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GLOBAL ROBUST DISSIPATIVITY OF INTEGRO-DIFFERENTIAL SYSTEMS MODELLING NEURAL NETWORKS WITH TIME DELAY

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Abstract. In this paper, the global robust dissipativity of integro-differential systems modeling neural networks with time delay is studied. Several sufficient conditions are derived to ensure the global robust dissipativity of the neural networks proposed by using proper Lyapunov functionals and some analytic techniques. The results are shown to improve the previous global dissipativity results derived in the literature. Some examples are given to illustrate the correctness of our results.

1. Introduction

Stability of neural networks with delays has been a popular problem in this decade because the stability properties of equilibrium points of neural systems play an important role in some practical problems, such as optimization solvers, associative memories, image compression, speed detection of moving objects, processing of moving images, and pattern classification. Therefore, the stability of neural networks with or without delay has received much attention in the literature [2, 3, 4, 5, 9, 10, 11, 14, 15]. As pointed out in [1, 6, 8], the global dissipativity is also an important concept in dynamical neural networks. The concept of global dissipativity in dynamical systems is a more general concept and it has found applications in the areas such as stability theory, chaos and synchronization theory, system norm estimation, and robust control [8]. The global dissipativity of several classes of neural networks were discussed, and some sufficient conditions for the global dissipativity of neural networks with constant delays are derived in [8]. The authors of [1] analyzed the global dissipation of continuous-time dynamical neural networks with time delay. However, parameter fluctuation in neural network implementation on very large scale integration (VLSI) chips is unavoidable. To the best of our knowledge, there does not seem to be much (if any) study on global robust

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dissipativity of delayed integro-differential systems modelling neural networks with uncertainties.

Motivated by above discussion, in this paper, we will study the global robust dissipativity of delayed integro-differential systems modelling neural networks with uncertainties and derive some criteria for the delayed neural network with uncertainties via Lyapunov functionals and analysis techniques.

The organization of this paper is as follows. In Section 2, problem formulation and preliminaries are given. In Section 3, some new results are given to ascertain the global robust dissipativity of the delayed neural networks based on Lyapunov method and we give concluding remarks of results. Section 4 gives some examples to illustrate the effectiveness of our results. A conclusion is drawn in Section 5.

2. Problem formulations and preliminaries

Consider the model of continuous-time neural networks described by the following integro-differential systems

$$\begin{aligned}\frac{dx_i(t)}{dt} &= -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau)) \\ &\quad + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} K_{ij}(t-s) f_j(x_j(s))ds + u_i,\ i = 1, 2, \dots, n\end{aligned} \tag{2.1}$$

or

$$\begin{aligned}\frac{dx(t)}{dt} &= -Dx(t) + Af(x(t)) + Bf(x(t-\tau)) \\ &\quad + C\int_{-\infty}^{t} K(t-s) f(x(s))ds + u,\end{aligned} \tag{2.2}$$

where n denotes the number of the neurons in the network, $x_i(t)$ is the state of the ith neuron at time t, $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \dots, f_n(x_n(t))]^T$ denote the activation functions of the j-th neuron at time t, $D = \operatorname{diag}(d_1, d_2, \dots, d_n) > 0$ is a positive diagonal matrix, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $C = (c_{ij})_{n\times n}$ are the feedback matrix and the delayed feedback matrix, respectively, τ is the transmission delay, $u = (u_1, u_2, \dots, u_n)^T \in \mathbb{R}^n$ be a constant external input vector.

In our analysis, we will employ the following two classes of activation functions:

(1) The general set of monotone nondecreasing activation functions is defined as

$$\mathcal{G} = \{f(x) | D^+ f_i(x_i) \ge 0,\ i = 1, 2, \dots, n\}.$$

(2) The set of Lipschitz-continuous activation functions is defined as

$$\mathcal{L} = \{f(x) | 0 \le \frac{f_i(x_i) - f_i(y_i)}{x_i - y_i} \le L_i,\ L_i > 0, \}$$

for all $x_i, y_i \in R, x_i \ne y_i$, $i = 1, 2, \dots, n$.

$K(\cdot) = (K_{ij}(\cdot))_{n\times n}$, $i, j = 1, 2, \dots, n$ are the delay kernels which are assumed to satisfy the following conditions simultaneously:

(i) $K_{ij} : [0, \infty) \to [0, \infty)$;
(ii) K_{ij} are bounded and continuous on $[0, \infty)$;
(iii) $\int_0^\infty K_{ij}(s)ds = 1$;
(iv) there exists a positive number ε such that $\int_0^\infty K_{ij}(s)e^{\varepsilon s}ds < \infty$.

Some examples of functions that meet the above conditions can be found in [12].

