation, she remarked how she thought it was good that Conner came up with a reason for negative five, but wished they could have explored it more (*fostering perseverance*) saying,

> I think it could have been played a little bit more, I think he was just, as a whole, just using the chips in like a different way. Whenever he said you just away 1, not thinking them as like a, the positive negatives are kind of like, not together but, you know what I mean? Separating those out and then seeing how much you have left. You take away 1 aspect of it. I think that he was kind of on the right track so

> that's why I was like, eh okay because yes, you're taking away one because you have that positive 1 so it kind of cancels out so then you have negative 5 instead. (Becky, Day 6 Interview)

This statement and Becky's following thought about wanting to explore this conversation more with Conner highlights that Becky wanted Conner to better understand this *single representation* and *link the representation to the idea* of positives, negatives, and zero pairs.

Moreover, when I asked Becky to further reflect on what she found positive about the interaction, she continued to elaborate on Conner's explanation.

> by him saying that kind of makes me interpret okay well he understands that by him saying this one less means, because it is one less that he's including that positive and that's why it's one less because you're gaining and moving towards that, I'm using the number line, moving towards that zero. So, I think that that was good. (Becky, Day 6 Interaction)

This elaboration on Conner's thinking suggests that Becky was considering *multiple representations* in Conner's justification. Similarly, she noted that because Conner preferred the car model, she would still incorporate the car model into the problem first and then try to work with him using the Chip model (*multiple representations*).

***Interaction 5.*** Similar to Conversation 4, Becky worked with Conner on the problem $6 + {}^{-}8$. Conner had already completed the problem with his number line, so Becky asked him to show her using the chips instead (*multiple representations*). It looked as if Conner had been trying to solve this problem previously because he had the proper amount of chips placed on his desk. Conner had placed six yellow chips in a row and eight red chips directly above them, with the two remaining red chips extending to the right.

However, Ms. Fray was headed towards their direction, and Conner had become distracted. Conner wanted to play a card game and was determined to ask Ms. Fray when he could play but did respond to Becky's prompt for an explanation using the chips (*explain how* and *fostering perseverance*). Becky then asked Conner if he remembered what the pairs of yellow and red chips were called (press for *formal language*), to which he responded, "zero". However, when Becky pushed further and asked Conner what three yellow and three red chips would equal, he said six, which sparked a conversation about whether the color of the chips mattered (*linking a representation to an idea*). Through this conversation, Becky aided Conner in correcting his misunderstanding (*Teacher assisted in student correction*).

Although similar to the previous interaction, I chose to include this conversation because it focused on Becky's press for justification from Conner. She pressed Conner to fully explain his thinking and reasoning for why his answer was negative two. Therefore, the primary focus of this conversation was justification, but themes of perseverance, mathematical troubles, and representations were also present.

***Becky's Reflection: Interaction 5.*** When I asked Becky to reflect on the interaction, she described how she was trying to push Conner to make the connection (*linking a representation to an idea*) and vocalize the reasoning for his answer (*justify why*) using his representation (*sing representation*) and proper vocabulary (*formal language*). She also remarked how Conner was very distracted throughout the entire interaction and was asking questions to keep his attention on the problem (*fostering perseverance*).

Becky continued to discuss how she thought she should have stressed the idea of zero pairs more, and that the vocabulary was essential in advancing the students' understanding. She noted that her first goal was to have the students first identify the fact that the chips zero out and that their value was zero (*bridging language*).

> Yeah so, he's on ... yeah that's just when I was wanting him to just first identify that these are zero. Their value is zero. Once I hit that mark I was just like okay, so therefore he's kind of understanding it a little bit more. (Becky, Day 6 Interview)

However, the next step in understanding was to use the proper vocabulary term (*formal language*) when identifying when the chips pair to values of zero.

> Yeah, the whole zero pairs I think should definitely be, should have been pushed a little bit more because yeah there's two things that are encompassing those zeros which is positive and negative. (Becky, Day 6 Interview)

I then asked Becky to reflect more on vocabulary, and she remarked how "the language of math is so important to make sure that you're saying the right things because it builds on itself." (Becky, Day 6 Interview) Additionally, she thought that this was the cause of a few students' misunderstandings, including the instance of the final interaction for the day where Meryl did not understand Becky's question regarding the value of her chips.

***Interaction 6.*** The final interaction I had recorded for the interview stemmed from a more extended conversation involving Becky asking Meryl to explain her answer to the problem $6 + {}^{-}4$ (*explain how*). However, the part of the conversation involving Meryl's confusion about the word value (*formal language*) happened off-camera. Meryl had illustrated the problem with her chips correctly and had come to the correct conclusion. Still, when Becky prompted Meryl to explain why (*justify why*) she could "take away" the paired chips (*linking a representation with an idea*), Meryl didn't understand what Becky was asking. After explaining her question in another way, Meryl was able to demonstrate that the two negatives took away the two positives, and after a little more probing from Becky, said that they were zero pairs (*press for formal language*).

***Becky's Reflection: Interaction 6.*** Even though the conversation we had been speaking about involving Meryl's misunderstanding of the word value did not happen in the video, Becky recalled the interaction because it revolved around that set of problems they had been working on, and someone in the background was asking similar questions.

Becky remarked that she thought Meryl's misunderstanding was because a vocabulary word had a rushed introduction before the class started using it. However, she felt it was good that Meryl realized her own confusion regarding the word saying, "it was good that [Meryl] asked and she didn't understand cause it might throw her off when we're getting to high level stuff." (Becky, Day 6 Interview) Becky notes that "whenever we use the common terms it makes more sense for them…But I feel that it's important for us to use the proper vocabulary" (Becky, Day 6 Interview). This suggests that Becky finds both *bridging language* and *formal language* important for student growth and understanding. Additionally, she remarked keeping a vocabulary list on the board and constantly talking about the vocabulary words and meanings are ways to help the students remember them and understand them better.

Becky continued this thought by mentioning how, when they introduced absolute value to the class, they were continually relating the idea of absolute value to distance and would use both words when talking about it. Thus, when they started doing more advanced material, the students were able to make the connection to absolute value because the idea was familiar. Moreover, Becky stated, “they have to understand what it means for us to ask some questions about it too or else it just confuses them as a whole…If they don’t really understand what it means it just confuses them more.” (Becky, Day 6 Interview)

***Becky’s Reflection: Multiple Interactions.*** After reflecting on both of Conner’s and Meryl’s interactions, I asked Becky to return to Conner’s interaction; specifically, to Conner’s use of the number line. I asked Becky if she found it beneficial that Conner had used the car model, something that he was comfortable with, first. Becky reflected that she did not find this beneficial because it allowed Conner to become answer driven instead of process focused. Becky stated, “It just helps him get the right answer, which isn’t important, in like the big picture because he couldn’t figure out how to actually do the Chip model.” (Becky, Day 6 Interview) She thought that it would have been better if he would have just done the problem with the chips without an answer already in mind.

Becky also focused on how different the two models were (*connecting multiple representations*) saying, “the Chip model is like completely different and I think that by him doing the car model it’s just helping him get the right answer” (Becky, Day 6 Interview). Thus, I further asked Becky to reflect on the scenario and consider what if the order of the representations had been switched, and Conner had first used the Chip model but checked his answer with the Car model. Becky thought this ordering of the models would have been beneficial in that it would confirm his answer if he had come to the same answer two different ways. I continued to probe Becky more about this idea and asked about the idea of getting two different answers.

This scenario triggered Becky to recall a video she watched both during the camp PD and in her content course, which contained a little girl solving a subtraction problem using multiple representations and realizing she had made a mistake somewhere because she had arrived at different answers. However, with this video in mind and the conversations she has had around it, Becky found this to be a valuable idea. She continued to reiterate that the order of the representations used, in the case of Conner, did matter saying, "So yeah I feel like if it was opposite it would—I don't think that would be bad. I think, yeah just kind of like matters with the ordering."

Therefore, Becky acknowledged how she wanted students to *actively manipulate materials* and *work in their own way*, which are two ways to support student APS. Additionally, Becky reflected on asking questions to help students and *provide hints and answer questions in helpful ways*. When reflecting in general about her supports regarding the day's interactions, Becky spoke mainly of having students talking and using their own ideas but did not reflect on the use of representations or language as supports. Lastly, Becky noted that she tries to consider and build on students' prior knowledge saying, "I feel like I was trying to, but I feel like this is less so, something that I can try to do, build on prior knowledge." (Becky, Day 6 Interview) Thus, Becky notes that she tries to listen to students and provide hints that help to support and build on the students' prior knowledge, but it is ultimately up to the student to form those connections. This reflection provides evidence that Becky considers *students' perspectives* when working with them.

***Becky's Day 6 Reflection Overview.*** We can see Becky acknowledging and supporting student APS allowing students to manipulate the representations, fostering perseverance, listening carefully to what the students are saying, and providing questions and hints for the students. Becky also reflected on supporting student APS by considering how to ask better open-ended questions that allow more freedom in student answers. Moreover, Becky self-critiqued a few hints she had provided and how they limited some of the student APS. Table 16 summarizes Becky's Day 6 reflections.

*Table 16.* Selected evidence from Becky's day 6 interview

| Mathematical Practice | Category | Evidence from Reflection (Day 6 Interview) |
|---|---|---|
| Mathematical Language | Bridging Language<br>(want*) Official Language | *I feel like it ... whenever we use the common terms it makes more sense for them, you know? Day to day. But I feel that it's important for us to use the proper vocabulary so that their being introduced to it at a younger level but maybe we should just put it on the board. We're talking about value and then talk it about a little bit more. (Interaction 6) |
| | | Again, just kind of like making sure to associate okay well since these are red these are negative and since you have ... these are your zero pairs it's just like associating the chips with the vocabulary because then it kind of makes a full circle if that makes sense. |
| Perseverance | Fostering Perseverance | He was very interested in playing war at this point. It was like every question I'd ask he'd turn around and then come back and then answer and then turn around. (Interaction 5) |
| Visual Representations | Single Representation<br>Linking a Representation to an Idea | I think it was good that he recognized that because he had that positive it's just one less and so that's how he got the negative 5. I thought that that was good but I just, you know, again, he's not associating it the way that I was hoping and the way we're trying to use the Chip model I think. (Interaction 4) |
| | Multiple Representations | by him saying that kind of makes me interpret okay well he understands that by him saying this one less means, because it is one less that he's including that positive and that's why it's one less because you're gaining and moving towards that, I'm using the number line, moving towards that zero. So, I think that that was good. (Interaction 4) |

*Table 16.* Continued

| Mathematical Practice | Category | Evidence from Reflection (Day 6 Interview) |
|---|---|---|
| Visual Representations | (none) Connecting Multiple Representations | cause those are just such different and their mixing them so I don't even know if it'd be beneficial to introduce and have him do it on the car model and be like okay now let's try it on the Chips type thing. |
| Justification | Justify Why | [Conner], he just doesn’t want to explain why, you know? |
| | Explain How | Or it’s hard to ask him questions. Like, so how did you get there? It’s hard to ask him questions and for him to answer those without being ... and he’s so like is it right? Is it right? (Interaction 5) |
| Mathematical Troubles | Teacher Assisted in Student Correction | I think it helps them whenever they have them on top of each other so that they can see those zero pairs. Like oh, okay these I can cancel out and then I’ll have this much left over. I don’t know of course that’s how I think of it and I think that, but he wasn’t thinking about it that way or I think a lot of them weren’t thinking about it that way but it helps and that’s how we’re trying to set it up. By putting the zero pairs on top and it’s establishing that those zero pairs are zero.<br><br>I think he had them side by side and so that’s when he was counting up the 7. Then, so I was like, and I shouldn’t have done this, but moving the one on top to help him visualize it that way so then he found 6 cause he just counted them ... well first he counted them and then he goes “7”. (Interaction 4) |
| Student APS | (1) listen carefully, (2) create opportunities for students to work in their own ways, (4) manipulate objects, (6) provide helpful hints and answer questions, (7) acknowledging students perspectives | |

**Day 8: Fourth Reflection Interview.** On the eighth day of camp, the students had begun to work with variables and solving equations. The card game *War* had become a classroom favorite and put the students' addition and subtraction of integer skills to the test. I chose two interaction videos from this day for Becky to reflect on. One video contained a spontaneous conversation about adding integers that was prompted by an incorrect answer during a game of *War*. This video highlighted a student using multiple representations and techniques from outside of camp to try and solve the problem. The second video contained a group of students working together to solve an equation and bouncing ideas off of each other, and Becky fostering perseverance in the students' problem-solving.

***Interaction 7.*** Becky had noticed that Pedro got an answer wrong during a card game and brought the problem to Pedro's attention (*teacher identification*). She presented Pedro with an expression written on a personal whiteboard, saying, "Okay [Pedro], show me negative eleven plus a positive three." While spinning the marker in his hand, Becky remarked, "The marker could be your car" (*hint* and press for *single representation*), but Pedro replied that he didn't understand the number line and preferred to do it on paper. Becky did not push back, but said, "Do it how you think that you could solve it" (*allowing students to work in their own way*).

First Pedro wrote $\begin{array}{r} 11 \\ \underline{-\quad 3} \end{array}$, but then realized that the borrowing technique didn't help, so then he drew eleven tally marks and erased three. However, Pedro still wasn't sure, and Conner had been bouncing in his chair because he knew the answer. Becky told Conner not to share his answer but prompted Pedro to think about the number line in a different way, saying, "we are all the way over here, we are all the way at the negative eleven, okay? Why do you think that you subtracted?" To this prompt to *justify why* Pedro credited his idea to subtract the numbers to something Ms. Fray had said that day about subtracting when the signs were opposite. Becky then questioned Pedro, "So if this is your car, you move eleven spaces, but I'm adding three more, three positive. So, where should I go? Should I keep going this way, or should I go back this way?" (press to use a *single representation*), which helped Pedro to arrive at the correct answer.

This interaction showed Becky allowing Pedro to use multiple representations and work in his own way. It also illustrated a way Becky helped students to correct their understandings. Thus, Becky supported APS by *listening to the student*, allowing him to work in his own way, *manipulate the marker and number line*, and provide helpful hints when Pedro was stuck.

