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A REGULARIZED TRACE FORMULA FOR A DISCONTINUOUS STURM-LIOUVILLE OPERATOR WITH DELAYED ARGUMENT

MEHMET BAYRAMOGLU, AZAD BAYRAMOV, ERDOĞAN ŞEN

Abstract. In this study, we obtain a formula for the regularized sums of eigenvalues for a Sturm-Liouville problem with delayed argument at two points of discontinuity.

1. Introduction

We consider the boundary value problem for the differential equation

$$y''(x) + q(x)y(x - \Delta(x)) + \lambda^2 y(x) = 0, \tag{1.1}$$

on $[0, h_1) \cup (h_1, h_2) \cup (h_2, \pi]$, with boundary conditions

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \tag{1.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0 \tag{1.3}$$

and transmission conditions

$$y(h_1 - 0) = \delta y(h_1 + 0), \tag{1.4}$$

$$y'(h_1 - 0) = \delta y'(h_1 + 0), \tag{1.5}$$

$$y(h_2 - 0) = \gamma y(h_2 + 0), \tag{1.6}$$

$$y'(h_2 - 0) = \gamma y'(h_2 + 0) \tag{1.7}$$

where the real-valued function $q(x)$ is continuous in $[0, h_1) \cup (h_1, h_2) \cup (h_2, \pi]$ and has finite limits

$$q(h_1 \pm 0) = \lim_{x \to h_1 \pm 0} q(x), \quad q(h_2 \pm 0) = \lim_{x \to h_2 \pm 0} q(x),$$

the real-valued function $\Delta(x) \geq 0$ is continuous in $[0, h_1) \cup (h_1, h_2) \cup (h_2, \pi]$ has finite limits

$$\Delta(h_1 \pm 0) = \lim_{x \to h_1 \pm 0} \Delta(x), \quad \Delta(h_2 \pm 0) = \lim_{x \to h_2 \pm 0} \Delta(x),$$

if $x \in [0, h_1)$, then $x - \Delta(x) \geq 0$; if $x \in (h_1, h_2)$, then $x - \Delta(x) \geq h_1$; if $x \in (h_2, \pi]$, then $x - \Delta(x) \geq h_2$; where λ is a spectral parameter; $h_1, h_2, \alpha, \beta, \delta \neq 0$, $\gamma \neq 0$ are arbitrary real numbers such that $0 < h_1 < h_2 < \pi$ and $\sin\alpha \sin\beta \neq 0$.

The goal of this article is to calculate the regularized trace for the problem (1.1)–(1.7).

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The theory of regularized trace of ordinary differential operators has long history. Gelfand and Levitan [4] first obtained the trace formula for the Sturm-Liouville differential equation. After this study several mathematicians were interested in developing trace formulas for different differential operators. The current situation of this subject and studies related to it are presented in the comprehensive survey paper [10]. The regularized trace formulas for differential equations with the operator coefficients are found in [1, 2, 5, 6, 7, 12]. However, there are a small number of works on the regularized trace for the differential equations with retarded argument; see [9, 13].

Pikula [9] investigated the problem of second order:

$$\begin{gathered} -y'' + q(x)y(x-\tau) = \lambda y, \\ y'(0) - hy(0) = y'(\pi) + Hy(\pi) = 0, \\ y(x-\tau) = y(0)\varphi(x-\tau), \quad x \le \tau,\ \varphi(0) = 1 \end{gathered}$$

and obtained trace formula of first order: if $\tau \ge \pi$, then

$$\sum_{n=1}^{\infty}(\lambda_n(\tau) - n^2) = q(0)\varphi(-\tau) - \frac{h^2+H^2}{2}$$

and if $\tau \le \pi$, then

$$\begin{aligned} &\sum_{n=1}^{\infty}\Big[\lambda_n(\tau) - n^2 - \frac{2}{\pi}\Big(h + H + \frac{\cos n\tau}{2}\int_\tau^\pi q(t)dt)\Big)\Big] \\ &= q(0)\varphi(-\tau) - \frac{h^2+H^2}{2} + \frac{h+H}{2\pi}\int_\tau^\pi q(t)dt + \Big[\frac{q(\tau)+q(\pi)}{4} - q(\tau)\Big]\frac{\pi-\tau}{\pi} \\ &\quad + \Big[\frac{b}{4} - \Big(\frac{1}{\sqrt{8}}\int_\tau^\pi q(t)dt\Big)^2\Big]\frac{\pi - 2\tau}{\pi}. \end{aligned}$$

Here the definition of b can be found in [9]. Yang [13] obtained formulas of the first regularized trace for a discontinuous boundary value problem with retarded argument and with transmission conditions at the one point of discontinuity. As in this problem, if the retardation function $\Delta \equiv 0$ in (1.1) and $\delta = 1, \gamma = 1$ we have the formula of the first regularized trace for the classical Sturm-Liouville operator (see [4]) which is called Gelfand-Levitan formula [10].