Clearly, if one assume that $c_{ij} = 0$ $(i, j = 1, 2, \dots, n)$, the system (2.1) reduces to the dynamical neural networks model

$$\frac{dx_i(t)}{dt} = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau)) + u_i, \tag{2.3}$$

whose global dissipativity was investigated in [1].

The initial conditions associated with the system (2.1) are

$$x_i(s) = \phi_i(s), \quad -\tau \le s \le 0,\ i = 1, 2, \dots, n, \tag{2.4}$$

where $\phi_i(\cdot)$ is bounded and continuous on $[-\tau, 0]$.

In electronic implementation of neural networks, the values of the constant and weight coefficients depend on the resistance and capacitance values which are subject to uncertainties. This may lead some deviations in the values of d_i, a_{ij}, b_{ij} and c_{ij}. Hence, it is important to ensure the global robust dissipativity of the designed network against such parameter deviations. Since these deviations are bounded in practice, the quantities d_i, a_{ij}, b_{ij} and c_{ij} may be intervalized as follows: for $i = 1, 2, \dots n$,

$$\begin{gathered}
D_I := [\underline{D}, \overline{D}] = D = \operatorname{diag}(d_i) : \underline{D} \le D \le \overline{D}, \quad \text{i.e., } \underline{c}_i \le c_i \le \bar{c}_i, \\
A_I := [\underline{A}, \overline{A}] = A = (a_{ij})_{n\times n} : \underline{A} \le A \le \overline{A}, \quad \text{i.e., } \underline{a}_{ij} \le a_{ij} \le \bar{a}_{ij}, \\
B_I := [\underline{B}, \overline{B}] = B = (b_{ij})_{n\times n} : \underline{B} \le B \le \overline{B}, \quad \text{i.e., } \underline{b}_{ij} \le b_{ij} \le \bar{b}_{ij}, \\
C_I := [\underline{C}, \overline{C}] = C = (c_{ij})_{n\times n} : \underline{C} \le C \le \overline{C}, \quad \text{i.e., } \underline{c}_{ij} \le c_{ij} \le \bar{c}_{ij}.
\end{gathered}$$

Definition 2.1. The neural network defined by (2.1) or (2.2) is said to be a dissipative system, if there exists a compact set $S \subset R^n$, such that $\forall x_0 \in \mathbb{R}^n$, $\exists T > 0$, when $t \ge t_0 + T$, $x(t, t_0, x_0) \subseteq S$, where $x(t, t_0, x_0)$ denotes the solution of (2.1) from initial state x_0 and initial time t_0. In this case, S is called a globally attractive set. A set S is called positive invariant if $\forall x_0 \in S$ implies $x(t, t_0, x_0) \subseteq S$ for $t \ge t_0$.

Definition 2.2. The neural network defined by (2.1) or (2.2) is globally robust dissipative if the system is globally dissipative for all $D \in D_I$, $A \in A_I$, $B \in B_I$, $C \in C_I$.

Definition 2.3 ([15])**.** A vector $v = [v_1, \dots, v_n]^T > 0$, if and only if every $v_i > 0$, $i = 1, \dots, n$.

Definition 2.4. If $R \to R$ is a continuous function, then the upper right derivative $\frac{D^+ f(t)}{dt}$ of $f(t)$ is defined as

$$D^+ f(t) = \lim_{\theta \to 0^+} \frac{f(t+\theta) - f(t)}{\theta}. \tag{2.5}$$

3. Global robust dissipativity

In this section, we present the following results:

Theorem 3.1. *Let $f(\cdot) \in \mathcal{G}$, $f(0) = 0$ and $f_i(x_i) \to \infty$ as $|x_i| \to \infty$. the neural network defined by* (2.1) *is a globally robust dissipative system and the set $S_1 =$*

$\{x : |x_i(t)| \le |u_i|/\underline{d}_i, i = 1, 2, \dots, n\}$ is a positive invariant and globally attractive set, if there exist positive constants $p_i > 0$, $i = 1, 2, \dots, n$ such that

$$p_i(-\overline{a}_{ii} - b_{ii}^* - c_{ii}^*) - \sum_{j=1, j\neq i}^{n} p_j(a_{ji}^* + b_{ji}^* + c_{ji}^*) \ge 0, \tag{3.1}$$

where $i = 1, \dots, n$, $a_{ji}^* = \max(|\underline{a}_{ji}|, \overline{a}_{ji})$, $b_{ji}^* = \max(|\underline{b}_{ji}|, \overline{b}_{ji})$, $c_{ji}^* = \max(|\underline{c}_{ji}|, \overline{c}_{ji})$.