***Becky's Reflection: Interaction 7.*** I interviewed Becky on the morning of the ninth day of camp, and she remembered this interaction being challenging because Pedro was a student that did not typically talk. She also commented on how she found Pedro's use of the vertically aligned subtraction and tally marks unusual. Becky reflected on her interaction with Pedro, and although noting that those ways could work, she also wanted Pedro to understand the way they had been using.

> [pointing to the vertically aligned problem of 11-3] He was like zero and then did 11. It's the same thing.
>
> And so that didn't help him enough. I'm like, I'm surprised you just didn't use fingers. So, then he, you know, whatever and then used that way to find out that the answer was eight, but then still wasn't remembering that it was a negative.

> So, I guess he got it wrong or something. And then when he was doing it, he wasn't registering the fact that it was a negative. And so that's why I wanted him to use a number line of some sort so that he could see, okay, well this is a negative. You're moving back. (Becky, Day 8 Interview)

This statement shows how Becky reflected on creating opport*unities for the student to work in his own way*, but also *providing hints* to aid the student in his thinking after she *listened carefully*. In bringing in the number line (*providing access to manipulatives*), Becky was providing Pedro with *multiple representations* and *assisting him in his correction* of the problem. This reflection highlights Becky's press to explore the problem using multiple strategies to *foster perseverance* in problem-solving.

Thus, Becky reflected on how she wanted Pedro to keep thinking about the problem and extend his known strategies to include the number line. Additionally, she commented on how she was providing the hints and prompting to look at the number line because she wanted him to notice the missing negative sign.

Becky also reflected on how difficult it was to get Pedro, and the other quiet students, to explain their thinking. She stated, "it was like tougher for me to have him show me" (Becky, Day 8 Interview). She asserted that is why she asked so many questions because she wanted to break it down more for them to answer more. However, she also noted that she could have backed off on the questions some. She remarks that she thought Pedro was starting to understand and notes, "Maybe I should have backed off with the question cause I feel like he could have…I thought it would help if I fed it to him a little bit more" (Becky, Day 8 Interview). However, she continued this thought by saying, “He doesn’t want to say anything, so in order for me to get a response out of him, then I have to ask these questions” (Becky, Day 8 Interview).

Becky also mentioned that she does this to not only get the students to explain but for them not to feel discouraged. Thus, in addition to *fostering perseverance* by extending the strategies and focus on the problem, Becky also takes into account the student’s feelings when working on an issue. Becky noted that Amy did well with the students because “she will like break it down” and not get frustrated with the students. Moreover, Becky said she also tried to be patient and help the students so they wouldn’t become discouraged (*encouragement for effort*). She attributed this idea to her camp experience and her work with the students from the previous days.

Throughout this interaction, Becky focused on multiple representations and ways to foster Pedro’s perseverance in problem-solving. During her reflection, Becky showed that she tried to foster student APS by listening to students’ ideas, having them work in their own way, manipulate instead of passively listen, offer encouragement when the student showed effort, and provide helpful hints and questions.

***Interaction 8.*** Interaction 8 involved Becky working with Conner and Nigel on the problem $5 + \_ = 7$. Becky approached Conner because he was confused; however, Conner quickly asked if Nigel could work with them. Becky agreed and then explained the problem.

After a small amount of time, Conner claimed to know the answer, saying that it was two. Becky responded, “Okay, how did you get two? Can you write it on your board, and then we can talk about it.” She furthered this *explanation on how* with a *justify why* prompt of, “what makes you think it would be two?” However, the goal of the exercise was to push the students' thinking further and to extend it to variables, which they had been discussing. Thus, Becky pushed Conner to think more about the problem and to consider another strategy (*foster perseverance*). Soon, Nigel quickly chimed in, so Becky also began to prompt Nigel to contribute ideas (*student talk*) saying, "Okay, let's hear Nigel…let's hear him. What he has to say." (*listening carefully*).

Nigel asked about the idea of subtraction, to which Becky said, "I don't know. Let's find out. How would you use subtraction?" (*explain how* and *allowing students to work in their own way*). Conner mentioned that seven minus nine would be two, but Nigel did not agree, saying, "No, seven minus five" (*student identification* and *correction*), which both Becky and Conner agreed.

This interaction showed how Becky fostered perseverance by focusing the attention on multiple strategies and facilitating Nigel's ideas into the conversation, which changed the participation format. She also pushed the students to explain their thinking so she could listen to the students' ideas and see how they worked in their own ways. In addition to fostering students APS in those ways, she also created opportunities for partner talk, replied to student questions in helpful ways, and provided hints.

***Becky's Reflection: Interaction 8.*** Becky first reflected on her press for Conner to extend his thinking, noting that what he did was good but wouldn't be ideal for larger numbers. She also said that she was hoping Nigel would contribute to the conversation (*student talk*) and help Conner think about different ways (*foster perseverance*). Thus, not only did Becky reflect on how she extended the problem, but also the change in participation format, which attributed to the creation of student talk.

> I think that he identified the easy way because he had five, then he was like, okay, seven. He was like, "Oh, if I just add two, I'll get to seven." So, then I was like, "Okay, so that's one way, but I wanted them to ... because that can work with the smaller ones, but that can't work when it gets bigger. If you have bigger numbers because that's going to be hard for you to be like ... you'll run out of fingers.
>
> So, I wanted to see if they could pick up on the fact that you could also subtract.
>
> And so, Nigel ... what did I ask, can you think of another way? (Becky, Day 8 Interview)

Next, Becky focused on the conversation the boys had. She noted that Conner bounced off of Nigel's subtraction idea but with the wrong number, which she noticed that she and Nigel had identified (*teacher identification* and *student identification*). However, she thought that Nigel had found the mistake to have been helpful because he came up with a correct idea next (*student correction*).

> I think Conner was just ... I think maybe he might've gotten confused cause he was like, "Oh two"
>
> So, then I was like, "No, no, no."
>
> I think that helped Nigel because he heard that he was like, "No, that's not right."
>
> And then he was like, "Oh, it could be five. You take the five away from seven and that's what gets you the two which is like what you're missing." (Becky, Day 8 Interview)

I also asked Becky to reflect on her press for Conner to explain his original answer, which had happened before the collaborative exchange. She noted that she wanted to see *how* he was thinking and to see if he was making the connection.

Additionally, Becky remarked how that she was “trying to not jump in so much…trying to stop that and seeing if they make those connections” (Becky, Day 8 Interview); noting that this helps her help them (*hints*).

From this reflection, one can see that Becky focused on listening to the students’ ideas, wanting them *to work in their own way* and *explain how* they are thinking. Additionally, Becky noted that she intended to use the information gathered to help her provide hints, answer questions, or guide the students. Becky later noted that sometimes having the students work together had the opposite effect than it did here, in that sometimes the students “shut down” and became answer focused. However, this time, they kept working on the process, and she remarked, “I liked that because that was the first interaction that I’ve seen, one of the first, where they’re bouncing back those ideas within a small group. So yeah, I liked that.” (Becky, Day 8 Interview)

***Becky’s Reflection: Multiple Interactions.*** Lastly, I asked Becky to reflect on both videos and the supports she used with the students. The first thing she mentioned was how she would prefer the students to *work with their own ideas* saying that she tried to not “shut down” an idea or answer because it was wrong, but would have preferred for the student to fully explain their thinking first before addressing any errors or misconceptions. However, Becky made the exception that she would *identify* (*teacher*) an error if the answer was really wrong.

> As a whole, I don’t think that I’ve ever really done this, but just because, unless they’re going like really far into, “no, this is really wrong” because I don’t want to like confuse them, but, if they put down an answer that isn’t right or something, I’m trying or we’re trying to push it in a different direction. (Becky, Day 8

> Interview)

Conversely, she also notes that she is trying not to “shut down” answers, but having the students *explain* and *identify* (*student*) the errors so they can fix them together (*teacher assisted in student correction*).

> So not discouraging their previous work or shutting them down for their answer. And then of course, I’m still trying to do the communication stuff…It’s tough, me trying to get Pedro and the rest to try to work together to get somewhere. I feel like that could have been utilized in that situation, it just didn’t really get there. I feel like the thing that ... the most thing that I’m trying to do with a teaching practice, if that makes sense, is to have their answer, you know, however way that they’re answering it and then working from there. (Becky, Day 8 Interview)

After this, she continued to describe how she struggled with how to *praise* the students and move on to something else without focusing on the fact that the students reached the correct answer. She also considered how far to extend a problem or focus on an idea before it was no longer beneficial, saying, “I bet I could have gone maybe even longer, but I don’t know if that would have been beneficial or not.” However, she found the length of the interaction between Nigel and Conner, although it could have potentially been longer, to be sufficient since it allowed Nigel time to think about Conner’s ideas and modify his own from the information he gained. Therefore, again reflecting on and thinking about the benefits of student talk.

***Becky’s Day 8 Reflection Overview.*** Thus, Becky reflected on ways she fostered perseverance, required students to explain their thinking, and how she handled students’ mathematical troubles. Throughout this reflection, Becky also focused on several ways she fostered student APS too. Becky mentioned and reflected on instances where she listened carefully, had the students work in their own ways, created opportunities for students to talk, and provided hints and answers to student questions. Additionally, Becky reflected on how she wanted to “talk less” and ask fewer questions so she wouldn’t be guiding the students so much. Table 17 illustrates the mathematical practices and student APS supports that Becky reflected on during Day 8.

*Table 17.* Selected evidence from Becky’s day 8 interview

| Mathematical Practice | Category | Evidence from Reflection (Day 8 Interview) |
|---|---|---|
| Justification | Explain How | So, I wanted to see how he was thinking of it. Yeah, I wanted to see how he was thinking of it. (Interaction 7) |
| | (notice) Justify Why | I think that helped Nigel because he heard that he was like, “No, that’s not right.”<br><br>And then he was like, “Oh, it could be five. You take the five away from seven and that’s what gets you the two which is like what you’re missing.” (Interaction 8) |
| Mathematical Troubles | (notice) Student Correction | |
| | (notice) Student Identification | |
| | Teacher Identification | As a whole, I don’t think that I’ve ever really done this, but just because, unless they’re going like really far into, “no, this is really wrong” because I don’t want to like confuse them (Multiple Interactions) |
| | Assisted correction | [pointing to the vertically aligned problem of 11-3] He was like zero and then did 11. It’s the same thing.<br><br>And so that didn’t help him enough. I’m like, I’m surprised you just didn’t use fingers. So, then he, you know, whatever and then used that way to find out that the answer was eight, but then still wasn’t remembering that it was a negative.<br><br>So, I guess he got it wrong or something. And then when he was doing it, he wasn’t registering the fact that it was a negative. And so that’s why I wanted him to use a number line of some sort so that he could see, okay, well this is a negative. You’re moving back. (Interaction 7) |
| Visual Representations | Multiple Representations | |

*Table 17.* Continued

| Mathematical Practice | Category | Evidence from Reflection (Day 8 Interview) |
|---|---|---|
| Perseverance | Foster Perseverance | he was like, okay, seven. He was like, "Oh, if I just add two, I'll get to seven." So, then I was like, "Okay, so that's one way, but I wanted them to ... because that can work with the smaller ones, but that can't work when it gets bigger. If you have bigger numbers because that's going to be hard for you to be like ... you'll run out of fingers.<br><br>So, I wanted to see if they could pick up on the fact that you could also subtract. And so, Nigel ... what did I ask, can you think of another way? And then Nigel ... |
| Perseverance | Foster Perseverance | I'm trying not to like shut down that answer, but just have them like, rethink about it or like show them, "Okay. Okay." And then push them into a different direction. (Interaction 8) |
| Mathematical Trouble | (want) Student Identification | |
| Student APS | (1) listen carefully, (2) create opportunities for students to work in their own ways, (3) create opportunities for student talk: group work, (4) manipulate objects, (5) praise and encouragements, (6) provide helpful hints and answer questions | |

***Conclusion***

In conclusion, there is evidence that Becky reflected on ways to foster student APS in all seven aspects from the analytic framework. Becky focused much effort on listening to the students by having them explain their thinking. Additionally, Becky wanted the students to work in their own ways but noticed that she occasionally guided the students' strategies through her use of hints and questions. She also made it a point for the students to manipulate objects or animate the problem while explaining, often asking the students to "show her" what they did.

Becky reflected on how she liked when students worked together and would have liked to foster that type of interaction more often because it allowed the students to continue working on the problems and build on each other's ideas. Similarly, she also reflected on how she would praise and encourage students when they became frustrated with a problem, or they were slowly making progress. Lastly, Becky did not reflect on acknowledging students' perspectives. However, she did mention that she tried to build on students' prior mathematical knowledge but struggled with this practice. Thus, Becky reflected multiple times on seven of the eight ways to foster student APS and remarked on ideas that could align with how she would foster the eight-way of acknowledging students' perspectives.

Becky attributed her success and reflectional improvements to her current camp experiences with teachers, Amy, and students. Becky also mentioned some of the material from the camp PD, such as IMAP videos, practitioner articles involving questioning and communication, and group discussions. Similarly, Becky frequently brought up her experience from her university content course and would relate it to the

PD material or camp experience. Furthermore, Becky called upon were her own experiences as a student, regarding the differences between her K-12 teachers and her university content teacher when it came to questioning and prompting for students thinking. Lastly, Becky recalled her experiences working as a babysitter and as a retail clothing employee when reflecting on catering to students' frustrations, struggles, and other emotional needs.

Table 18 highlights the findings related to the research questions of this study.