Note that, the trace formulas can be used in the inverse problems of spectral analysis of differential equations [10] and have applications in the approximate calculation of the eigenvalues of the related operator [3, 10].

2. A formula for the regularized trace

First we give the asymptotic behavior of solutions for large values of the spectral parameter. Let $\omega_1(x,\lambda)$ be a solution of (1.1) on $[0,h_1]$, satisfying the initial conditions

$$\omega_1(0,\lambda) = \sin\alpha \quad \text{and} \quad \omega_1'(0,\lambda) = -\cos\alpha. \tag{2.1}$$

These conditions define a unique solution of (1.1) on $[0,h_1]$ [8, p. 12].

After defining the above solution, then we will define the solution $\omega_2(x,\lambda)$ of (1.1) on $[h_1,h_2]$ by means of the solution $\omega_1(x,\lambda)$ using the initial conditions

$$\omega_2(h_1,\lambda) = \delta^{-1}\omega_1(h_1,\lambda) \quad \text{and} \quad \omega_2'(h_1,\ \lambda) = \delta^{-1}\omega_1'(h_1,\lambda). \tag{2.2}$$

These conditions define a unique solution of (1.1) on $[h_1,h_2]$.

After describing the above solution, then we will give the solution $\omega_3(x,\lambda)$ of (1.1) on $[h_2,\pi]$ by means of the solution $\omega_2(x,\lambda)$ using the initial conditions

$$\omega_3(h_2,\lambda)=\gamma^{-1}\omega_2(h_2,\lambda) \quad \text{and} \quad \omega_3'(h_2,\lambda)=\gamma^{-1}\omega_2'(h_2,\lambda). \tag{2.3}$$

These conditions define a unique solution of (1.1) on $[h_2,\pi]$.

Consequently, the function $\omega(x,\lambda)$ is defined on $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$ by

$$\omega(x,\lambda)=\begin{cases}\omega_1(x,\lambda), & x\in[0,h_1),\\ \omega_2(x,\lambda), & x\in(h_1,h_2),\\ \omega_3(x,\lambda), & x\in(h_2,\pi]\end{cases} \tag{2.4}$$

is a solution of (1.1) on $[0,h_1)\cup(h_1,h_2)\cup(h_2,\pi]$, which satisfies one of the boundary conditions and the transmission conditions (1.4)–(1.7) Then the following integral equations hold:

$$\begin{aligned}\omega_1(x,\lambda)&=\sin\alpha\cos\lambda x-\frac{\cos\alpha}{\lambda}\sin\lambda x\\ &\quad-\frac{1}{\lambda}\int_0^x q(\tau)\sin\lambda(x-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau,\end{aligned} \tag{2.5}$$

$$\begin{aligned}\omega_2(x,\lambda)&=\frac{1}{\delta}\omega_1(h_1,\lambda)\cos\lambda(x-h_1)+\frac{\omega_1'(h_1,\lambda)}{\lambda\delta}\sin\lambda(x-h_1)\\ &\quad-\frac{1}{\lambda}\int_{h_1}^x q(\tau)\sin\lambda(x-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau,\end{aligned} \tag{2.6}$$

$$\begin{aligned}\omega_3(x,\lambda)&=\frac{1}{\gamma}\omega_2(h_2,\lambda)\cos\lambda(x-h_2)+\frac{\omega_2'(h_2,\lambda)}{\lambda\gamma}\sin\lambda(x-h_2)\\ &\quad-\frac{1}{\lambda}\int_{h_2}^x q(\tau)\sin\lambda(x-\tau)\omega_3(\tau-\Delta(\tau),\lambda)d\tau.\end{aligned} \tag{2.7}$$