Proof. Let us use the positive and radially unbounded Lyapunov functional

$$\begin{aligned} V(x(t)) = \sum_{i=1}^{n} p_i \big\{ |x_i(t)| &+ \sum_{j=1}^{n} \int_{t-\tau}^{t} b_{ij}^* |f_j(x_j(s))| ds \\ &+ \sum_{j=1}^{n} c_{ij}^* \int_0^\infty K_{ij}(s) \Big(\int_{t-s}^{t} |f_j(x_j(\xi))| d\xi \Big) ds \big\}. \end{aligned} \tag{3.2}$$

Calculating the upper right derivative D^+V of the Lyapunov functional V along the solution of (2.1), we have that

$$\begin{aligned} &D^+V(x(t)) \\ &\le -\sum_{i=1}^{n} \{ p_i d_i |x_i(t)| - p_i(a_{ii} + b_{ii}^*)|f_i(x_i(t))| - \sum_{j=1, j\neq i}^{n} p_i(|a_{ij}| + b_{ij}^*)|f_j(x_j(t))| \\ &\quad - p_i \sum_{j=1}^{n} |c_{ij}| \int_{-\infty}^{t} K_{ij}(t-s)|f_j(x_j(s))| ds - p_i \sum_{j=1}^{n} c_{ij}^* \int_0^\infty K_{ij}(s)|f_j(x_j(t))| ds \\ &\quad + p_i \sum_{j=1}^{n} c_{ij}^* \int_0^\infty K_{ij}(s)|f_j(x_j(t-s))| ds - p_i|u_i| \} \\ &\le -\sum_{i=1}^{n} \{ p_i d_i |x_i(t)| - p_i(a_{ii} + b_{ii}^* + c_{ii}^*)|f_i(x_i(t))| \\ &\quad - \sum_{j=1, j\neq i}^{n} p_i(a_{ij}^* + b_{ij}^* + c_{ij}^*)|f_j(x_j(t))| - p_i|u_i| \} \\ &\le -\sum_{i=1}^{n} p_i(\underline{d}_i |x_i(t)| - |u_i|) - \sum_{i=1}^{n} \{ p_i(-\overline{a}_{ii} - b_{ii}^* - c_{ii}^*) \\ &- \sum_{j=1, j\neq i}^{n} p_j(a_{ji}^* + b_{ji}^* + c_{ji}^*) \} |f_i(x_i(t))| \\ &\le -\sum_{i=1}^{n} p_i \underline{d}_i \big(|x_i(t)| - \frac{|u_i|}{\underline{d}_i} \big) < 0, \end{aligned}$$

when $x_i \in \mathbb{R}^n \setminus S_1$; i.e., $x \overline{\in} S_1$. Which implies that for all $x_0 \in S_1$ holds $x(t, t_0, x_0) \subseteq S_1$, $t \ge t_0$. For $x \overline{\in} S_1$, there exists $T > 0$ such that

$$x(t, t_0, x_0) \subseteq S_1, \quad \forall t \ge t_0 + T,$$

meaning that the neural network defined by (2.1) is a dissipative system and the set S_1 is a positive invariant set and globally attractive set. $\square$

Remark 3.2. As we can see, if vary the parameters $p_i > 0$, $(i = 1, 2, \dots, n)$ properly, one can obtain a series of corollaries. Specially, If choose $p_i = 1$, the following corollary can be derived.

Corollary 3.3. *Let $f(\cdot) \in \mathcal{G}$, $f(0) = 0$ and $f_i(x_i) \to \infty$ as $|x_i| \to \infty$. the neural network defined by (2.1) is a globally robust dissipative system and the set $S_1 = \{x||x_i(t)| \le |u_i|/\underline{d}_i, i = 1, 2, \dots, n\}$ is a positive invariant and globally attractive set, if the following condition holds*

$$-\overline{a}_{ii} - b^*_{ii} - c^*_{ii} - \sum_{j=1, j\neq i}^{n} (a^*_{ji} + b^*_{ji} + c^*_{ji}) \ge 0, \tag{3.3}$$

where $i = 1, \dots, n$, $a^*_{ji} = \max(|\underline{a}_{ji}|, \overline{a}_{ji})$, $b^*_{ji} = \max(|\underline{b}_{ji}|, \overline{b}_{ji})$, $c^*_{ji} = \max(|\underline{c}_{ji}|, \overline{c}_{ji})$.

Theorem 3.4. *Let $f(\cdot) \in \mathcal{G}$, $f(0) = 0$ and $f_i(x_i) \to \infty$ as $|x_i| \to \infty$. If the following condition holds*

$$\overline{A} + \overline{A}^T + (\|B^*\|_\infty + \|B^*\|_1 + \|C^*\|_\infty + \|C^*\|_1)I \le 0, \tag{3.4}$$

where $B^* = (b^*_{ji})_{n\times n}, C^* = (c^*_{ji})_{n\times n}$, then, the neural network defined by (2.2) is a robust dissipative system and the set $S_2 = \{x||x_i(t)| \le |u_i|/\underline{d}_i, i = 1, 2, \dots, n\}$ is a positive invariant and globally attractive set.