*Table 18*. Summary of Becky's results

| | Conceptions | Enacted Supports | ePSTs' Reflections |
|---|---|---|---|
| Justifications | How | How | How |
| | Why | Why | Why |
| Perseverance | | Praise Process | Praise Process |
| | Fostering Perseverance | Fostering Perseverance | Fostering Perseverance |
| Mathematical Language | | Contextual Language | Contextual Language |
| | Bridging Language | Bridging Language | Bridging Language |
| | Formal Language | Formal Language | Formal Language |
| Visual Representation | Single Representation | Single Representation | Single Representation |
| | Multiple Representations | Multiple Representations | Multiple Representations |
| | Linking a Representation to an Idea | Linking a Representation to an Idea | Linking a Representation to an Idea |
| | | | (none) Connecting Multiple Representations |

*Table 18*. Continued

| | Conceptions | Enacted Supports | ePSTs' Reflections |
|---|---|---|---|
| Mathematical Troubles | Student Identification | | (want/notice) Student Identification |
| | | Teacher Identification | Teacher Identification |
| | Student Correction | | (want/notice) Student Correction |
| | Teacher Assisted in Student Correction | Teacher Assisted in Student Correction | Teacher Assisted in Student Correction |
| | | Teacher Correction | Teacher Correction |
| Student APS | Listening Carefully | Listening Carefully | Listening Carefully |
| | Allowing students to work in their own ways | Allowing students to work in their own ways | Allowing students to work in their own ways |
| | | Student talk | Student talk |
| | Manipulating objects instead of passively listening | Manipulating objects instead of passively listening | Manipulating objects instead of passively listening |
| | | Praising & Encouraging | Praising & Encouraging |
| | helpful hints & questions | helpful hints & questions | helpful hints & questions |
| | Acknowledging students' perspectives and interests | | (want) Acknowledging students' perspectives and interests |

**Cross-Case Analysis**

In this section, I summarize some key findings regarding the similarities and differences across the three ePSTs. I first start by examining the differences in the ePSTs conceptions and purposes of the five mathematical practices and student APS. Next, I move on to discuss which mathematical practices were supported by the ePSTs and the ways they fostered student APS. I then focus on the ePSTs reflections of the five practices and student APS during the stimulated-recall interviews, including things they noticed or did, and what they would have done differently. Lastly, I look at the lived experiences the ePSTs mentioned while reflecting on the video interactions.

***Conceptions***

In comparing Amy, Linda, and Becky's conceptions of the five mathematical practices, they all conceived their definitions in a similar enough way to be categorized the same (see figure 12).

| **Students Mathematical Practices** | **Categories of Teacher's Moves to Support the Students' Mathematical Practices** | | | |
|---|---|---|---|---|
| Justification | No Explanation | Explaining How | Justifying Why | |
| Visual Representations | Single Visual Representation | Multiple Visual Representations | Linking Visual Representation to an Idea/Concept | Connecting Multiple Visual Representations |
| Mathematical Troubles | Teacher Correction | Teacher Assisted in Student Correction | Student Correction | |
| | No Identification | Student Identification | Teacher Identification | |
| Mathematical Language | Contextual Language | Bridging Language | Formal Language | |
| Perseverance | Praises Unsuccessful Effort of Answer | Praises Process | Fosters Perseverance | |

*Figure 12.* Summary of the ePSTs' conceptions of the mathematical practices

However, there were slight differences in what they thought was the purpose of the practices. All three ePSTs mentioned the idea of justification as a way to survey or understand their students' thinking, but Amy and Linda also noted that they could use the knowledge gained from their students' justifications to help them when a mathematical trouble occurred. Conversely, all three ePSTs pointed out that the purpose of mathematical troubles was to have students create a more robust conceptual understanding of the mathematics.

Another significant difference in the ePSTs' conceptualizations was in their definitions of *formal language*. Although the thought of using formal language as a way to communicate mathematical ideas was evident across all three conceptions, Becky took it slightly further and mentioned that using "non-precise language, that is not clear, can create misunderstanding and confusion".

Finally, perseverance seemed to be viewed uniquely across the ePSTs, but all thought of it as a way to overcome struggles and confusion. Amy thought of perseverance as being synonymous with a growth mindset, which is similar to Linda's definition of "viewing the problems as a journey" and not being afraid of getting the wrong answer. Becky's definition was slightly different from Amy and Linda's, but this may be because Amy and Linda's perception of perseverance had been affected by the camp's governing precepts since persistence is a key component. Becky's ideas revolved around turning a negative situation into a positive one, and not giving up on the problem.

Interestingly, all three ePSTs had similar definitions for visual representations. They all noted that students did not think alike, and this was a way to help their thinking and figure out which method works best for them.

It is also important to note that both Amy and Linda had worked for the camp previously and mentioned that they try to improve or focus on different elements each year. Thus, this year's focus may be slightly different from when they first attended camp. When looking at Amy's conceptions of the five practices, she puts a significant emphasis on justification. For Amy, justifications tied into almost every practice, except for perseverance. Linda, although having a significant focus on justification, focused more on *formal language* and even mentioned it in her definition of justification. Lastly, Becky had somewhat of a split focus between justification and mathematical troubles; however, her focus seemed to lean more towards mathematical troubles as she stated, "it's important for – well, working with mistakes in general, just exploring the mistakes that students make to like help them learn from it."

The ePSTs also connected the practices in various ways and would often mention overlapping ideas when discussing the five practices. Figure 13 illustrates the connections the ePSTs made between the five practices when they conceptualize them.

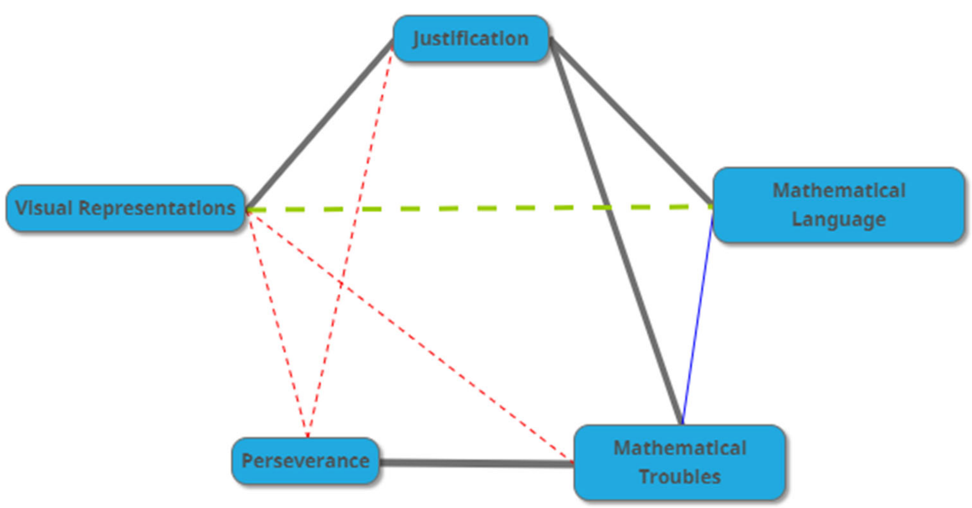


*Figure 13.* ePSTs' connections

The bolded connections between visual representations and justification, perseverance and mathematical troubles, justification and mathematical troubles, and mathematical language and justification are connections made by all the ePSTs. The dotted red connections between visual representation and perseverance, perseverance and justification, and visual representations and justification were connections made by Amy. In contrast, the thin solid blue connection between mathematical language and mathematical troubles is a connection that was made by Becky, and the thick dashed connection between mathematical language and visual representations was made by Linda. From Figure 13, one can see that overall, there was a large focus on justification by all the ePSTs, especially Amy, who made an additional connection to perseverance.

I now turn to explore each of these connections a little deeper. I will first examine the ePSTs' similarities, and then follow by exploring the differences amongst the ePSTs.

**The Five Practices.** All three ePSTs supported the five practices in similar ways and were able to identify elements of supporting those practices in their stimulated-recall interviews. The ePSTs' conceptualization matched both their supports and reflections for justification and representations. In addition to their conceptualization of perseverance as *fostering perseverance*, all three of the ePSTs supported and reflected on *praising* the students process and mastery. Similarly, in addition to the ePSTs conceptions about *bridging language* and *formal language*, the ePSTs supported and reflected on these categories as well as *contextual language.* However, when it came to mathematical troubles things did not always align.

The ePSTs conceptions about identifying troubles aligned with the *student identification* and *student correction* of the mathematical troubles, but evidence from this study suggests that they only *teacher identifications and corrections* were utilized, with the exception of *teacher assistance in student correction*. Conversely, when watching the interactions during the stimulated-recall interviews, majority of the *teacher identifications and corrections* were pointed out by the ePSTs as something they wanted to change.

Moreover, Linda identified an instance of not requiring a justification from a student and accepting their answer. At the same time, Amy reflected on how the students didn't press themselves or the teachers for justifications and would just accept the answer. Additionally, there was one instance across all the stimulated-recall videos that contained an unidentified mistake. Still, it was pointed out and reflected on by Linda, who said that she didn't intentionally let the mistake go unidentified.

***Justification.*** All three ePSTs pressed for reasoning from the students in the form of both *explaining how* and *justifying why*. However, the majority of the presses for justification by both Amy and Becky were for *explaining how*, while Linda pressed for students to *justify why*. That said, due to the nature of the curriculum and consideration of "Why?" as a press for *justifying why*, it was difficult to distinguish the difference between the two categories of justification. In fact, the two categories were often used interchangeably by the ePSTs, and a process-based answer ("I did this") was considered a justification of why something worked.

***Mathematical Language.*** *Formal language* was popular in both the ePSTs' conception and in their support of the practice for mathematical language. However, *contextual language* and *bridging language* were both used and reflected on by all of the ePSTs. Interestingly, all three ePSTs reflected that they would have liked to have provided better hints in situations that dealt with *contextual language*, for instance, sea-level, AD-BC timelines, and temperature. Similarly, all three participants also reflected on the importance of using *formal language* and modeling the language for the students but realized that students would adopt the language in their own time.

***Visual Representations.*** At some time throughout the camp, all the participants reflected on the use of *multiple representations* to help aid in students' understanding; however, a secondary representation was only used as a way to check an answer. The primary focus of visual representations was the ability to use a single representation to help visualize a problem's process, thus *linking a representation to an idea*. In fact, Becky mentioned that she believed the primary representations used in the class (chip model and car model) were so different that they couldn't be used to understand one another. Additionally, both Linda and Becky recalled the same professional development video regarding the use of multiple representations, but Becky thought that the order in which the representations were used affected the productivity of the struggle and could make the struggle more answer focused instead of exploratory. Similarly, Amy worried that the chip model could be confusing in certain situations and was hesitant even to consider mentioning it as an idea to the students.

***Mathematical Troubles.*** In the case of mathematical troubles, it was fairly normal for the ePSTs to reflect on how they handled and supported the students' mistakes and would reflect on how they would like the student to be the one to *identify* (*student*) the mistake and make the *correction* (*student*). Similarly, the ePSTs reflected on their questioning during the identification process and remarked how they would have liked to have done things differently. For instance, they would have liked the students to explain their thinking more, have asked fewer leading questions, been less hands-on, or to have assisted better in the students' problem-solving process. Although all of the participants both identified and corrected student troubles at some point throughout camp, they rarely saw this as a good way of working with mistakes.

***Perseverance.*** Perseverance was the one category that seemed to be very different for all the participants. Overall, the evidence provided by the ePSTs during their interviews suggests that they overlooked and were unaware of the supports for *fostering perseverance* and *praising process*. Moreover, it was almost always mentioned as something the student did, and not something the teacher could support. Contrary to this view, the ePSTs supported and reflected on ways to better support the students' perseverance in problem-solving frequently, and even conceptualized ways of supporting it. Amy and Linda thought of perseverance as remaining positive, and process and strategy focused. Conversely, Becky focused on taking negative situations and turning them positive. Furthermore, Becky and Linda both mentioned perseverance as a way for students to learn from their mistakes.

Lastly, all three participants recognized the benefits of questioning in helping *foster perseverance*. However, all three also recognized or supported *fostering perseverance* by pressing for a focus on strategies, recognizing emotions, creating extensions for the problem, and changing participation format. Additionally, *praising process* was not always reflected on; Linda reflected on three instances, Becky reflected on two, and Amy reflected on one instance.

**Student APS.** All of the ePSTs fostered student APS, or reflected on fostering it, in all seven ways from the analytic framework. The primary ways were through *listening* *carefully to students*, *allowing them to work in their own ways*, and *providing helpful hints and questions*. Often, the ePSTs would answer the students' questions with another question. Both Linda and Becky also focused on allowing the students to *manipulate objects or actively work out the problem*, while Amy was less focused on this aspect. Amy tended to focus her reflections of student APS on providing *encouragement when the students showed progress, effort, or persistence in problem-solving*. However, this encouragement was not typically in the form of praise, but a recognition of the difficulties or struggles the students were having.

The ePSTs also supported *student collaborations and student talk*, but also reflected on how they could have incorporated it more or could have facilitated it better. Similarly, *acknowledging students' perspectives and interests* was not a major reflection point. Amy focused her reflection on using students' interests to drive the conversations, Becky remarked on building on the students' prior knowledge to guide the discussions, and Linda reflected using a mixture of ideas to foster curiosity and to use "questioning to see where they're coming from".

Although supporting student APS in all seven ways, the ePSTs had some difficulty in their supports, primarily *providing hints and questions*. The ePSTs remarked how they thought they were asking too many questions, were leading the discussions too much, wanted to use less questions, wanted to use more open-ended questions, and provide better hints for the students. Similarly, the ePSTs also mentioned that they all wanted to have the *students use more of their own strategies* and to explain their thinking more. Many of these reflections revolved around interactions with students when they were struggling with a mathematical trouble. Thus, it was the in-the-moment need to respond in a satisfying way to a student that caused the ePST to struggle. Sometimes this struggle was due to the language of the question, the model being used, or a confusion presented in the students' strategy.

***Lived Experiences***

Throughout the interviews, the ePSTs remarked on lived experiences that they said affected and informed their teaching. These experiences ranged from the camp and professional development settings to outside academic work experiences had by the ePSTs. However, all of the ePSTs remarked on their current camp experiences and information gained from the PD, talking to their current lead teacher, and working with the students.

Amy mentioned how she would ask Ms. Fray's insight into the way she set up a problem or would handle certain situations. Additionally, Amy noted that Ms. Fray pointed out that the students will naturally get the correct answer if you guide them with questions, so questions should be more open if you want to explore the students' thinking. Amy also remarked how Ms. Fray focused on timing her lessons through a student majority understanding, which seemed more realistic to the traditional school setting but also caused some students to get behind and struggle.

Becky commented on how modeling vocabulary helped the students understanding by recalling how Ms. Fray modeled vocabulary words, their meanings, and kept a list of vocabulary words on the board for the first week of camp. Similarly, she noted that Ms. Fray was really good about probing the students for their reasoning and allowing the students to be the ones to identify the errors or misunderstandings.