Differentiating (2.5) and (2.6) with respect to x, we obtain

$$\omega_1'(x,\lambda)=-\lambda\sin\alpha\sin\lambda x-\cos\alpha\cos\lambda x-\int_0^x q(\tau)\cos\lambda(x-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau, \tag{2.8}$$

$$\begin{aligned}\omega_2'(x,\lambda)&=-\frac{\lambda}{\delta}\omega_1(h_1,\lambda)\sin\lambda(x-h_1)+\frac{\omega_1'(h_1,\lambda)}{\delta}\cos\lambda(x-h_1)\\ &\quad-\int_{h_1}^x q(\tau)\cos\lambda(x-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\omega_3'(x,\lambda)&=-\frac{\lambda}{\gamma}\omega_2(h_2,\lambda)\sin\lambda(x-h_2)+\frac{\omega_2'(h_2,\lambda)}{\gamma}\cos\lambda(x-h_2)\\ &\quad-\int_{h_2}^x q(\tau)\cos\lambda(x-\tau)\omega_3(\tau-\Delta(\tau),\lambda)d\tau,\end{aligned} \tag{2.10}$$

Moreover

$$|\omega_1(x,\lambda)|<\text{const.},\quad x\in[0,h_1],$$

see [11]. Then from (2.5) we have

$$
\begin{aligned}
&\omega_1(x,\lambda)\\
&= \sin\alpha\cos\lambda x - \frac{\cos\alpha}{\lambda}\sin\lambda x - \frac{\sin\alpha}{\lambda}\int_0^x q(\tau)\sin\lambda(x-\tau)\cos\lambda(\tau-\Delta(\tau))d\tau\\
&\quad + \frac{\cos\alpha}{\lambda^2}\int_0^x q(\tau)\sin\lambda(x-\tau)\sin\lambda(\tau-\Delta(\tau))d\tau + O(\frac{1}{\lambda^3})\\
&= \sin\alpha\cos\lambda x - \frac{\cos\alpha}{\lambda}\sin\lambda x - \frac{\sin\alpha}{2\lambda}\int_0^x q(\tau)\sin\lambda(x-\Delta(\tau))d\tau\\
&\quad - \frac{\sin\alpha}{2\lambda}\int_0^x q(\tau)\sin\lambda(x-2\tau+\Delta(\tau))d\tau - \frac{\cos\alpha}{2\lambda^2}\int_0^x q(\tau)\cos\lambda(x-\Delta(\tau))d\tau\\
&\quad + \frac{\cos\alpha}{2\lambda^2}\int_0^x q(\tau)\cos\lambda(x-2\tau+\Delta(\tau))d\tau + O(\frac{1}{\lambda^3}).
\end{aligned}
\tag{2.11}
$$

Therefore,

$$
\begin{aligned}
&\omega_2(x,\lambda)\\
&= \frac{\sin\alpha}{\delta}\cos\lambda x - \frac{\cos\alpha}{\lambda\delta}\sin\lambda x - \frac{\sin\alpha}{2\lambda\delta}\int_0^x q(\tau)\sin\lambda(x-\Delta(\tau))d\tau\\
&\quad - \frac{\sin\alpha}{2\lambda\delta}\int_0^x q(\tau)\sin\lambda(x-2\tau+\Delta(\tau))d\tau - \frac{\cos\alpha}{2\lambda^2\delta}\int_0^x q(\tau)\cos\lambda(x-\Delta(\tau))d\tau\\
&\quad + \frac{\cos\alpha}{2\lambda^2\delta}\int_0^x q(\tau)\cos\lambda(x-2\tau+\Delta(\tau))d\tau + O(\frac{1}{\lambda^3}).
\end{aligned}
\tag{2.12}
$$

and

$$
\begin{aligned}
&\omega_3(x,\lambda)\\
&= \frac{\sin\alpha}{\delta\gamma}\cos\lambda x - \frac{\cos\alpha}{\lambda\delta\gamma}\sin\lambda x - \frac{\sin\alpha}{2\lambda\delta\gamma}\int_0^x q(\tau)\sin\lambda(x-\Delta(\tau))d\tau\\
&\quad - \frac{\sin\alpha}{2\lambda\delta\gamma}\int_0^x q(\tau)\sin\lambda(x-2\tau+\Delta(\tau))d\tau - \frac{\cos\alpha}{2\lambda^2\delta\gamma}\int_0^x q(\tau)\cos\lambda(x-\Delta(\tau))d\tau\\
&\quad + \frac{\cos\alpha}{2\lambda^2\delta\gamma}\int_0^x q(\tau)\cos\lambda(x-2\tau+\Delta(\tau))d\tau + O(\frac{1}{\lambda^3}).
\end{aligned}
\tag{2.13}
$$