Proof. Consider radially unbounded Lyapunov functional

$$\begin{aligned} V(x(t)) = 2\sum_{i=1}^n \int_0^{x_i(t)} f_i(s)ds + \sum_{i=1}^n \sum_{j=1}^n \int_{t-\tau}^t b^*_{ji} f_i^2(x_i(s))ds \\ + \sum_{i=1}^n \sum_{j=1}^n c^*_{ji} \int_0^\infty K_{ji}(s)\Big(\int_{t-s}^t f_i^2(x_i(\xi))d\xi\Big)ds. \end{aligned}$$

Calculating the upper right derivative D^+V of the Lyapunov functional V along the solution of (2.1), we have that

$$\begin{aligned} D^+V(x(t)) &= -2\sum_{i=1}^n d_i f_i(x_i(t))x_i(t) + 2\sum_{i=1}^n\sum_{j=1}^n a_{ij} f_i(x_i(t)) f_j(x_j(t)) \\ &\quad + 2\sum_{i=1}^n\sum_{j=1}^n b_{ij} f_i(x_i(t)) f_j(x_j(t-\tau)) + 2\sum_{i=1}^n f_i(x_i(t))u_i \\ &\quad + 2\sum_{i=1}^n\sum_{j=1}^n c_{ij} f_i(x_i(t)) \int_{-\infty}^t K_{ij}(t-s) f_j(x_j(s))ds \\ &\quad + \sum_{i=1}^n\sum_{j=1}^n b^*_{ji} f_i^2(x_i(t)) - \sum_{i=1}^n\sum_{j=1}^n b^*_{ji} f_i^2(x_i(t-\tau)) \\ &\quad + \sum_{i=1}^n\sum_{j=1}^n c^*_{ji} \int_0^\infty K_{ji}(s)[f_i^2(x_i(t)) - f_i^2(x_i(t-s))]ds \\ &\le -2\sum_{i=1}^n d_i |f_i(x_i(t))||x_i(t)| + 2\sum_{i=1}^n\sum_{j=1}^n a_{ij} f_i(x_i(t)) f_j(x_j(t)) \\ &\quad + 2\sum_{i=1}^n\sum_{j=1}^n |b_{ij}||f_i(x_i(t))||f_j(x_j(t-\tau))| + 2\sum_{i=1}^n |f_i(x_i(t))||u_i| \end{aligned}$$

$$
\begin{aligned}
&+ 2\sum_{i=1}^{n}\sum_{j=1}^{n}|c_{ij}|\int_0^{\infty} K_{ij}(s)|f_i(x_i(t))||f_j(x_j(t-s))|ds \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} b^*_{ji} f_i^2(x_i(t)) - \sum_{i=1}^{n}\sum_{j=1}^{n} b^*_{ji} f_i^2(x_i(t-\tau)) \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} c^*_{ji}\int_0^{\infty} K_{ji}(s)[f_i^2(x_i(t)) - f_i^2(x_i(t-s))]ds.
\end{aligned}
$$

Since

$$
2|f_i(x_i(t))||f_j(x_j(t-\tau))| \le f_i^2(x_i(t)) + f_j^2(x_j(t-\tau)), \tag{3.5}
$$

$$
\begin{aligned}
&2\int_0^{\infty} K_{ij}(s)|f_i(x_i(t))||f_j(x_j(t-s))|ds \\
&\le \int_0^{\infty} K_{ij}(s) f_i^2(x_i(t))ds + \int_0^{\infty} K_{ij}(s) f_j^2(x_j(t-s))ds.
\end{aligned} \tag{3.6}
$$

Substitute (3.5) and (3.6) into the inequality above these two, we obtain

$$
\begin{aligned}
&D^+V(x(t)) \\
&\le -2\sum_{i=1}^{n}\underline{d}_i|f_i(x_i(t))||x_i(t)| + 2\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{a}_{ij} f_i(x_i(t)) f_j(x_j(t)) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} b^*_{ij} f_i^2(x_i(t)) + 2\sum_{i=1}^{n}|f_i(x_i(t))||u_i| \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} c^*_{ij}\int_0^{\infty} K_{ij}(s) f_i^2(x_i(t))ds \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} b^*_{ji} f_i^2(x_i(t)) + \sum_{i=1}^{n}\sum_{j=1}^{n} c^*_{ji}\int_0^{\infty} K_{ji}(s) f_i^2(x_i(t))ds \\
&\le -2\sum_{i=1}^{n}\underline{d}_i|f_i(x_i(t))||x_i(t)| + 2\sum_{i=1}^{n}|f_i(x_i(t))||u_i| \\
&\quad + f^T(x(t))\Big(\overline{A} + \overline{A}^T + (\|B^*\|_\infty + \|B^*\|_1 + \|C^*\|_\infty + \|C^*\|_1)I\Big) f(x(t)) \\
&\le -2\sum_{i=1}^{n}\underline{d}_i|f_i(x_i(t))||x_i(t)| + 2\sum_{i=1}^{n}|f_i(x_i(t))||u_i| < 0,
\end{aligned}
$$

when $x_i \in \mathbb{R}^n \setminus S_2$. This implies that the set S_2 is a positive invariant and globally attractive set. $\square$