Additionally, Becky mentioned on how she used her experiences with the students at camp to help her with other camp interactions. Similarly,

Linda remarked on watching Ms. Berry model vocabulary for the students, create a chart, and discuss the different words used to mean the same things. Linda, like Amy, also commented about conversations they had with the lead teacher. Linda remarked that Ms. Berry pointed out that students would often use a hypothesized answer to guide their thinking and strategy, but this was only beneficial if the hypothesized answer was correct.

Other attributes of the camp, such as the PD, reflections, and governing precepts were also mentioned in the ePSTs reflections as something that helped inform their teaching. For instance, both Linda and Becky recalled a specific PD video regarding the use of multiple representations to help aid in productive struggle. Also, all of the ePSTs mentioned how the camp has helped them learn how to teach, use questioning techniques to gain students' understandings, and the importance of listening to students' ideas and having them provide their justifications. Becky also remarked how the PD seminars showed how questioning to understand students thinking and having the students arrive at a student identification of an error was more beneficial to the students than a teacher identifying the error and saying it was wrong. She also recalled this study's survey and interviews and how the practices and videos were related to her current situation. Thus, the study itself made her more aware of practices and supports being used around her and would bring in the information she had thought about or remembered from previous interviews or the survey.

In addition to their current camp experiences, all the ePSTs reflected on their university course work too. Amy and Becky remarked that their content course is where they first saw the car model, and the course helped enhance their understanding of mathematics. Although Amy and Becky had different content teachers, their instructors used a similar curriculum. Thus, they both mentioned the benefits of seeing questioning techniques and the content teachers' constant presses for the students to justify and explain their thinking. Linda did not mention a specific class but said that her coursework taught her about different learning styles and gave her the tools to help individualize learning to work for each student depending on their needs as a learner. Similarly, each participant commented that as a student themselves, they could not justify their answers. Thinking about K-12 experiences, Linda and Amy said they would go through the process and arrive at an answer but could not say why it was correct. Additionally, Amy added that because of this, she could not look at an incorrect answer and realize that it wasn't true.

Amy and Linda also reflected on their previous years working at the camp. Amy indicated that she was out of her comfort zone during her first year at camp because she wasn't familiar with the mathematics (she was in an advanced topics classroom) and had to follow the students' strategies. She reflected how this was very helpful because in her second year she had practice with following students' strategies, but this time it was more familiar content (same content as the current camp year). Both ePSTs remarked that their previous year at camp introduced them to the importance of vocabulary in students' understanding.

Additionally, Amy reflected on the supports she recalled Ms. Berry utilizing with her students. Amy thought about how Ms. Berry gave helpful hints and demonstrated with a question instead of adding information or directing students, pressed the students to explain their thinking for both correct and incorrect ideas, and had students use multiple methods to check their work. Amy remarked how in understanding the students' work and interests, she could better provide hints and questions to help the students or keep them engaged. Similarly, Linda noted that because it was the same material as the previous year, she knew from the prior year what some of the typical mathematical troubles were and how to help the students. Moreover, Linda remarked that if the students were having difficulty working at their desk, she found it helpful for students to manipulate or model the situation in a larger context, such as walking a problem out on the life-sized number line.

However, not all lived experiences were shared experiences. Amy mentioned that she used her knowledge from camp and the university courses and practiced that in her Pre-K observation setting. The Pre-K observation provided her with more practice and a different environment. Linda recalled her volleyball experience and said that it helped her focus when multiple things were occurring in the classroom and allowed her to pull relevant outside information into the conversations. Lastly, Becky thought about her experiences babysitting and working at a clothing store. She said that working with people or children when they are upset or discouraged helped her when encouraging students while working with their mathematical troubles.

# V. DISCUSION AND CONCLUSION

## Discussion

This case study took place over two weeks at a summer math camp for elementary and middle school students. The participants in this study were elementary preservice teachers who were majoring in interdisciplinary studies and had not yet completed their elementary classroom observations yet. The goal of this study was to see the ePSTs' conceptions of the mathematical practices favored by the setting and how the ePSTs' supported them. Thanheiser and colleagues (2013) note that for ePSTs to develop mathematical content knowledge for teaching, mathematics teacher educator must first understand two things:

> (a) the conceptions PSTs bring to teacher education (Bransford, Brown, & Cocking, 1999), since "the key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they do know" (Conference Board of the Mathematical Sciences, 2012, p. 17), and (b) how those conceptions can be further developed. (p. 5)

Thus, I considered two research questions to guide this study.

1. What are the elementary preservice teachers' conceptions and rationales of
    a. Student *autonomous problem-solving* (APS), and
    b. the mathematical practices related to student autonomous problems-solving that are emphasized by the Governing Precepts of math camp?
        i. Fostering student *perseverance*
        ii. Pressing for student *justifications*
        iii. Supporting students' development of *mathematical language*

iv. Allowing students to work with *mathematical troubles*

v. Supporting students use of *visual representations*

2. What do the elementary preservice teachers notice about their fostering of student APS, regarding their supports of these five practices, during focused stimulated-recall reflection interviews?

In the previous chapter, I highlighted the findings of this study for the three ePSTs, Amy, Linda, and Becky. I described each case according to the videos they reflected on that day, with a daily summary to conclude the day's reflection. Before the ePST's reflections, I reported their conceptions regarding the five practices and student APS. I also highlighted how their conceptions of APS were connected to their conceptions of the five practices, and how their conceptions of the five practices support APS in other ways that were not articulated for APS. Additionally, I reported which lived experiences the ePSTs recalled when reflecting on their supports for that day. Lastly, after all the case findings were reported, I presented a cross-case analysis of commonalities and differences among the three participants.

In chapter 2, I described both my theoretical and conceptual framing for this study and relevant literature that led to the analytic framework presented in Chapter 3. I used the analytic framework to analyze my data and answer my first two research questions. I was also able to relate my participants lived experiences, the focus of my third research question, back to the topics or categories used in my analytic framework. Surveys, video recordings, survey interviews, stimulated-recall interviews, and clinical interviews constituted my primary data sources. Surveys, survey interviews, and clinical interviews served as the primary sources to answer research question one. Video recordings and stimulated-recall interviews were used to answer question two.

In this chapter, I conclude with (1) a discussion regarding the ePSTs' conceptions, supports, and reflections regarding each of the five mathematical practices and student APS related to literature, (2) limitations of the current study, (3) concluding remarks and implications from the study, and (4) ideas for future research in this area.

***Conceptions, Supports, and Reflections***

This study looked at the five mathematical practices of justification, visual representations, mathematical language, perseverance, and mathematical troubles, as similar in importance to student autonomous problem-solving (APS). It did not deliberately distinguish the connection from the practices to APS. Instead, this study looked at these six topics equally and as individual components, with the caveat that the ePSTs could connect them to the other topics as they saw fit. This is both a strength and a limitation of this study, which I discuss later.

The participants in this study were able to identify mathematical supports that connected the practices to student APS, which agreed with what literature tells us. In the following section, I discuss how my findings of the ePSTs' conceptions, reflections, and supports of these practices and student APS relate to the relevant literature.

**Perseverance.** The ePSTs conceptualized and supported perseverance in ways that aligned with Lewis and Özgün-Koca (2016) *fostering perseverance.* However, they also supported students' perseverance in ways that matched Dweck's (2006) ideas of praise and encouragement, which matched their conceptions that perseverance looks like students having a *growth mindset*. This finding turned out to be the opposite of what I had hypothesized since the preservice teachers in this study related perseverance to the ideas of *growth mindset* (Dweck, 2006) and *productive struggle* (Hiebert & Grouws, 2007; Warshauer, 2015). The ePSTs conceptions about perseverance were not surprising since two of the ePSTs in the study had already participated in the camp and professional development setting, which includes models and discussions around both of these ideas. Becky, who was new to the camp experience, had taken a content course that contained work on *productive struggle*.

Moreover, this suggests that introduction to and involvement in such work altered the ePSTs beliefs to include components of perseverance. This finding agrees with that of Russo and associates, who found that recent emphasis on growth mindset helped shift "teachers' willingness to embrace struggle and view it as a necessary aspect of learning mathematics" (Russo et al., 2020, p. 8). Additionally, the ePSTs views on perseverance fit mostly into Russo and colleagues (2020) ideas of *conditionally positive responses*, in that they held positive beliefs, but mentioned teacher involvement in the struggle. Linda's conception was the only one that fell into the perspective of *a positive belief*; however, in her reflections, she would often structure struggle with questions.

A possible explanation for why the ePSTs viewed supporting perseverance in terms of *fostering perseverance* instead of both *fostering perseverance* and *praising productive process* could be influenced by how the ePSTs perceived their role in students' perseverance. Evidence suggests that the ePSTs saw perseverance as something that the students were responsible for, and the ePSTs often had difficulty noticing their supports of perseverance as related to the practice. This was especially evident for Amy, who remarked about students' confidence and willingness to explain ideas from previous problems as attributed to past teachers. She said, "[t]hat wasn't necessarily anything I put in them, but whoever they had in the past, teachers and stuff, they've given them that sense of confidence" (Amy, Day 4). This finding agreed with what Warshauer and colleagues (2019) found in their preservice teachers' understandings of *productive struggle*.

Furthermore, the statement made by Amy sheds light on what could be seen as a potential contradiction in ePSTs views of perseverance. Most evident in Amy's comments, the ePSTs viewed perseverance as connected to confidence. This connection seemed contradictory on the surface since perseverance was deeply embedded in the notion of struggle. However, the ePSTs, like existing literature, see perseverance as a way to strengthen confidence. Dweck (2000) noted that students with a growth mindset associated with their mathematical ability are more likely to have greater confidence that they will succeed. Dweck's ideas also explained Amy's connections of perseverance to both justification and visual representations.

The notion of explaining, justifying, and exploring a completed problem as a way to extend or reflect on a problem can cause an emotional trigger on a students' confidence if they attribute prompts for an explanation as synonymous with negative assessments (Steuer et al., 2013). Similarly, visual representations could be used to explore or justify a problem further, especially if there was a mathematical trouble. Thus, the connection between mathematical troubles and perseverance by all the participants was not surprising, given how both the ePSTs and literature defined perseverance.

**Mathematical Trouble.** In general, the ePSTs viewed mathematical troubles as beneficial for students, and that working with these troubles allowed students to create better understandings. ePSTs' conceptions matched ideas of student identification and correction of troubles, except for the occasional teacher assistance in student correction. This outlook on mathematical troubles aligned highly with research that rationalizes working with and addressing mistakes as extremely beneficial for students' learning (Anghileri, 2006; Borasi, 1996; Brodie, 2011, 2014). However, as seen in research, the ePSTs had difficulty supporting the students in exploring their mathematical troubles. The ePSTs would often guide the students by funneling their questions or make the correction or repair themselves. This finding supported Rach, Ufer, and Heinze's (2013) idea that mistakes are often teacher-directed.

Notably, the ePSTs were aware of their struggles, and the misalignment with their beliefs. Often, the ePSTs would reflect on ways to be less "hands-on" or using questions that were more open-ended and less directive. One of the most apparent reflections on this was when Linda recognized that she had pointed out and corrected the error for the student. She reflected that was something she would have liked to change saying, "She's [the student] smart enough, she'll figure it out that that wasn't the right answer" and "I would've like to try and see if she could have figured that out by herself" (Linda, Day 5). The idea of ePSTs' conceptions being different from their enactments was not very shocking since research has told us that teachers' (preservice, inservice, novice, or experienced) conceptions may vary from their practice (Chazan & Ball, 1999; Philipp, 2007; Thompson 1984). Additionally, this literature also tells us that the inconsistencies are typically due to other conceptions serving a more prominent role, such as time restrictions, which was the most evident cause for the ePSTs in this study. Another consideration is the ePSTs comfortability at the moment and the possibility that they reverted to ways that they had experienced learning, which is a difficult habit to break (Cohen, 1990; Pajares, 1992; Stohlmann et al., 2014).

The ePSTs also found mathematical troubles involving fewer familiar struggles or concepts to be the most challenging. Becky noted that she was new to the models, and Linda said that subtraction using the chip model was a “whole other beast” that confused the students and her when she first learned it. Amy stated that the chip model was something she still wasn’t entirely comfortable with and was working towards a better understanding. To this effect, Amy said, “I need to work on the Chip Model myself and just really understanding” and “having a better understanding of the chips and Chip Model to guide those questions better and have better wording” (Amy, Day 6).

This realization was beneficial in more than one way. Analysis of children’s mathematical thinking has been shown to help increase PSTs content knowledge (Philipp et al., 2007), but also the analysis and identification of errors are considered part of *specialized content knowledge* (Thanheiser et al., 2013). One might also conjecture that the ePSTs’ added awareness of their inadequate cognition would likely drive their desire for the acquisition of this knowledge. This conjecture has merit, based on the fact that PSTs are more likely to increase and adapt their noticing of salient matters when becoming aware of their existence (Star & Strickland, 2007). Thus, ePSTs may become more aware of these topics while planning and pay special attention to them. However, I also realize the less optimistic approach to this awareness would be that the avoidance or direct instruction of said topics could be possible.

Overall, the realization of inadequate knowledge needed for teaching also agrees with Thanheiser and associates (2013) remark that

> [t]he majority of the PSTs do not possess adequate conceptual understanding of numbers and operations that they would need for their teaching (Li & Kulm, 2008; Newton, 2008; Thanheiser, 2009, 2010). They have difficulty in carrying out teacher-like tasks, such as modeling operations with multiple representations (Luo, 2009; Rizvi & Lawson, 2007), interpreting students' alternative algorithms (Li & Kulm, 2008; Son & Crespo, 2009), and identifying the roots of student errors (Tirosh, 2000). (Thanheiser et al., 2013, p. 16)

This remark also highlights a curious but not surprising finding; the ePSTs vaguely relate visual representations with the idea of mathematical trouble. Amy mentioned the concept of using multiple representations as a prompt for students to check their work for right/wrong answers in her conceptions of mathematical troubles. In contrast, the others did not mention this, but they supported the students in this same manner.