The solution $\omega(x,\lambda)$ defined above is a nontrivial solution of (1.1) satisfying conditions (1.2) and (1.4)–(1.7). Putting $\omega(x,\lambda)$ in (1.3), we obtain the characteristic equation

$$
W(\lambda) \equiv \omega(\pi,\lambda)\cos\beta + \omega'(\pi,\lambda)\sin\beta = 0. \tag{2.14}
$$

The set of eigenvalues of boundary value problem (1.1)–(1.7) coincides with the set of the squares of roots of (2.14), and eigenvalues are simple. From (2.5)-(2.10) and (2.14) we obtain

$$
\begin{aligned}
&W(\lambda)\\
&\equiv \Big\{\frac{1}{\gamma}\Big[\frac{1}{\delta}(\sin\alpha\cos\lambda h_1 - \frac{\cos\alpha}{\lambda}\sin\lambda h_1\\
&\quad - \frac{1}{\lambda}\int_0^{h_1} q(\tau)\sin\lambda(h_1-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau)\cos\lambda(h_2-h_1)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\lambda\delta}\Big(\lambda\sin\alpha\sin\lambda h_1+\cos\alpha\cos\lambda h_1\\
&+\int_0^{h_1}q(\tau)\cos\lambda(h_1-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\Big)\sin\lambda(h_2-h_1)\\
&-\frac{1}{\lambda}\int_{h_1}^{h_2}q(\tau)\sin\lambda(h_2-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau\Big]\cos\lambda(\pi-h_2)\\
&+\frac{1}{\lambda\gamma}\Big[-\frac{\lambda}{\delta}\Big(\sin\alpha\cos\lambda h_1-\frac{\cos\alpha}{\lambda}\sin\lambda h_1\\
&-\frac{1}{\lambda}\int_0^{h_1}q(\tau)\sin\lambda(h_1-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\Big)\sin\lambda(h_2-h_1)\\
&-\frac{1}{\delta}(\lambda\sin\alpha\sin\lambda h_1+\cos\alpha\cos\lambda h_1\\
&+\int_0^{h_1}q(\tau)\cos\lambda(h_1-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau)\cos\lambda(h_2-h_1)\\
&-\int_{h_1}^{h_2}q(\tau)\cos\lambda(h_2-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau\Big]\sin\lambda(\pi-h_2)\\
&-\frac{1}{\lambda}\int_{h_2}^{\pi}q(\tau)\sin\lambda(\pi-\tau)\omega_3(\tau-\Delta(\tau),\lambda)d\tau\Big\}\cos\beta\\
&+\Big\{-\frac{\lambda}{\gamma}\Big[\frac{1}{\delta}\Big(\sin\alpha\cos\lambda h_1-\frac{\cos\alpha}{\lambda}\sin\lambda h_1\\
&-\frac{1}{\lambda}\int_0^{h_1}q(\tau)\sin\lambda(h_1-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\Big)\cos\lambda(h_2-h_1)\\
&-\frac{1}{\lambda\delta}\Big(\lambda\sin\alpha\sin\lambda h_1+\cos\alpha\cos\lambda h_1+\int_0^{h_1}q(\tau)\cos\lambda(h_1-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\Big)\\
&\times\sin\lambda(h_2-h_1)-\frac{1}{\lambda}\int_{h_1}^{h_2}q(\tau)\sin\lambda(h_2-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau\Big]\sin\lambda(\pi-h_2)\\
&+\frac{1}{\gamma}\Big[-\frac{\lambda}{\delta}\Big(\sin\alpha\cos\lambda h_1-\frac{\cos\alpha}{\lambda}\sin\lambda h_1\\
&-\frac{1}{\lambda}\int_0^{h_1}q(\tau)\sin\lambda(h_1-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\Big)\sin\lambda(h_2-h_1)\\
&-\frac{1}{\delta}\Big(\lambda\sin\alpha\sin\lambda h_1+\cos\alpha\cos\lambda h_1\\
&+\int_0^{h_1}q(\tau)\cos\lambda(h_1-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau\Big)\cos\lambda(h_2-h_1)\\
&-\int_{h_1}^{h_2}q(\tau)\cos\lambda(h_2-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau\Big]\cos\lambda(\pi-h_2)\\
&-\int_{h_2}^{\pi}q(\tau)\cos\lambda(\pi-\tau)\omega_3(\tau-\Delta(\tau),\lambda)d\tau\Big\}\sin\beta,
\end{aligned}
$$