Theorem 3.5. *Let $f(\cdot) \in \mathcal{L}$, $f(0) = 0$ and $f_i(x_i) \to \infty$ as $|x_i| \to \infty$. If the following condition holds*

$$
\overline{A} + \overline{A}^T + \overline{B}\overline{B}^T + (1 + \|C^*\|_\infty + \|C^*\|_1)I \le 0, \tag{3.7}
$$

where $B^* = (b^*_{ji})_{n\times n}, C^* = (c^*_{ji})_{n\times n}$, then, the neural network defined by (2.2) is a robust dissipative system and the set

$$
S_3 = \{x||f_i(x_i(t))| \le L_i|u_i|/d_i, i = 1, 2, \dots, n\}
$$

is a positive invariant and globally attractive set.

Proof. Let us use the following positive definite and radially unbounded Lyapunov functional

$$
\begin{aligned}
V(x(t)) &= 2\sum_{i=1}^{n}\int_{0}^{x_i(t)} f_i(s)ds + \sum_{i=1}^{n}\int_{t-\tau}^{t} f_i^2(x_i(s))ds \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^* \int_0^\infty K_{ji}(s)\left(\int_{t-s}^{t} f_i^2(x_i(\xi))d\xi\right) ds.
\end{aligned}
$$

Calculating the upper right derivative D^+V of the Lyapunov functional V along the solution of (2.1), we have that

$$
\begin{aligned}
D^+V(x(t)) &= -2\sum_{i=1}^{n} d_i f_i(x_i(t))x_i(t) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} f_i(x_i(t))f_j(x_j(t)) \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} f_i(x_i(t))f_j(x_j(t-\tau)) + 2\sum_{i=1}^{n} f_i(x_i(t))u_i \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} f_i(x_i(t))\int_{-\infty}^{t} K_{ij}(t-s)f_j(x_j(s))ds \\
&\quad + \sum_{j=1}^{n} f_i^2(x_i(t)) - \sum_{j=1}^{n} f_i^2(x_i(t-\tau)) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^* \int_0^\infty K_{ji}(s)[f_i^2(x_i(t)) - f_i^2(x_i(t-s))]ds \qquad (3.8)\\
&\quad - 2\sum_{i=1}^{n} \frac{d_i}{L_i} f_i^2(x_i(t)) + 2\sum_{i=1}^{n} |f_i(x_i(t))||u_i| \\
&\quad \times f^T(x(t))(A+A^T)f(x(t)) + 2f^T(x(t-\tau))B^Tf(x(t)) \\
&\quad + 2\sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^* \int_0^\infty K_{ij}(s)|f_i(x_i(t))||f_j(x_j(t-s))|ds \\
&\quad + f^T(x(t))f(x(t)) - f^T(x(t-\tau))f(x(t-\tau)) \\
&\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}^* \int_0^\infty K_{ji}(s)[f_i^2(x_i(t)) - f_i^2(x_i(t-s))]ds.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
&-f^T(x(t-\tau))f(x(t-\tau)) + 2f^T(x(t-\tau))B^Tf(x(t)) \\
&= -[f(x(t-\tau)) - B^Tf(x(t))]^T[f(x(t-\tau)) - B^Tf(x(t))] + f^T(x(t))BB^Tf(x(t)),
\end{aligned}
$$

from which it follows that

$$
-f^T(x(t-\tau))f(x(t-\tau)) + 2f^T(x(t-\tau))B^Tf(x(t)) \le f^T(x(t))BB^Tf(x(t)).
$$

Also we have

$$
\begin{aligned}
&2\int_0^\infty K_{ij}(s)|f_i(x_i(t))||f_j(x_j(t-s))|ds \\
&\le \int_0^\infty K_{ij}(s)f_i^2(x_i(t))ds + \int_0^\infty K_{ij}(s)f_j^2(x_j(t-s))ds.
\end{aligned}
$$