**Visual Representations.** The ePSTs in the study both conceptualized and supported the mathematical practice of using visual representations in their presses for employing single and multiple representations, but also their drive to connect the representation(s) to the mathematical idea. However, evidence suggests that the ePSTs had not yet developed plans or supports to help connect or relate multiple representations together. Evidence from this study indicates that the ePSTs saw visual representations as an additional component to be learned or another part of the problem. Not only did the ePSTs want the students to get the correct answer, but to be able to use both the car model and the chip model to demonstrate the solution. The ePSTs would often use the number line as a way for the students to check their work with the chip model to make sure the students achieved the correct answer.

Similarly, the ePSTs conceptualized the idea of using multiple representations as beneficial because all students work differently and can choose which way works best for them. This finding partially agrees with Jao (2013), in that it allows students access to the mathematics to form their own understanding. However, this finding also revealed that ePSTs were viewing the representations as completed objects, and not a composition of smaller parts. I argue that this is why the ePSTs did not relate the models to each other or had found the models to be unrelated. All the participants used the completed models as a way to check answers or the final product.

These ideas align with the *individual cognition* roles of representations from Stylianou (2010), which endorsed students' use of representations to understand information, reduce cognitive load, facilitate exploration of the given information, and detect wrong approaches. However, in viewing representations as part of the problem or complete products, the role of *social practice* became limited to communication. Thus, if a process-step was the source of trouble or when trying to expand upon an existing idea, multiple representations were not considered. This finding agrees with Stylianou's (2010) finding that when conducting a discussion, "teachers appeared to limit their approach to a 'sharing of solutions'" (p. 339), and the representations served as a visual for the solution and highlighted different approaches students could have taken. Thus, the middle school teachers in Stylianou's study held "relatively narrow conceptions of representations," which, as a result, meant that representations played a "peripheral role in their instruction" (p. 339). Moreover, like the ePSTs in my study and Stylianou's findings,

> often students are instructed to represent a given task in multiple ways, but are only instructed how to meaningfully manipulate the standard representational

> forms (usually a symbolic representation) and are encouraged to use these standard representations throughout their solutions. There is little discussion regarding a fluid use of representations throughout the problem-solving process and the advantages of the ability to translate (as discussed by Janvier, 1987) among these different representations. (Stylianou, 2010, p. 339)

Thus, it is not surprising that the ePSTs did not press the students to connect across multiple representations when they were present, but only used them as a fact check. On a positive note, this idea of fact-checking and assessment correspond to views Stylianou (2010) found in the middle school teachers who held a "more flexible view of representation." The middle school teachers, like my ePSTs, used representations as a way to understand their students' thinking. This also aligned with the connections the ePSTs made to justifications when forming their conceptions of visual representations.

**Justification.** Justifications were seen as a critical practice by all of the ePSTs and were considered a valuable way to understand students' thinking. Moreover, the ePSTs mentioned and supported ideas of *explaining how* and *justifying why* when trying to understand the students' thinking. The ePSTs' conception of *justification* as an important way to understand students' thinking coincides with Staples, Bartlo, and Thanheiser's (2012) remark that teachers used student justifications to aid in formative assessments of students' understanding and knowledge. Possibly due to increased awareness through their university coursework and camp professional development, the ePSTs were especially attuned to requiring their students to reason, explain, and justify their thinking.

However, most of the justifications prompted by the ePSTs were generic "why" justifications or to explain how a procedure worked. Prompts for deeper meaning often received lower-level responses that did not use definitions or mathematical properties; alternatively, they obtained restated facts, statements, or rules. Therefore, it was unclear if the ePSTs were aware of the varying levels that can happen within justification. This is not unexpected, considering Melhuish and associates' (2020) findings that elementary teachers typically did not notice justification at a deeper level and would focus on surface-level details. They also found that the elementary teachers' definitions of justifying differed considerably from the research field. Thus, the finding that the ePSTs did not consider differing justification levels is consistent with Melhuish and collaborators' findings. Moreover, it corresponds with Lo, Grant, and Flowers's (2004) finding that ePSTs justifications were less formal and often more explanatory, descriptive, and included things such as diagrams and storylines.

Although all the ePSTs said that the purpose of justifications was to understand students' thinking better. They also related it to the ability to help students better when a mistake occurred, use visual representations to help illustrate justifications, and use formal mathematical language to articulate the students' justifications. Meaning, the ePSTs related the mathematical practice of justification to all the other practices except for perseverance. Amy was the only participant to form a clear but vague connection between justification and perseverance. As discussed above, Amy saw perseverance as related to confidence in justifying both correct and incorrect answers. Although it wasn't explicit, there is evidence to suggest that the other ePSTs viewed perseverance as having a transitive connection to justification through mathematical troubles. In other words, Linda and Becky's connection between justifications and perseverance was contingent on the inclusion of mathematical troubles. However, both of these ePSTs mentioned ideas similar to Amy in their reflections, wanting to make it a point to ask students to justify both their incorrect and correct answers so as not to create a negative association between justifying and having an incorrect answer.

Considering the view of justification as intertwined with the communication of ideas, all of the ePSTs connected the use of mathematical language to justification. It was a common remark that mathematical language was a way to articulate student ideas, with the caveat that Becky also thought that the misuse of mathematical language students' justifications could lead to mathematical troubles.

**Mathematical Language.** Except for Becky's link to mathematical troubles, all of the ePSTs saw mathematical language as a way to communicate ideas. They viewed language to be an important component in the classroom, especially *formal language*. However, the ePSTs were aware that it takes time for students to develop *formal language* and would use, or allow the students to use, non-formal language such as *bridging language* and *contextual language* until they became more familiar with the formal terms. However, outside of mentioning a constant need for the teachers to demonstrate the use and meanings of *formal language,* and use visuals such as word walls and descriptive charts, the ePSTs did not have other ideas of how to support students in their understanding of formal terminology. This is not surprising considering how "teachers have reported limited knowledge of instructional strategies for teaching mathematics vocabulary (Institute of Education Sciences, 2014)" (Turner et al., 2019, p. 2).

This finding shows the need for ePSTs induction into more educational work on the mathematical practice of language, which has found to be lacking in our field (Riccomini et al., 2015; Turner et al., 2019). Although the ePSTs conceptions of mathematical language were consistent with the three purposes for teaching vocabulary (Riccomini et al., 2015), their views seemed to most align with a *lexicon perspective* instead of using a compilation from the various perspectives regarding vocabulary which is recommended by Turner and associates (2019). Additionally, the supports they noted to help aid students in their acquisition of formal terminology are helpful but limited. The supports pointed out in this study are only a few and part of the recommended instruction by Marzano (2004). To form deep meanings of *formal language*, students must consistently work with vocabulary in a variety of ways.

***Summary of Connections among Mathematical Practices***

As previously stated above and in the findings, connections emerged among the ePSTs conceptions, which were based on their surveys, survey interviews, and clinical interviews. Here I discuss the conceptualized connects of the mathematics practices from each ePST's as a map, instead of looking at the individual mathematical practice connections as I had done above. Additionally, I discuss how the ePSTs connections among the practices align with literature. The ePSTs all formed conceptual connections between justification and visual representations, mathematical language, and mathematical troubles. They also formed a conceptual connection between perseverance and mathematical troubles.

These connections aligned with the ePSTs views that justifications can be used to understand students' thinking, which, as we saw above, agrees with literature (Staples, Bartlo, & Thanheiser, 2012). Additionally, representations were used as a way to aid in students' understanding (Jao, 2013) and served as a visual for the students' strategies and reasoning (Stylianou, 2010). Different types of mathematical language were used in students' justifications, which allowed the students to frequently use and build their language (Riccomini et al., 2015). This could be seen in the ePSTs conceptions regarding mathematical language, in that they saw language as a way for students to communicate their ideas. Similarly, ePSTs thought that students would benefit from finding, exploring, and correcting their own mistakes, which literature has proven to be valid (Anghileri, 2006; Borasi, 1996; Brodie, 2011, 2014). However, the ePSTs also thought that students would need to view the process "as a journey", related it to productive struggle, and a growth mindset. Thus, the ePSTs connected the ideas of perseverance to mathematical troubles by wanting their students to hold an *adaptive reaction pattern* (Steuer et al., 2013). These connections are illustrated in figure 14.

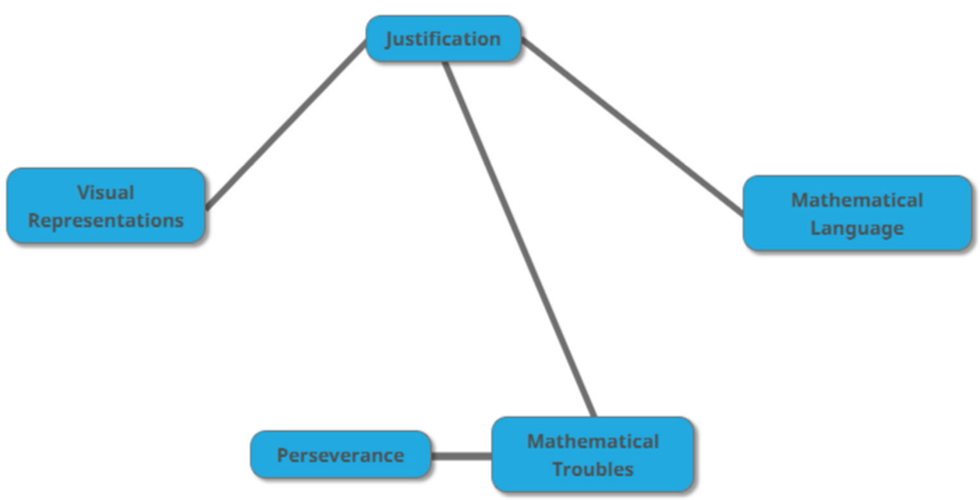


*Figure 14.* Similar ePSTs' conceptual connections

Thus, as seen in figure 14, the ePSTs primary connection between the practices was through justification. This finding was an interesting surprise because it suggested that the ePSTs viewed all the practices as being connected in some way.

Additionally, each ePST formed other connections among the practices. Amy formed two triangle structures between justifications, perseverance, and visual representations and visual representations, mathematical troubles, and perseverance. This can be seen in figure 15 as a red triangle between justification, visual representations, and perseverance and the blue triangle between visual representations perseverance, and mathematical troubles, with the dotted lines representing Amy's conceptions and the solid lines of any color representing all three of the ePSTs' conceptions.

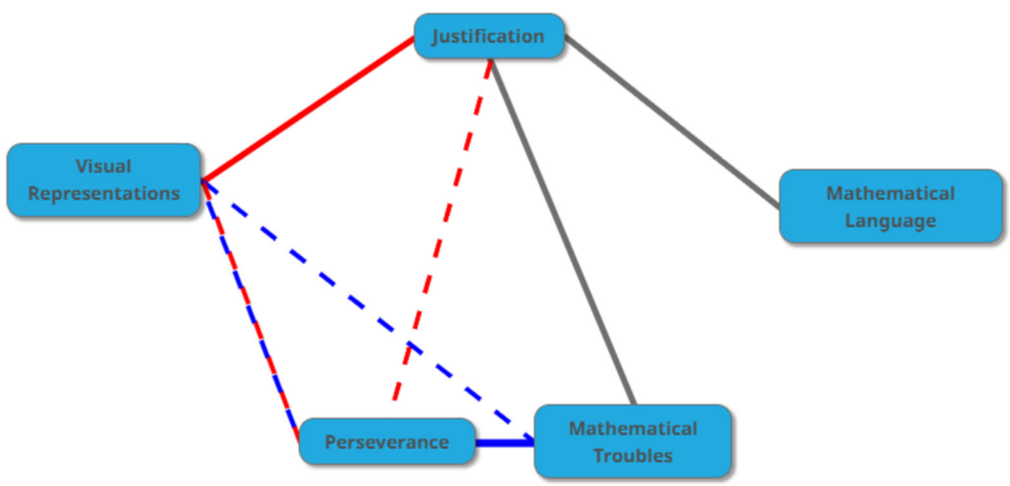


*Figure 15.* Amy's additional conceptual connections

However, although Becky and Linda did not articulate conceptions that formed these connections, both ePSTs supported the mathematical practices in ways that align with how Amy made these connections. As noted above, Amy connected justifications to perseverance through supporting students' confidence in exploring previously solved problems, which was an idea vocalized in both Becky and Linda's stimulated-recall reflections regarding justifications. This highlights the ePSTs views that correspond to ideas of *growth mindset* (Dweck, 2006). Moreover, this exploration was often done using representations; hence, the red triangle. This red triangle agrees with Stylianou's (2010) finding that teachers would treat representations as visuals of different approaches, which, if students were paying attention, could foster perseverance by exploring differing strategies (Lewis & Özgün-Koca, 2016).

Similarly, the blue triangle was established through Amy's vocalization of using multiple visual representations to aid in students' mathematical troubles. Again, although not vocalizing this in their conceptions, Becky and Linda mentioned similar thoughts in their stimulated-recall reflections regarding mathematical troubles and multiple visual representations. This blue triangle illustrates the ePSTs connections to the literature regarding teachers' use of multiple representations as a way to assess their students' thinking (Stylianou, 2010), as the ePSTs would often use multiple representations as a way for the students to assess the correctness of their answer or create opportunities for *productive struggle* (Warshauer, 2015).

Linda vocalized conceptions that connected the mathematical practices of visual representations and mathematical language. Neither Amy nor Becky vocalized this in their conceptions but did support the students in ways that establish this connection. Figure 16 illustrates the connection.