which implies

$$
\begin{aligned}
W(\lambda) = & -\frac{\sin\alpha\sin\beta}{\delta\gamma}\lambda\sin\lambda\pi + \frac{\sin(\alpha-\beta)}{\delta\gamma}\cos\lambda\pi - \frac{\cos\alpha\cos\beta}{\lambda\delta}\sin\lambda\pi \\
& - \frac{\sin\beta}{\delta\gamma}\int_0^{h_1} q(\tau)\cos\lambda(\pi-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau \\
& - \frac{\sin\beta}{\gamma}\int_{h_1}^{h_2} q(\tau)\cos\lambda(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau \\
& - \sin\beta\int_{h_2}^{\pi} q(\tau)\cos\lambda(\pi-\tau)\omega_3(\tau-\Delta(\tau),\lambda)d\tau \\
& - \frac{\cos\beta}{\lambda\delta\gamma}\int_0^{h_1} q(\tau)\sin\lambda(\pi-\tau)\omega_1(\tau-\Delta(\tau),\lambda)d\tau \\
& - \frac{\cos\beta}{\lambda\gamma}\int_{h_1}^{h_2} q(\tau)\sin\lambda(\pi-\tau)\omega_2(\tau-\Delta(\tau),\lambda)d\tau \\
& - \frac{\cos\beta}{\lambda}\int_{h_2}^{\pi} q(\tau)\sin\lambda(\pi-\tau)\omega_3(\tau-\Delta(\tau),\lambda)d\tau.
\end{aligned} \tag{2.15}
$$

From (2.11)-(2.13), we obtain

$$
\begin{aligned}
& \omega_1(\tau-\Delta(\tau),\lambda) \\
& = \sin\alpha\cos\lambda(\tau-\Delta(\tau)) - \frac{\cos\alpha}{\lambda}\sin\lambda(\tau-\Delta(\tau)) \\
& \quad - \frac{\sin\alpha}{2\lambda}\int_0^{\tau-\Delta(\tau)} q(t_1)\sin\lambda(\tau-\Delta(\tau)-\Delta(t_1))dt_1 \\
& \quad - \frac{\sin\alpha}{2\lambda}\int_0^{\tau-\Delta(\tau)} q(t_1)\sin\lambda(\tau-\Delta(\tau)-2t_1+\Delta(t_1))dt_1 + O(\frac{1}{\lambda^3}),
\end{aligned} \tag{2.16}
$$

$$
\begin{aligned}
\omega_2(\tau-\Delta(\tau),\lambda) = & \frac{\sin\alpha}{\delta}\cos\lambda(\tau-\Delta(\tau)) - \frac{\cos\alpha}{\lambda\delta}\sin\lambda(\tau-\Delta(\tau)) \\
& - \frac{\sin\alpha}{2\lambda\delta}\int_0^{\tau-\Delta(\tau)} q(t_1)\sin\lambda(\tau-\Delta(\tau)-\Delta(t_1))dt_1,
\end{aligned} \tag{2.17}
$$

$$
-\frac{\sin\alpha}{2\lambda\delta}\int_0^{\tau-\Delta(\tau)} q(t_1)\sin\lambda(\tau-\Delta(\tau)-2t_1+\Delta(t_1))dt_1 + O(\frac{1}{\lambda^3}), \tag{2.18}
$$

$$
\begin{aligned}
& \omega_3(\tau-\Delta(\tau),\lambda) \\
& = \frac{\sin\alpha}{\delta\gamma}\cos\lambda(\tau-\Delta(\tau)) - \frac{\cos\alpha}{\lambda\delta\gamma}\sin\lambda(\tau-\Delta(\tau)) \\
& \quad - \frac{\sin\alpha}{2\lambda\delta\gamma}\int_0^{\tau-\Delta(\tau)} q(t_1)\sin\lambda(\tau-\Delta(\tau)-\Delta(t_1))dt_1 \\
& \quad - \frac{\sin\alpha}{2\lambda\delta\gamma}\int_0^{\tau-\Delta(\tau)} q(t_1)\sin\lambda(\tau-\Delta(\tau)-2t_1+\Delta(t_1))dt_1 + O(\frac{1}{\lambda^3}),
\end{aligned} \tag{2.19}
$$