Using the above inequality in (3.8) results in:

$$\begin{aligned}
D^+V(x(t)) \le & -2\sum_{i=1}^n \frac{d_i}{L_i} f_i^2(x_i(t)) + 2\sum_{i=1}^n |f_i(x_i(t))||u_i| \\
& + f^T(x(t))(A + A^T + BB^T + I)f(x(t)) \\
& + \sum_{i=1}^n\sum_{j=1}^n c_{ij}^* \int_0^\infty K_{ij}(s) f_i^2(x_i(t))ds \\
& + \sum_{i=1}^n\sum_{j=1}^n c_{ji}^* \int_0^\infty K_{ji}(s) f_i^2(x_i(t))ds \\
\le & -2\sum_{i=1}^n \frac{\underline{d}_i}{L_i} f_i^2(x_i(t)) + 2\sum_{i=1}^n |f_i(x_i(t))||u_i| \\
& + f^T(x(t))\big[\overline{A} + \overline{A}^T + \overline{B}\overline{B}^T + (1 + \|C^*\|_\infty + \|C^*\|_1)\, I\big] f(x(t)) \\
\le & -2\sum_{i=1}^n \frac{\underline{d}_i}{L_i} f_i^2(x_i(t)) + 2\sum_{i=1}^n |f_i(x_i(t))||u_i| < 0,
\end{aligned}$$

when $x_i \in \mathbb{R}^n \setminus S_3$. This implies that the set S_3 is a positive invariant and globally attractive set. □

Theorem 3.6. *Let $f(\cdot) \in \mathcal{L}$, $f(0) = 0$ and $f_i(x_i) \to \infty$ as $|x_i| \to \infty$. If the matrix Q given by*

$$Q = P(\overline{A} - L^{-1}\underline{D}) + (\overline{A} - L^{-1}\underline{D})^T P + PBB^T P + (1 + \|PC^*\|_\infty + \|PC^*\|_1)I \tag{3.9}$$

is negative definite, then the neural network defined by (2.2) is a robust dissipative system and the set

$$S_4 = \big\{x : [f_i(x_i(t)) + \frac{p_i u_i}{\lambda_M(Q)}]^2 \le \big(\frac{p_i u_i}{\lambda_M(Q)}\big)^2,\ i = 1, 2, \dots, n\big\}$$

is a positive invariant and globally attractive set, where $L = \mathrm{diag}(L_1, L_2, \dots, L_n)$, $P = \mathrm{diag}(p_1, p_2, \dots, p_n)$ and $\lambda_M(Q)$ is the maximum eigenvalue of the matrix Q.

Proof. Let us use the following positive definite and radially unbounded Lyapunov functional

$$\begin{aligned}
V(x(t)) = & 2\sum_{i=1}^n p_i \int_0^{x_i(t)} f_i(s)ds + \sum_{i=1}^n \int_{t-\tau}^t f_i^2(x_i(s))ds \\
& + \sum_{i=1}^n\sum_{j=1}^n p_i c_{ji}^* \int_0^\infty K_{ji}(s)\Big(\int_{t-s}^t f_i^2(x_i(\xi))d\xi\Big)ds.
\end{aligned}$$

Calculating the upper right derivative D^+V of the Lyapunov functional V along the solution of (2.1), it follows

$$\begin{aligned}
D^+V(x(t)) = & -2\sum_{i=1}^n p_i d_i f_i(x_i(t)) x_i(t) + 2\sum_{i=1}^n\sum_{j=1}^n p_i a_{ij} f_i(x_i(t)) f_j(x_j(t)) \\
& + 2\sum_{i=1}^n\sum_{j=1}^n p_i b_{ij} f_i(x_i(t)) f_j(x_j(t-\tau)) + 2\sum_{i=1}^n p_i f_i(x_i(t)) u_i
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{i=1}^{n}\sum_{j=1}^{n}p_i c_{ij} f_i(x_i(t))\int_{-\infty}^{t}K_{ij}(t-s)f_j(x_j(s))ds\\
&+\sum_{i=1}^{n}f_i^2(x_i(t))-\sum_{i=1}^{n}f_i^2(x_i(t-\tau))\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}p_i c_{ji}^*\int_0^\infty K_{ji}(s)\left[f_i^2(x_i(t))-f_i^2(x_i(t-s))\right]ds\\
\le& -2\sum_{i=1}^{n}p_i\frac{d_i}{L_i}f_i^2(x_i(t))+2\sum_{i=1}^{n}p_i f_i(x_i(t))u_i\\
&+f^T(x(t))(PA+A^TP)f(x(t))+2f^T(x(t-\tau))B^TPf(x(t))\\
&+2\sum_{i=1}^{n}\sum_{j=1}^{n}p_i c_{ij}^*\int_0^\infty K_{ij}(s)|f_i(x_i(t))||f_j(x_j(t-s))|ds\\
&+f^T(x(t))f(x(t))-f^T(x(t-\tau))f(x(t-\tau))\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}p_i c_{ji}^*\int_0^\infty K_{ji}(s)[f_i^2(x_i(t))-f_i^2(x_i(t-s))]ds. \qquad (3.10)
\end{aligned}
$$