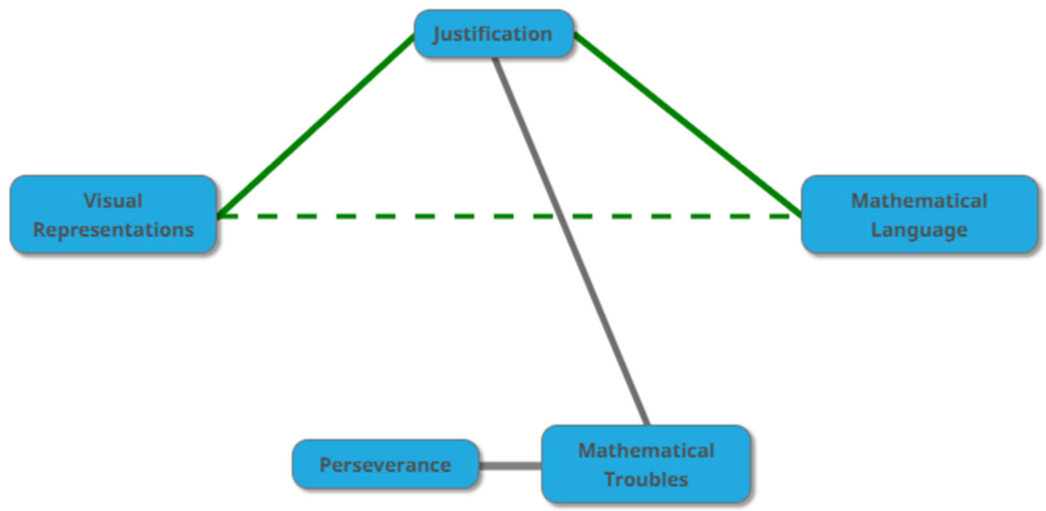


*Figure 16.* Linda’s additional conceptual connection

It is worth noting that a triangle was formed between visual representations, justifications, and mathematical language. Linda's conceptualization of visual representations included “coming to a conclusion/answer and explaining their reasoning. This can include using precise language to differentiate different strategies to reach an answer.” Thus, Linda thought of representations as a way to reason about a strategy and articulate strategies and representations using precise language. She also mentioned the idea of language as differentiating between students’ strategies. Similarly, Amy reflected on her use of language and use of a number line when working with students and their justifications regarding a problem involving the timing of two events occurring in years before the common era (BC). Becky had a similar instance to Amy, but it involved a number line and surface-level. Both Amy and Becky stated reflective remarks in their stimulated-recall interviews for these instances that would support the connection of these practices. These views highlight some of the supports that were used or were the source of trouble when supporting students' vocabulary development. However, these supports are examples of recommendations made by Marzano (2004).

Becky conceptualized the connection between mathematical language and mathematical troubles, noting that misuse of language could lead to troubles. This, again, created a triangular relation involving justification as the misuse of language would occur in a justification. Figure 17 shows the connection below.

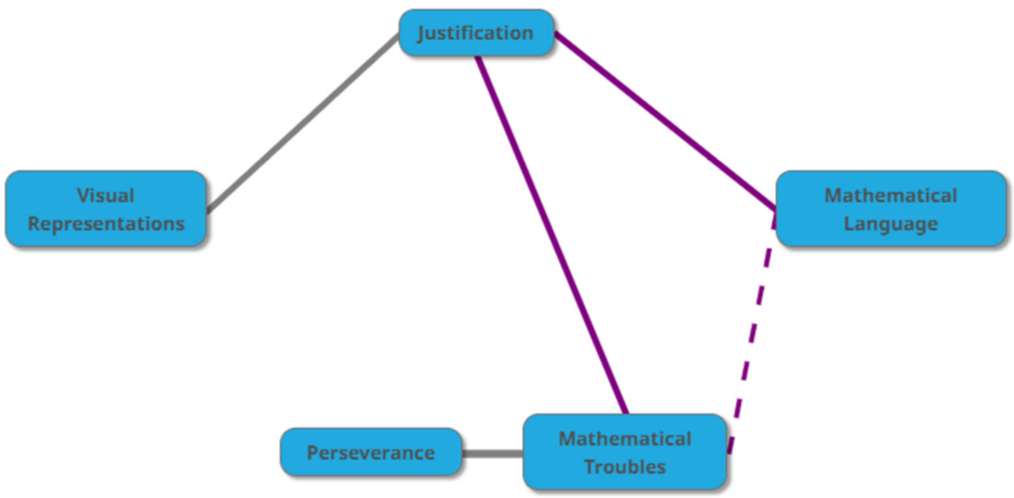


*Figure 17.* Becky's additional conceptual connection

Although both Amy and Linda did not vocalize this connection in their conceptions, they both reflected on instances where language was the source of trouble during a student's justification. Thus, this connection speaks to the ePSTs understanding that agrees with literature; that is, language is an important component of understanding (Riccomini et al., 2015; Turner et al., 2019).

Lastly, although all the ePSTs had slightly different views when it came to supporting mathematical language, none of the ePSTs made a connection between perseverance and mathematical language. This is of particular interest considering at least one ePST typically made a connection between practices, but none of the ePSTs made a connection here. One possible explanation for this is that both of these practices seemed to be difficult for the ePSTs to support and conceptualize. Additionally, in looking at the ePSTs conceptualization of perseverance and mathematical language, the two practices seem disconnected. Figure 18 illustrates the connections formed by all of the ePSTs and the ones formed on an individual level.

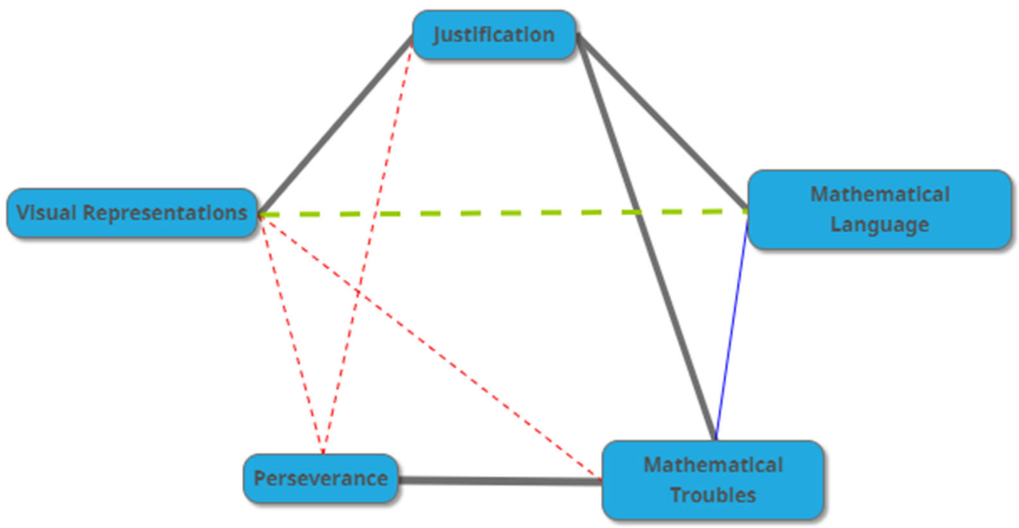


*Figure 18.* ePSTs' conceptual connections

However, the ePSTs did make moves to support these practices simultaneously, such as changing the participation format during a discussion regarding terminology and notation, using different strategies, tuning in to students' emotions, and praising students for their use of *formal language*. Thus, a connection could be formed from these supports between the two practices through the lens of encouraging, motivating, and exploring the use of language development. This finding, and all the findings in this summary section, are part of a more extensive discussion below, which highlights ePSTs potential to have made the connections between practice but an inability to articulate them.

**Student APS.** The ePSTs were able to support student autonomous problem-solving in all seven ways related to this study, which stem from the eight ways to foster student autonomy recommended by Bozack and colleagues (2008). Thus, in supporting the mathematical practices of this study, the ePSTs were able to support and reflect on APS. Moreover, the ePSTs were able to conceptualize how student APS was related to the five practices of this study or how they viewed the five practices as related to ways of supporting APS. In agreement with earlier findings regarding the mathematical practices, the evidence regarding APS revealed that the ePST supported APS in ways that they had not conceptualized. Thus, although these moves were purposeful, they were not done with the sole intent to foster APS.

Moreover, the ePSTs reflected on the difficulties they encountered when supporting the students. As seen in the discussion regarding mathematical troubles, ePSTs had difficulty in providing helpful hints and appropriate questions when working with a mathematical trouble they were not comfortable with. This situation typically led to a more teacher-driven approach which took away from the students' autonomy by not allowing them to work and manipulate in their own way. This finding seems to follow logically, considering Rach, Ufer, and Heinze's (2013) research regarding mistakes often being teacher-driven when explored as a whole class or explored with students in such a way that made in near impossible for students to respond incorrectly. This teacher-driven idea can also be extended to removing opportunities for student autonomy using visual representations. As seen in research by Stylianou (2010), the ePSTs often supported the use of specific common representations. Although the students had opportunities to engage actively instead of passively listening, they did not have a choice in choosing a manipulation that fit their own strategy.

Similar to the ePSTs difficulty with hints and questions, they often reflected that they would have liked to have had facilitated student discussions more. The ePSTs were sometimes unsure of this move, because they were scared that the students would “shut down,” become answer-driven, or would lose interest if discussing strategies that were different from their own. Thus, this shows that ePSTs are aware of the benefits of student-to-student discourse, and align with current reform efforts, but acknowledge that facilitation of such interactions can be scary or complicated. These fears are not new to teachers, novice or experienced, as several studies focus on facilitating student discourse (Fennema & Nelson, 1997; Sherin, 2002; Stein, Engle, Smith, & Hughes,2008).

Additionally, the ePTSs supported the students in ways that were unexpected in that they encouraged the students’ efforts, so the students didn’t “shut down” and would continue to work. I relate this to Bozack and associates’ ideas of offering encouragement when students showed effort. However, as this information was unknown to the ePSTs, they more than likely attributed this to the more familiar ideas of *growth mindset*. This move worked as an attempt to try and change a *fixed mindset* to a *growth mindset* and motivated the student not to give up (Dweck, 2006). Moreover, the ePSTs would praise the students for their work and successful progress, which is an important part of Bozack’s framework but was often disregarded as a teaching move by the ePSTs. They saw this move as a means to offer emotional support for the students. Therefore, although research tells us that attending to emotions is an important teaching move (Bozack et al., 2008; Lewis & Özgün-Koca, 2016), the ePSTs saw this as something unrelated.

### *Lived Experiences*

We know that preservice teachers enter their teaching programs with preconceived conceptions from their own experiences (Daniel, 2020; Stohlmann, Cramer, Moore, Maiorca, 2014; Valentine and Bolyard, 2019). Although conceptions are known to change, they are also known to be influential towards a teacher's action (Ambrose et al., 2004; Stohlmann et al., 2014; Thompson, 1984, 1992). Thus, it is worth noting some of the lived experiences that the ePSTs in this study remarked as being influential in their practice.

All of the ePSTs remarked about their camp experiences as something that was influential in their decision for certain supports. They referenced prior and current camp teachers' moves and reflective conversations, as well as their experiences with the students. This aligns with findings from Valentine and Bolyard (2019), who found that ePSTs were influenced by family, teachers, peers, and learning communities. Daniel (2020) noted that "[c]ertain judgments [by preservice teachers] appeared to be evidence-based ones," where the evidence was based on "the knowledge they held about their students" (p. 16).

Additionally, the ePSTs mentioned the lack of justification and conceptual understanding that was required of them as K-12 learners but had learned how these ideas were important. Thus, the ePSTs had entered the program with this knowledge (Stohlmann et al., 2014) and stated how they did not want that experience for their students. Therefore, there is evidence to suggest that the ePSTs of this study had created a negative relation (Valentine & Bolyard, 2019) with their K-12 learning and thus turned away from teacher moves like they experienced, which did not require justification.

Moreover, evidence suggested a positive relation (Valentine & Bolyard, 2019) formed about justification and understanding between the ePSTs and their experiences from their university courses.

Similarly, experiences from the camp professional development (PD) were called upon in the ePSTs reflections as being influential. These PD influences were primarily in the form of research literature readings, student artifacts, and videos. The recollection of these PD items relates to the fields call to include more research-based knowledge and genuine student thinking in courses (Philipp, 2007; Philipp, Thanheiser, & Clement, 2002). The ePSTs also mentioned their reflection time with the lead teachers, and Becky recalled conversations involved in participation for this study. These outcomes suggest that the ePSTs were not only given time to reflect but began reflecting on events themselves, which is a useful skill towards improving their practice (AMTE, 2017).

Lastly, some lived experience was not mathematical, role model, or school-related, but were experiences that influenced the ePSTs supports in other ways. Linda recalled her volleyball experience. She noted that her volleyball training helped her deal with the complexities of the classroom and focus her attention on the conversation in front of her. Furthermore, she remarked that it allowed her to include peripheral distractions as helpful additions to a conversation. This description coincides with the findings of professional volleyball players' ability to modulate their spatial and attentional focus. Anzeneder and Bosel (1998) found that the "ability of [high-level professional] volleyball players to perform attentional zooming operations is presumably attributable to the assiduous nature of the sport" (Anzeneder & Bosel, 1998, p. 264). Similarly, they found that the volleyball players were able to focus their attention more efficiently within and around the cued area and were still able to attend to unexpected cues outside of their attention focus. Although Linda was not a professional volleyball player, she had spent a significant amount of time practicing the sport and the skills needed to be a successful player.

Becky remarked on her experiences babysitting and working at a retail clothing store. She attributed these experiences as influential on how she thought about working with emotional situations. Although I was not able to find any literature that directly linked babysitting to teaching supports, Philipp's (2008) *Circles of Caring* highlights ePSTs first concern as a child's, happiness and comfortability. Becky related the emotions she dealt with in her experiences to the children's emotional needs during the camp.

**Limitations**

These findings are not intended as representative of the general population of ePSTs' knowledge. This study observed and studied three ePSTs over the course of two weeks, and these two weeks provided a snapshot of the ePSTs' conceptions, supports, and reflections over that timeframe. Additionally, these ePSTs were self-selecting in the nature of wanting to work at a summer math camp, thus holding a positive view towards the benefits an early approximated field-experience can have for their teaching. Due to the nature of the ePSTs coursework and their involvement in the camp and professional development setting, these participants were exposed to certain values and beliefs regarding *productive struggle*, *growth mindset*, and *doing mathematics*.

This exposure and camp curriculum also created some limitations regarding the use of certain models. The chip model and car model (number line use) were both built into the camp curriculum to be taught and used to understand operations on integers. Thus, *doing mathematics* was also seen as modeling the problems with these representations, and *productive struggle* also included the use of these representations to facilitate understanding. Lastly, the camp content for my study focused mainly on operations on integers but also included an introduction to variables, expressions, and equations. Therefore, the findings in this study may not extend to other content areas.