Using (2.15)–(2.19) we obtain

$$
\begin{aligned}
&W(\lambda)\\
&\equiv -\frac{\sin\alpha\sin\beta}{\delta\gamma}\lambda\sin\lambda\pi+\frac{\sin(\alpha-\beta)}{\delta\gamma}\cos\lambda\pi-\frac{\cos\alpha\cos\beta}{\lambda\delta\gamma}\sin\lambda\pi\\
&\quad-\frac{\sin\alpha\sin\beta}{2\delta\gamma}\int_0^\pi q(\tau)[\cos\lambda(\pi-\Delta(\tau))+\cos\lambda(\pi-2\tau+\Delta(\tau))]d\tau\\
&\quad-\frac{\sin\alpha\cos\beta}{2\lambda\delta\gamma}\int_0^\pi q(\tau)[\sin\lambda(\pi-\Delta(\tau))+\sin\lambda(\pi-2\tau+\Delta(\tau))]d\tau\\
&\quad+\frac{\cos\alpha\sin\beta}{2\lambda\delta\gamma}\int_0^\pi q(\tau)[\sin\lambda(\pi-\Delta(\tau))-\sin\lambda(\pi-2\tau+\Delta(\tau))]d\tau\\
&\quad+\frac{\sin\alpha\sin\beta}{4\lambda\delta\gamma}\int_0^\pi q(\tau)\Big[\int_0^{\tau-\Delta(\tau)}q(t_1)\sin\lambda(\pi-\Delta(\tau)-\Delta(t_1))dt_1\Big]d\tau\\
&\quad-\frac{\sin\alpha\sin\beta}{4\lambda\delta\gamma}\int_0^\pi q(\tau)[\int_0^{\tau-\Delta(\tau)}q(t_1)\sin\lambda(\pi-2\tau+\Delta(\tau)+\Delta(t_1))dt_1]d\tau\\
&\quad+\frac{\sin\alpha\sin\beta}{4\lambda\delta\gamma}\int_0^\pi q(\tau)\Big[\int_0^{\tau-\Delta(\tau)}q(t_1)\sin\lambda(\pi-\Delta(\tau)-2t_1+\Delta(t_1))dt_1\Big]d\tau\\
&\quad-\frac{\sin\alpha\sin\beta}{4\lambda\delta\gamma}\int_0^\pi q(\tau)\Big[\int_0^{\tau-\Delta(\tau)}q(t_1)\sin\lambda(\pi-2\tau+\Delta(\tau)\\
&\quad+2t_1-\Delta(t_1))dt_1\Big]d\tau+O(\frac{1}{\lambda^3}).
\end{aligned}
\tag{2.20}
$$

Denote

$$
\begin{gathered}
A_1(\lambda)=\frac12\int_0^\pi q(\tau)\cos\lambda\Delta(\tau)d\tau,\quad A_2(\lambda)=\frac12\int_0^\pi q(\tau)\sin\lambda\Delta(\tau)d\tau,\\
B_1(\lambda)=\frac14\int_0^\pi q(\tau)\Big[\int_0^{\tau-\Delta(\tau)}q(t_1)\cos\lambda(2\tau-\Delta(\tau)-\Delta(t_1))dt_1\Big]d\tau,\\
B_2(\lambda)=\frac14\int_0^\pi q(\tau)\Big[\int_0^{\tau-\Delta(\tau)}q(t_1)\sin\lambda(2\tau-\Delta(\tau)-\Delta(t_1))dt_1\Big]d\tau,\\
C_1(\lambda)=\frac14\int_0^\pi q(\tau)\Big[\int_0^{\tau-\Delta(\tau)}q(t_1)\cos\lambda(\Delta(\tau)+\Delta(t_1))dt_1\Big]d\tau,\\
C_2(\lambda)=\frac14\int_0^\pi q(\tau)\Big[\int_0^{\tau-\Delta(\tau)}q(t_1)\sin\lambda(\Delta(\tau)+\Delta(t_1))dt_1\Big]d\tau,\\
D_1(\lambda)=\frac12\int_0^\pi q(\tau)\cos\lambda(2\tau-\Delta(\tau))d\tau,\\
D_2(\lambda)=\frac12\int_0^\pi q(\tau)\sin\lambda(2\tau-\Delta(\tau))d\tau,\\
E_1(\lambda)=\frac14\int_0^\pi q(\tau)\Big[\int_0^{\tau-\Delta(\tau)}q(t_1)\cos\lambda(\Delta(\tau)+2t_1-\Delta(t_1))dt_1\Big]d\tau,\\
E_2(\lambda)=\frac14\int_0^\pi q(\tau)\Big[\int_0^{\tau-\Delta(\tau)}q(t_1)\sin\lambda(\Delta(\tau)+2t_1-\Delta(t_1))dt_1\Big]d\tau,
\end{gathered}
$$

$$
\begin{aligned}
H_1(\lambda) &= \frac{1}{4}\int_0^{\pi} q(\tau)\Big[\int_0^{\tau-\Delta(\tau)} q(t_1)\cos\lambda(2\tau-\Delta(\tau)-2t_1+\Delta(t_1))dt_1\Big]d\tau,\\
H_2(\lambda) &= \frac{1}{4}\int_0^{\pi} q(\tau)\Big[\int_0^{\tau-\Delta(\tau)} q(t_1)\sin\lambda(2\tau-\Delta(\tau)-2t_1+\Delta(t_1))dt_1\Big]d\tau.
\end{aligned}
$$