We have

$$
\begin{aligned}
&-f^T(x(t-\tau))f(x(t-\tau))+2f^T(x(t-\tau))B^TPf(x(t))\\
&=-\left[f(x(t-\tau))-B^TPf(x(t))\right]^T\left[f(x(t-\tau))-B^TPf(x(t))\right]\\
&\quad+f^T(x(t))PBB^TPf(x(t)),
\end{aligned}
$$

from which it follows that

$$
-f^T(x(t-\tau))f(x(t-\tau))+2f^T(x(t-\tau))B^TPf(x(t))\le f^T(x(t))PBB^TPf(x(t)).
$$

Also we have

$$
\begin{aligned}
&2\int_0^\infty K_{ij}(s)|f_i(x_i(t))||f_j(x_j(t-s))|ds\\
&\le\int_0^\infty K_{ij}(s)f_i^2(x_i(t))ds+\int_0^\infty K_{ij}(s)f_j^2(x_j(t-s))ds.
\end{aligned}
$$

Using the above inequality in (3.10), it results in

$$
\begin{aligned}
D^+V(x(t))\le& 2\sum_{i=1}^{n}p_i f_i(x_i(t))u_i+f^T(x(t))P\overline{B}\overline{B}^TPf(x(t))+f^T(x(t))f(x(t))\\
&+f^T(x(t))(P(\overline{A}-L^{-1}\underline{D})+(\overline{A}-L^{-1}\underline{D})^TP)f(x(t))\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}|p_i c_{ij}^*|\int_0^\infty K_{ij}(s)f_i^2(x_i(t))ds\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}|p_i c_{ji}^*|\int_0^\infty K_{ji}(s)f_i^2(x_i(t))ds\\
=& 2\sum_{i=1}^{n}p_i f_i(x_i(t))u_i+f^T(x(t))Qf(x(t))
\end{aligned}
$$

$$\leq 2\sum_{i=1}^{n} p_i f_i(x_i(t))u_i + \sum_{i=1}^{n} \lambda_M(Q) f_i^2(x_i(t))$$
$$= \lambda_M(Q)\sum_{i=1}^{n}\big\{\big[f_i(x_i(t)) + \frac{p_i u_i}{\lambda_M(Q)}\big]^2 - \big(\frac{p_i u_i}{\lambda_M(Q)}\big)^2\big\} < 0$$

when $x_i \in \mathbb{R}^n \setminus S_4$. This imiplies that the set S_4 is a positive invariant and globally attractive set. □

Corollary 3.7. *Let $f(\cdot) \in \mathcal{L}$, $f(0) = 0$ and $f_i(x_i) \to \infty$ as $|x_i| \to \infty$. If the matrix*

$$Q = (\overline{A} - L^{-1}\underline{D}) + (\overline{A} - L^{-1}\underline{D})^T + BB^T + (1 + \|C^*\|_\infty + \|C^*\|_1)I \tag{3.11}$$

is negative definite, then the neural network defined by (2.2) *is a robust dissipative system and the set*

$$S_5 = \big\{x : \big[f_i(x_i(t)) + \frac{u_i}{\lambda_M(Q)}\big]^2 \leq \big(\frac{u_i}{\lambda_M(Q)}\big)^2,\ i = 1, 2, \dots, n\big\}$$

is a positive invariant and globally attractive set, where $L = \operatorname{diag}(L_1, L_2, \dots, L_n)$, and $\lambda_M(Q)$ is the maximum eigenvalue of the matrix Q.

Remark 3.8. In this paper, we derive several different conditions to check the global robust dissipativity of integro-differential systems modeling neural networks and the conditions in Theorems 3.1–3.6 are independent of each other.

Remark 3.9. For system (2.1) or (2.2), when the parameters are certain, the global robust dissipativity of the system has been studied in [13]. However, parameter fluctuation in neural network implementation on very large scale integration (VLSI) chips is unavoidable and it is important to ensure that system be dissipative with respect to these uncertainties in the design and applications of neural networks. Therefore, their conclusions can be included in our results as special cases.

Remark 3.10. If delay kernel functions $k_{ij}(t)$ are of the form

$$k_{ij}(t) = \delta(t - \tau_{ij}), \quad i, j = 1, 2, \dots, n, \tag{3.12}$$

then system (2.1) reduces to a system with time-varying delays which has been lucubrated in [1, 8]. And many crucial results for dynamics of the system have been obtained. Therefore the discrete delays can be included in our models by choosing suitable kernel functions.