Additionally, the design of this study created a few limitations by its nature. This study looked at how the ePSTs conceptualized and spontaneously supported and connected the practices to themselves and student APS. I did not prompt the ePSTs to make connections between the five practices and the five practices to APS; all connections made were spontaneously formed by the ePST. Thus, if prompted, the ePSTs may have vocalized more connections between the topics. Any connections made were out of their own definitions, conceptions, reflections, and noticings (i.e., connections were based on the ePSTs vocalizations during the survey, survey interview, and clinical interview, which were focused on the individual topics). Additionally, the ePSTs may have had more nuanced beliefs or supports that were not made visible because of the wide spectrum of the study.

Moreover, this study focused on small groups or individual interactions that typically took place after a problem or set of problems had been assigned to the students by the lead teacher for the class. Consequently, the ePSTs did not necessarily have any input in the task design, setup, or implementation. These features are especially important in fostering student autonomy and perseverance; thus, by the nature of this focus, I limited the possible supports. Because of this limitation, student questions were often problem-specific or did not happen very often, which led to the combination of hints and responses as a way to support APS. Additionally, this also created limited opportunities for the ePSTs to support APS by acknowledging the students' perspective, which was mostly done by acknowledging students' interests and prior knowledge.

**Conclusions and Implications**

The ePSTs were aware of most supports they made towards the mathematical practices, except in the cases of perseverance and mathematical language. The ePSTs supported the students with praise and emotional responses but did not view these moves as ways to support perseverance. Additionally, the ePSTs did, on occasion, change the participation format to include other students, these changes were often a natural response to students who were either already working together or were sitting directly next to each other and had similar difficulties. However, the ePSTs reflected that they would have liked to have created instances of student talk. The ePSTs in this study also tended not to consider reflective or metacognitive questions to foster perseverance.

The ePSTs would often say they wanted the students to have a growth mindset, which was something that came over time. They related their teaching moves of asking students to narrate or make their strategies explicit (Lewis & Özgün-Koca, 2016) by questioning the students correct or incorrect answers; however, this teaching move seemed to serve more the purpose of having students become confident in justifying their answers than perseverance, which seemed to be an afterthought or an accompanying outcome. Overall, the ePSTs did not view perseverance as something supported by the teacher, rather as something internal to the student. Thus, as consistent with the literature recommendations mentioned here (Lewis & Özgün-Koca, 2016; Warshauer et al., 2019), university coursework should also include an awareness of perseverance and ways of fostering it.

Similar difficulties occurred with mathematical language. The ePSTs were a bit more aware of the language used inside the classroom, as a focus towards formal language was a goal of MEC but were not as confident in the ways to support the students' development of it. The ePSTs reflected on ideas such as vocabulary lists and using the students' own language during their development, which aligns with parts of Marzano's (2004) ways of effective vocabulary instruction and Riccomini and colleagues (2015) explicit vocabulary instruction. "Educators recognize that children may naturally learn vocabulary through incidental or embedded learning experiences; however, for many students, these types of mathematics learning encounters are not sufficient" (Riccomini et al., 2015, p. 241). Moreover, the views held by the ePSTs were primarily centered around students learning the mathematical language, which most aligns with a *lexicon perspective*. The ePSTs views were not as focused on the multiple meanings a word could have or how the mathematical words are situated within everyday life, meaning that their views were not as inclusive of the *register* and *situated-sociocultural perspectives*. These views appear to parallel with the findings from Turner and colleagues (2019), who found this to be true of early career teachers. Thus, in agreeance with Turner and colleagues (2019), teacher education (and future teacher education) should increase attention towards the *register* and *situated-sociocultural perspectives*, since views inclusive of all three perspectives would be optimal.

An interesting, yet concerning finding, is that the mathematical practice of justification was the central focus for the ePSTs conceptualizations, meaning that the ePSTs related all conceptions of the mathematical practices to justification in some form. However, their conceptualization suggested that they did not distinguish between using procedures, facts, definitions, properties, or other known mathematical ideas when reasoning *how* something worked and justifying by using definitions, properties, or other mathematical ideas to reason *why* ideas or methods are effective (Melhuish et al., 2020). Although this is not surprising given what literature about justification has told us (Lo et al., 2004; Melhuish et al., 2020), it is concerning that such a highly foundational practice for the ePSTs has such a broad, nonstandard conception, often including both the *how* and the *why* as forms of justifying why.

Moreover, this could cause difficulty for the ePSTs when adhering to ambitious teaching practices such as selecting and sequencing a variety of reasoning and methods of proof for a class discussion, developing and evaluating mathematical arguments or conjectures, or even expressing to students what constitutes a valid and rigorous conclusion (NCTM, 2000). The ePSTs provided evidence to suggest that they understood reasoning to be a powerful mathematical practice but appeared to falter at what evidence was needed to constitute a full justification, which plays a key role in "deepening students' mathematical activity and mathematical understanding" (Melhuish et al., 2020, p. 64; NCTM, 2000). Therefore, mathematics teacher educators must focus on creating opportunities for ePSTs with explicit and extensive attention toward mathematical content and reasoning forms. This requires more than just providing a definition (Melhuish et al., 2020), but should include examples that clearly illustrate the definition, opportunities for the ePSTs to create their own justifications (Lo et al., 2004), and consequently see teaching moves to elicit justifications modeled.

Furthermore, visual representations were noted as important in students' understandings but were not utilized in more than a superficial or evaluative manner. The representations were thought of as a new process and disjoint from one another, as clearly seen in the case of Becky, and therefore were not used to help repair intermittent confusions or impasses. Additionally, the ePSTs would often suggest the use of a particular representation, which not only limited the students' autonomy in their choice of manipulative but confined them to a prescribed strategy.

This also leads to implications for working with mathematical troubles. The ePSTs had difficulty in supporting students to identify and correct their own mathematical troubles (errors). The ePSTs would often use multiple visual representations in the hopes that the students would realize their mistake, but this often leads to the student getting two different answers and simply redoing their process instead of exploring their mistakes or misunderstandings and correcting them. Thus, in agreement with the literature, the ePSTs in this study could benefit from more practice working with students' ideas and troubles. However, I stipulate that this work should begin early on in content courses and continue through their educational trajectory. This work should consist of a variety of student artifacts, as these have been proven to be helpful (Philipp, 2007, 2008; Thanheiser, Strand, & Mills, 2011), but also an increase in dialog-like conversations and approximated practices involving student troubles. Static artifacts show students' work in its entirety, but ePSTs also need work with incomplete or evolving work as it was the trouble-repair-assimilation-trouble cycle (Ingram, 2012) that challenged the ePSTs in this study.

Moreover, consistent with a variety of artifacts, the ePSTs should also be increasingly exposed to a variety of complex experiences. The findings from this study regarding the relationship formed between the five practices and their connection to supporting APS suggest that instances that required the support of multiple practices simultaneously were the most challenging. The ePSTs grounded their supports of the mathematical practices in justifications. Because of this grounding upon a not yet fully formed understanding of justifications and constrained content knowledge or specialized content knowledge, the ePSTs often had difficulty in using manipulatives, providing helpful hints, answering students' questions, and allowing the students to work in their own way. I highlight the importance of the connection between the practices here because these difficulties often occurred when a situation involved justifying and persevering through a mathematical trouble that included a visual representation and often times new or unfamiliar language or notation. Therefore, highlighting the need to include more complex and multidimensional artifacts of students' mathematical thinking in teacher preparation courses; artifacts that involve more than an exploration of a single facet, but intertwine instances of a students' productive struggles, or rehearsal scenarios similar to those used by Lampert and colleagues (2013).

More generally, all of the practices and APS are related in various ways, but the ePSTs did not find some practices to be connected, or the connections to be important enough to mention. Although the ePSTs did not vocalize these connections when stating their conceptions and noticings, they did support mathematical practices in ways they had not conceptualized and, therefore, may have also made connections in their supports. This idea would highlight the need for providing ePSTs with the necessary language to articulate and vocalize their conceptions more fully and accurately to how they view the mathematical practices. Especially considering the impact of a common language, which could fully describe how a teacher situates herself within supporting students' mathematical practices, can have for their community of practice (AMTE, 2017; NRC, 2002; Wenger, 1998). Therefore, future teachers not only need extensive time to understand and practice "carrying out the interactive work of teaching" (Ball & Forzani, 2009, p. 503), but should also emphasize learning a common language and how to articulate conceptions and supports about practice.

An implication for this finding lies in the preparation work of the ePSTs. Research notes that mathematical practices and the teaching practices used to support the mathematical practices form interrelationships that are intricately connected as they all serve the goal of problem-solving (AMTE, 2017; CCSSM, 2010; Lampert et al., 2013). However, if the ePSTs are only ever given opportunities to *decompose* and view *representations* of the practices, which not only separate and individualize the practices from each other but also mitigates the connections within the practice itself; then the ePSTs may not *recompose* (Grossman & McDonald, 2008) these connections without multiple *approximations* of various complexities. This is especially true for practices that are not as frequently studied, focused on, or thought about, like perseverance or mathematical language. Thus, teacher educators walk a fine line when including both mathematics education research literature regarding mathematical practices and centering teacher education around the general practice. "One challenge involved in centering teacher education in practice is careful deconstruction and articulation of the work of teaching with an eye toward making the most detailed elements of instruction learnable without reducing teaching practice to an atomized collection of discrete and unconnected tiny acts (Grossman & McDonald, 2008).” (Ball & Forzani, 2009, p. 507)

This final finding allows me to conclude on a singular note, which leads to the need for future work. Work has begun to transition the acts of *decomposition* and *representations* of working with children’s mathematical thinking into earlier courses, such as content courses. However, this transition is still new with much work yet to be done, but one must wonder where the *recomposition* of ideas could take place in order to fully bridge the gap between theory and practice? Thus, earlier courses must also begin to incorporate ideas of recomposition through approximations of practice with reduced complexities to allow ePSTs to begin forming unified connections among their understandings, conceptions, and enactments regarding mathematical practices and how they can support students autonomous problem-solving.

**Future Work**

This study provided insight into connections, or lack of conceptualized connections among mathematical practices, and conceptualizations ePSTs had towards these practices and student APS. Thus, future work would benefit from work on ePSTs conceptions and connections among mathematical practices. This work could involve increasing the awareness of lesser-known practices such as perseverance and mathematical language, strengthening the more conceived practices to more align with what research shows to be beneficial, or to dive deeper into how ePSTs see these mathematical practices as related through a purposeful exploration.

Also, as noted in the literature (Melhuish et al., 2020; Turner et al., 2019), there is more work needed to be done to bring what research has found into teacher education courses and professional development settings. Similar to the effects of exposure to ideas like *growth mindset* on teachers' beliefs, as seen by Russo and colleagues' (2020), how could extended exposure to the findings from mathematics education research effect ePSTs conceptions? Melhuish and colleagues (2020) noted that simply exposing teachers to the definition is not enough, how can content coursework be enhanced to focus ePSTs on the various levels of reasoning and what constitutes a valid justification? Similarly, there is a need to expose ePSTs to ways of supporting perseverance and the development of mathematical language.

Additionally, as artifacts of student thinking have proven to increase ePSTs' conceptual knowledge needed for teaching mathematics, research would suggest that more dynamic artifacts or engagement with students' mathematical thinking could also be influential for different purposes (Hunter et al., 2015). Most research studies, as seen throughout the literature review here, focus on a single mathematical practice and thus lead to teacher education development with disjoint single foci. Therefore, how can mathematics educators use and connect the knowledge gained from these in-depth, single-focused studies to better the understandings of the individual practices and their interconnections for teacher development? Could the ideas of coaching and rehearsals be utilized in content courses in such a way to establish smaller approximations of practice that will aid in the recomposition of practices?

Lastly, although this study has been informative as to what the ePSTs conceptions were and what was influential in their design, the question of how teacher educators can build on this knowledge and utilize the ePSTs outside experience still remains. This study has shown that outside work and sports experiences also play a role in the ePSTs' conceptions. Therefore, in accordance with AMTE's (2017) standards for preparing teachers of mathematics, further work should look at ways to "engage all partners productively" (p. 27). This could be other faculty members, cooperating schools, or partners outside of the school system such as families, communities, business and industry representatives, or other various outside experiences that the ePSTs could encounter.

In conclusion, the work presented in this dissertation provides a starting point for multiple different directions. Research Question 1 provides ideas about conceptions, which leads to questions of how do we build on that information, how do we strengthen it, and what are the underlying connections? Research Question 2 looked at the supports and reflections regarding the ePSTs conceptions, so now how can we create a similar experience in their coursework, what do these experiences afford, and how can we decompose and recompose through practice? Lastly, the ePSTs in this study reflected on some lived experience they considered as influential in their conceptions and enactments. These experiences were just some of the experiences that have impacted their thinking, the ones that they were conscious of and reflected on in these particular situations, but what other lived experiences are considered? How can we utilize those experiences in the ePSTs preparation, and what does that mean towards engagement in partnerships?