Then from (2.20), we obtain

$$
\begin{aligned}
&W(\lambda)\\
&\equiv -\frac{\sin\alpha\sin\beta}{\delta\gamma}\lambda\sin\lambda\pi+\frac{\sin(\alpha-\beta)}{\delta\gamma}\cos\lambda\pi-\frac{\cos\alpha\cos\beta}{\lambda\delta\gamma}\sin\lambda\pi\\
&\quad -\frac{\sin\alpha\sin\beta}{\delta\gamma}[A_1(\lambda)\cos\lambda\pi+A_2(\lambda)\sin\lambda\pi+D_1(\lambda)\cos\lambda\pi+D_2(\lambda)\sin\lambda\pi]\\
&\quad -\frac{\sin\alpha\cos\beta}{\lambda\delta\gamma}[A_1(\lambda)\sin\lambda\pi-A_2(\lambda)\cos\lambda\pi+D_1(\lambda)\sin\lambda\pi-D_2(\lambda)\cos\lambda\pi]\\
&\quad +\frac{\cos\alpha\sin\beta}{\lambda\delta\gamma}[A_1(\lambda)\sin\lambda\pi-A_2(\lambda)\cos\lambda\pi-D_1(\lambda)\sin\lambda\pi+D_2(\lambda)\cos\lambda\pi]\\
&\quad +\frac{\sin\alpha\sin\beta}{\lambda\delta\gamma}[C_1(\lambda)\sin\lambda\pi-C_2(\lambda)\cos\lambda\pi]\\
&\quad -\frac{\sin\alpha\sin\beta}{\lambda\delta\gamma}[B_1(\lambda)\sin\lambda\pi-B_2(\lambda)\cos\lambda\pi]\\
&\quad +\frac{\sin\alpha\sin\beta}{\lambda\delta\gamma}[E_1(\lambda)\sin\lambda\pi-E_2(\lambda)\cos\lambda\pi]\\
&\quad -\frac{\sin\alpha\sin\beta}{\lambda\delta\gamma}[H_1(\lambda)\sin\lambda\pi-H_2(\lambda)\cos\lambda\pi]+O(\frac{1}{\lambda^3}).
\end{aligned}
\tag{2.21}
$$

Define

$$
W_0(\lambda)\equiv -\frac{\sin\alpha\sin\beta}{\delta\gamma}\lambda\sin\lambda\pi. \tag{2.22}
$$

Denote by $\lambda_n^0, n\in\mathbb{Z}$, zeros of the function $W_0(\lambda)$, then we have $\lambda_n^0=n, n\in\mathbb{Z}$, and it is simple algebraically except for λ_0^0.

Denote by C_n the circle of radius, $0<\varepsilon<\frac{1}{2}$, centered at the origin $\lambda_n^0=n, n\in\mathbb{Z}$, and by Γ_{N_0} the counterclockwise square contours with four vertices

$$
\begin{aligned}
A=N_0+\varepsilon+N_0 i,\quad B=-N_0-\varepsilon+N_0 i,\\
C=-N_0-\varepsilon-N_0 i,\quad D=N_0+\varepsilon-N_0 i,
\end{aligned}
$$

where $i=\sqrt{-1}$ and N_0 is a natural number. Obviously, if $\lambda\in C_n$ or $\lambda\in\Gamma_{N_0}$, then $|W_0(\lambda)|\ge M|\lambda|e^{|\operatorname{Im}\lambda|\pi}$ $(M>0)$ by using a similar method in [14]. Thus, on