Remark 3.11. When $c_{ij} = 0$ $(i, j = 1, 2, \dots, n)$, the system (2.1) reduces to system (2.3) which was studied in [1, 8]. In this case, Theorems 3.1–3.6 turn out to be generalized results for those in [1, 8]. Moreover, the results are less conservative and more extensive than those in [1], which will be illustrated in Example 4.2.

4. Examples

In this section, we give illustrative examples for our results.

Example 4.1. Consider the system (2.1) or (2.2) with constant delays: $\tau = 2$ for $i, j = 1, 2$, with the initial values of the system as follows:

$$\phi(s) = 0.3, \quad t \in [-2, 0).$$

Let $f(\cdot) \in \mathcal{G}$, and all the kernel properties given in (i)–(iv) are satisfied.

$$\underline{D} = \begin{bmatrix} 0.6 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \underline{A} = \begin{bmatrix} -2 & -\frac{1}{3} \\ \frac{1}{5} & -3 \end{bmatrix},$$
$$\overline{A} = \begin{bmatrix} -\frac{5}{4} & \frac{1}{4} \\ \frac{2}{5} & -\frac{7}{3} \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 0.25 & -0.5 \\ -0.2 & -0.7 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0.5 & 0.25 \\ 0 & -0.5 \end{bmatrix},$$
$$\underline{C} = \overline{C} = 0, \quad u_1 = -3, \quad u_2 = 2.$$

Hence

$$\begin{gathered} -\overline{a}_{11} - b_{11}^* - c_{11}^* - (a_{21}^* + b_{21}^* + c_{21}^*) = 0.15 > 0, \\ -\overline{a}_{22} - b_{22}^* - c_{22}^* - (a_{12}^* + b_{12}^* + c_{12}^*) = 0.80 > 0, \end{gathered} \tag{4.1}$$

Then the conditions of Corollary 3.3 are satisfied, therefore the model (2.1) is a globally robust dissipative system, and the set

$$S_1 = \big\{(x_1(t), x_2(t)) \big| |x_1(t)| \leq 5, |x_2(t)| \leq \frac{4}{3}\big\}$$

is a positive invariant and globally attractive set. However, since the parameters are uncertain, the results in [13] are not applicable for this example.

Example 4.2. Consider the system (2.1) or (2.2) with constant delays: $\tau = 1$ for $i, j = 1, 2$, with the initial values of the system as follows:

$$\phi(s) = 0.5, \quad t \in [-1, 0).$$

Let $f(\cdot) \in \mathcal{G}$, and all the kernel properties given in (i-iv) are satisfied. And here we let

$$\underline{D} = \begin{bmatrix} 0.6 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \underline{A} = \begin{bmatrix} -2 & -\frac{1}{3} \\ \frac{1}{5} & -3 \end{bmatrix},$$
$$\overline{A} = \begin{bmatrix} -\frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{5}{4} \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & -0.25 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix},$$
$$\underline{C} = \begin{bmatrix} 0.25 & -0.25 \\ -0.25 & -0.25 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}, \quad u_1 = 1.5, \quad u_2 = -4.$$

Hence

$$\overline{A} + \overline{A}^T + (\|B^*\|_\infty + \|B^*\|_1 + \|C^*\|_\infty + \|C^*\|_1)I = \begin{bmatrix} -1.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \leq 0, \tag{4.2}$$

Then, the conditions of Theorem 3.4 are satisfied, therefore the model (2.1) is a globally robust dissipative system, and the set

$$S_1 = \big\{(x_1(t), x_2(t)) \big| |x_1(t)| \leq 2.5, |x_2(t)| \leq \frac{8}{3}\big\}$$

is a positive invariant and globally attractive set. However, when $c_{ij} = 0$ $(i, j = 1, 2, \dots, n)$, since

$$Q_{11} = \frac{\overline{A} + \overline{A}^T}{2} + I = \begin{bmatrix} -\frac{3}{4} & \frac{5}{4} \\ \frac{5}{4} & -\frac{1}{4} \end{bmatrix},$$
$$Q_{12} = \frac{B^*}{2} + I = \begin{bmatrix} \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} \end{bmatrix},$$
$$Q_{22} = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

so we get that

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$$

is not negative semidefinite. So the condition of Theorem 3.5 in [1] is not satisfied, one can not determine the dissipativity of the neural network. Therefore, our obtained criteria for the global dissipativity of integro-differential neural networks with time delay are new.

Conclusion. The global robust dissipativity problem of integro-differential neural networks with time delay is discussed. Several results are presented to characterize the global dissipation together with their sets of attraction, which might have an impact in the studying the uniqueness of equilibria, the global asymptotic stability, the instability and the existence of periodic solutions. Our results extend the earlier works. Two examples are given to illustrate the correctness of our results.

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