**APPENDIX SECTION**

## APPENDIX A

Pilot Study-Written Reflection Prompts

| Day | Question 1 | Question 2 |
|---|---|---|
| 1 | Write about 2 things that caught your eye or surprised you in today's camp class. Was it an interaction, comment, activity, etc.? Explain. Why did that instance stand out to you? | |
| 2 | Write about 2 things that caught your eye or surprised you in today's camp class. Was it an interaction, comment, activity, etc.? Explain. Why did that instance stand out to you? | Write about an episode where a student or students were "*Doing Mathematics*". Describe the episode context, what you noticed, and why you would classify this instance as, "*Doing Mathematics*". |
| 3 | Write about 2 things that caught your eye or surprised you in today's camp class. Was it an interaction, comment, activity, etc.? Explain. Why did that instance stand out to you? | Write about a particular aspect of the *classroom culture* in your class. Describe how the *classroom culture* appears to contribute to students' mathematical learning. |
| 4 | Write about 2 things that caught your eye or surprised you in today's camp class. Was it an interaction, comment, activity, etc.? Explain. Why did that instance stand out to you? | Write about what you observed related to *student beliefs and dispositions*. In what way(s) did teacher practice(s) support the development of *student beliefs and dispositions*? |
| 5 | Write about 2 things that caught your eye or surprised you in today's camp class. Was it an interaction, comment, activity, etc.? Explain. Why did that instance stand out to you? | Write about and episode where a student or students were engaged in a *productive struggle* episode. Describe the episode context and what you noticed in the student to students or student to teacher interaction. |
| 6 | Write about 2 things that caught your eye or surprised you in today's camp class. Was it an interaction, comment, activity, etc.? Explain. Why did that instance stand out to you? | Describe an aspect of your class today that addressed *"engaging all students in learning mathematics."* In addition, did you observe ways students were grouped or were called on in class that promoted and/or supported *"engagement for all"*. |
| 7 | Write about 2 things that caught your eye or surprised you in today's camp | How is *questioning* used in the classroom that support students' |

| | | |
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| | class. Was it an interaction, comment, activity, etc.? Explain. Why did that instance stand out to you? | reasoning and sense-making of important mathematics? What *questions* stand out to you and what did you think were the purposes of the questions – describe 2. |
| 8 | Write about 2 things that caught your eye or surprised you in today's camp class. Was it an interaction, comment, activity, etc.? Explain. Why did that instance stand out to you? | Describe instances where *communication* among students and teachers is supported in the classroom. Give examples that you've observed or were a part of where *communication* promoted students' "doing mathematics". |

**APPENDIX B**

Pilot Study-Interview Protocol

Interview Protocol Project: Connecting Practice To Theory

Time of interview: ______________

Date: ________________

Place: _______________________________________

Interviewer: _______________________________________

Interviewee: _____________________________________

Position of Interviewee:______________________________

This project is about discovering if, and how, MEC helps connect practice and theory for Elementary Pre-service teachers who participate in MEC. The purpose of this interview is to help answer the following research questions.

1. What, if any, mathematical connections and pedagogical connections do elementary pre-service teacher's make to their content courses during MEC?

2. How do these connections change the elementary pre-service teacher's perceptions of the importance of mathematical content knowledge and/or pedagogical content knowledge?

3. How do elementary pre-service teachers perceive MEC as beneficial, if at all, to their teaching practice preparation? What features of MEC do the PSTs identify, if any?

4. Does MEC influence the PSTs beliefs about the nature of mathematics and how it is learned? If so, how?

5. Does MEC affect the PSTs attitudes about mathematics? If so, how?

6. Does MEC influence the PSTs motivation for wanting to become a teacher? If so, how?

**This interview will be semi-structured with the following main questions.

Questions:

1. Do you think [MathKidz] helped you, or was beneficial to you as a pre-service teacher? If so, how?
2. Were there any specific techniques you remembered from your content courses that you tried out, or saw being used at camp?
3. Can you recall an instance from camp that made you think back to the things you learned from your content course? Can you describe the instance? How did the instance relate to a theory from class?
4. Looking back on the content courses, would you have done anything differently?
5. What would be some advice you would give to an incoming math content student?
6. What would you have liked to have seen more of in the content courses?
7. What was your overall impression of [MathKidz]?
8. What would be some advice you would give to an incoming [MathKidz] fellow?
9. Would it be more helpful to participate in [MathKidz] before or after math content?
10. Did camp have an impact on you wanting to become a teacher? If so, how?
11. How do you feel about teaching children mathematics? Has this changed since before camp?
12. Now that you have been through camp, do you think you would do anything differently when teaching? If so, what?

    Other Questions, Comments, and Conversations

**APPENDIX C**

Stimulated-recall interview questions

| Day Recorded | Person | Video 1 | Video 2 |
|---|---|---|---|
| 2 | Becky | Walk me through what you were thinking during this problem?<br><br>Do you remember some of the student suggestions y'all used to solve the problem?<br><br>Why do you think the little girl wanted to "turn it around"?<br><br>What was your thought towards that?<br><br>You told her she got it right. What made you want to say this? | Walk me through what you were thinking during this problem?<br><br>What did you think when you sat down with these two girls? |
| 2 | Amy | Walk me through what you were thinking during this problem?<br><br>Why do you think the student wanted to change his answer?<br><br>What was your thought process in responding to him?<br><br>What teaching practices did you see in this video? | Walk me through this interaction.<br><br>Was there a struggle going on here?<br><br>Do you thinking it was creating the number line that caused the challenge?<br><br>How did the boys space out the numbers on their number line?<br><br>Is there anything you would have done differently?<br><br>What teaching practices did you see in this video? |

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| | | What made you want to use the student's number line to help aid her thinking? | What had the students' done prior to you walking over to them? |
| 3 | Linda | Walk me through this interaction.<br><br>What did you think when she said that -7 was colder than -9?<br><br>What was your thoughts when walking on the number line? Walking towards the negatives.<br><br>Do you think the symbols helped her?<br><br>Did you draw on any previous experiences while interacting? MEC, , volleyball, etc.<br><br>Which teaching practices or Governing Precepts would you say matched this interaction? | |
| 4 | Amy | *What are your thoughts about the car model?<br><br>*Are other ideas incorporated too?<br><br>Walk me through these interactions.<br><br>These are similar instances; did you do anything different?<br><br>What was your thought when you saw the error?<br><br>What was their error?<br><br>Do you see any teaching practices in these videos?<br><br>Did any prior experience help you support the students in these interactions?<br><br>Would you do anything different? | Walk me through this interaction.<br><br>What was the error or source of confusion here?<br><br>Talk a bit about how you were supporting the student.<br><br>Do you see any teaching practices in these videos?<br><br>Did any prior experience help you support the students in these interactions?<br><br>Would you do anything different?<br><br>Was there something that you liked or are particularly proud of in these interactions? |

| | | | |
|---|---|---|---|
| | | Was there something that you liked or are particularly proud of in these interactions? | |
| 4 | Becky | *What do you think about the car model?<br><br>*Is there another way that you can help the students?<br><br>Walk me through this interaction.<br><br>What concept was the student having difficulty with?<br><br>Ordering changed from left to right to numbers then operation. Why?<br><br>Asked lots of questions. Can you elaborate on your questioning?<br><br>Did you draw on any previous experiences to support your responses during this interaction?<br><br>What teaching practices would you say this interaction supported?<br><br>What about this interaction did you like?<br><br>What would you change? | Walk me through this interaction.<br><br>Talk a little about the questions you asked.<br><br>What teaching practices would you say this interaction supported?<br><br>Did you draw on any previous experience to support your responses and questions during this interaction?<br><br>What about this interaction did you like?<br><br>What would you change? |
| 5 | Tori | *What do you think about the car model?<br><br>Is there another way that you can help the students? | Walk me through this interaction.<br><br>What concept was the student having difficulty with?<br><br>Can you elaborate on your questioning? |

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| | | | Did you draw on any previous experiences to support your responses during this interaction?<br><br>What teaching practices would you say this interaction supported?<br><br>What about this interaction did you like?<br><br>What would you change? |
| 5 | Linda | *What do you think about the car model?<br><br>*Is there another way that you can help the students?<br><br>Walk me through this interaction.<br><br>What concept was the student having difficulty with?<br><br>Can you elaborate on your questioning?<br><br>Did you draw on any previous experiences to support your responses during this interaction?<br><br>What teaching practices would you say this interaction supported?<br><br>What about this interaction did you like?<br><br>What would you change? | Walk me through this interaction.<br><br>What concept was the student having difficulty with?<br><br>Can you elaborate on your questioning?<br><br>Why was the word “operator” emphasized?<br><br>Did you draw on any previous experiences to support your responses during this interaction?<br><br>What teaching practices would you say this interaction supported?<br><br>What about this interaction did you like?<br><br>What would you change? |
| 6 | Becky | *What do you think about the chip model?<br><br>Walk me through this interaction. | |

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| | | What concept was the student having difficulty with?<br><br>Can you elaborate on your questioning?<br><br>Did you draw on any previous experiences to support your responses during this interaction?<br><br>What teaching practices would you say this interaction supported?<br><br>What about this interaction did you like?<br><br>What would you change?<br><br>Why was zero pair emphasized? | |
| 6 | Amy | *What do you think about the chip model?<br><br>Walk me through this interaction.<br><br>What concept was the student having difficulty with?<br><br>Can you elaborate on your questioning?<br><br>What did you think about Meryl wanting to check with the number line?<br><br>Did you draw on any previous experiences to support your responses during this interaction?<br><br>What teaching practices would you say this interaction supported?<br><br>What about this interaction did you like? | |

| | | | |
|---|---|---|---|
| | | What would you change? | |
| 7 | Linda | Walk me through this interaction.<br>What concept was the student having difficulty with?<br>Can you elaborate on your questioning?<br>Talk to me about the student's explanations and wording.<br>Did you draw on any previous experiences to support your responses during this interaction?<br>What teaching practices would you say this interaction supported?<br>What about this interaction did you like?<br>What would you change? | Walk me through this interaction.<br>What concept was the student having difficulty with?<br>Talk to me about the usage of the negative in this problem.<br>Elaborate on bringing in the number line.<br>Can you elaborate on your questioning?<br>Talk to me about how you left the problem.<br>Did you draw on any previous experiences to support your responses during this interaction?<br>What teaching practices would you say this interaction supported?<br>What about this interaction did you like?<br>What would you change? |
| 8 | Amy | Walk me through this interaction.<br>What concept were the students having difficulty with? Start with the two girls, then Cedric.<br>What can you tell me about Cedric's strategy and explanation?<br>What did you think of the "any number" idea?<br>Can you elaborate on your questioning? | |

| | | | |
|---|---|---|---|
| | | Did you draw on any previous experiences to support your responses during this interaction?<br><br>What teaching practices would you say this interaction supported?<br><br>What about this interaction did you like?<br><br>What would you change? | |
| 8 | Becky | Walk me through this interaction.<br><br>Can you elaborate on your questioning?<br><br>What did you think of Pedro's use of 11-3 instead of the number line?<br><br>Did you draw on any previous experiences to support your responses during this interaction?<br><br>What teaching practices would you say this interaction supported?<br><br>What about this interaction did you like?<br><br>What would you change? | |
| 9 | Linda | Walk me through this interaction.<br><br>What concept was the student having difficulty with?<br><br>Can you elaborate on your questioning?<br><br>What do you think the benefit of the number line was? | |

| | | | |
|---|---|---|---|
| | | Did you draw on any previous experiences to support your responses during this interaction?<br><br>What teaching practices would you say this interaction supported?<br><br>What about this interaction did you like?<br><br>What would you change? | |

*These questions were asked before the videos were shown.

**APPENDIX D**

Clinical Interview Tasks, Time, Topic, Descriptions, and Questions

| Task No. & Time | Topic | Video Description | Questions |
|---|---|---|---|
| Task 1.1<br>(Video length 2:35) | Student Autonomy | Teacher-centered solution | What do you think about how this teacher is supporting the student?<br>Would you do anything differently?<br>What do you notice about the questions she uses?<br>Do you think the student is benefiting from her questions? |
| Task 1.2<br>(Video length 1:15) | Student Autonomy | Student-centered solution | What do you think about how this teacher is supporting the student?<br>Would you do anything differently?<br>What do you notice about the questions she uses?<br>Do you think the student is benefiting from her questions? |
| Task 1.3<br>(Compare videos) | Student Autonomy | | How do these two videos compare?<br>Do you think the students benefited equally? |
| Task 2<br>(Video length 1:56) | Justification | $4^{th}$ grade student explains his reasoning for why he thinks $-5 + -1$ is $-6$ | What do you notice about the video?<br>What do you think about how he answered the question?<br>What if this video stopped after he said ${}^{-}5 + {}^{-}1 = {}^{-}6$? |

| | | | |
|---|---|---|---|
| Task 3<br>(Video length 7:21) | Mathematical Troubles<br>Perseverance<br>Student Autonomy | The student solves the problem $638 + 476$ to be 224, realizes her error after being prompted by a researcher to read the problem and solution aloud. She proceeds to work with the researchers to correct her misconception. | What did you notice about this video?<br>What teaching practices are being supported here?<br>What did you think about how the error was discovered?<br>How did the student think through this problem?<br>What do you think helped her work through it?<br>How do you feel about the teachers questioning in this video?<br>Do you think the student understood the answer or what she did wrong?<br>What would you say the student did well with, if anything? |
| Task 4<br>(Video length 2:45) | Mathematical Language<br>Justification | The student was given the conjecture $a + b - b = a$, and was asked to state if she believed it was true or not and why. She proceeds to use her own language and understanding of the number system to justify her reasoning. | What do you notice about this video clip?<br>Does anything stand out to you about her explanation?<br>What do you think of her vocabulary?<br>Does she always use the correct terms?<br>Is that okay? |

## APPENDIX E

Picture of Governing Precepts of MEC

1. **Doing mathematics** is about making sense of and thinking deeply about fundamental concepts. Students should learn to "think deeply of simple things," (Arnold Ross). Students need to:
   a. Build on prior knowledge by making connections that follow the flow of ideas from what they previously understood to new ideas being studied
   b. Promote a deep understanding for why things work using visual models
   c. Focus on the math problems, not the answers
   d. Reflect on what they have learned to make sense of the mathematics

2. **Persistence** is critical to success in problem solving and doing mathematics. Students need to:
   a. Develop a "growth mindset," understand and believe that ability can be developed with hard work
   b. Be willing to take risks and understand that mistakes present opportunities for learning
   c. Take ownership of their own learning
   d. Develop confidence to tackle new situations without giving up easily

3. Teachers need to establish a **classroom culture** that develops students' curiosity and imagination. The keys to establishing this culture are to:
   a. Make math interesting, fun and relevant with challenging, well-sequenced problems
   b. Support student's productive struggle by responding to student questions with appropriate guidance
   c. Allow sufficient time for learning ideas deeply
   d. Use techniques to engage all students
   e. Balance individual and group work; both can be appropriate depending on the task

4. **Communication** between students and teachers is critical for learning. To facilitate this, teachers should:
   a. Ask probing questions to develop student understanding, and encourage students to question why things work
   b. Expect students to present their work and defend their reasoning using precise mathematical language
   c. Take student attempts seriously, and examine both right and wrong approaches
   d. Expect students to articulate and explain the key math concepts

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