$\lambda \in C_n$ or $\lambda \in \Gamma_{N_0}$, from (2.21) and (2.22), we have

$$
\begin{aligned}
&\frac{W(\lambda)}{W_0(\lambda)} \\
&= 1 + \frac{\cot\alpha - \cot\beta + A_1(\lambda) + D_1(\lambda)}{\lambda}\cot\lambda\pi + \frac{A_2(\lambda) + D_2(\lambda)}{\lambda} \\
&\quad + \Big((\cot\alpha - \cot\beta)A_2(\lambda) - (\cot\alpha + \cot\beta)D_2(\lambda) + C_2(\lambda) - B_2(\lambda) \\
&\quad + E_2(\lambda) - H_2(\lambda)\Big), \cot(\lambda\pi)/\lambda^2 \\
&\quad + \frac{\cot\alpha\cot\beta + (\cot\beta - \cot\alpha)A_1(\lambda) + (\cot\alpha + \cot\beta)D_1(\lambda)}{\lambda^2} \\
&\quad + \frac{-C_1(\lambda) + B_1(\lambda) - E_1(\lambda) + H_1(\lambda)}{\lambda^2} + O(\frac{1}{\lambda^3}).
\end{aligned} \tag{2.23}
$$

Expanding $\ln\frac{W(\lambda)}{W_0(\lambda)}$ by the Maclaurin formula, we find that

$$
\begin{aligned}
&\ln\frac{W(\lambda)}{W_0(\lambda)} \\
&= \frac{\cot\alpha - \cot\beta + A_1(\lambda) + D_1(\lambda)}{\lambda}\cot\lambda\pi + \frac{A_2(\lambda) + D_2(\lambda)}{\lambda} \\
&\quad + \Big((\cot\alpha - \cot\beta)A_2(\lambda) - (\cot\alpha + \cot\beta)D_2(\lambda) + C_2(\lambda) - B_2(\lambda) \\
&\quad + E_2(\lambda) - H_2(\lambda)\Big)\cot(\lambda\pi)/\lambda^2 \\
&\quad + \frac{\cot\alpha\cot\beta + (\cot\beta - \cot\alpha)A_1(\lambda) + (\cot\alpha + \cot\beta)D_1(\lambda)}{\lambda^2} \\
&\quad + \frac{-C_1(\lambda) + B_1(\lambda) - E_1(\lambda) + H_1(\lambda)}{\lambda^2} \\
&\quad - \frac{1}{2}\Big[\frac{(\cot\alpha - \cot\beta + A_1(\lambda) + D_1(\lambda))^2}{\lambda^2}\cot^2\lambda\pi + \frac{(A_2(\lambda) + D_2(\lambda))^2}{\lambda^2} \\
&\quad + 2\frac{(\cot\alpha - \cot\beta + A_1(\lambda) + D_1(\lambda))(A_2(\lambda) + D_2(\lambda))}{\lambda^2}\cot\lambda\pi\Big] + O(\frac{1}{\lambda^3}).
\end{aligned} \tag{2.24}
$$

It is well known (see [11]) that the spectrum of problem (1.1)–(1.7) is discrete and

$$
\lambda_n \sim n + O(1) \quad \text{as } |n| \to \infty. \tag{2.25}
$$

Next we present the more exact asymptotic distribution of the spectrum. Using the residue theorem we have

$$
\begin{aligned}
\lambda_n - n &= -\frac{1}{2\pi i}\oint_{C_n}\ln\frac{W(\lambda)}{W_0(\lambda)}d\lambda \\
&= -\frac{1}{2\pi i}\oint_{C_n}\frac{(\cot\alpha - \cot\beta + A_1(\lambda) + D_1(\lambda))\cot\lambda\pi}{\lambda}d\lambda \\
&\quad - \frac{1}{2\pi i}\oint_{C_n}\frac{A_2(\lambda) + D_2(\lambda)}{\lambda}d\lambda \\
&\quad - \frac{1}{2\pi i}\oint_{C_n}\Big[(\cot\alpha - \cot\beta)A_2(\lambda) - (\cot\alpha + \cot\beta)D_2(\lambda) \\
&\quad + C_2(\lambda) - B_2(\lambda) + E_2(\lambda) - H_2(\lambda)\Big]\frac{\cot\lambda\pi}{\lambda^2}d\lambda
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2\pi i}\oint_{C_n}\Big[\cot\alpha\cot\beta+(\cot\beta-\cot\alpha)A_1(\lambda)+(\cot\alpha+\cot\beta)D_1(\lambda)\\
&-C_1(\lambda)+B_1(\lambda)-E_1(\lambda)+H_1(\lambda)\Big]\frac{d\lambda}{\lambda^2}\\
&+\frac{1}{2\pi i}\oint_{C_n}\frac{(\cot\alpha-\cot\beta+A_1(\lambda)+D_1(\lambda))^2\cot^2\lambda\pi}{2\lambda^2}d\lambda\\
&+\frac{1}{2\pi i}\oint_{C_n}\frac{(A_2(\lambda)+D_2(\lambda))^2}{2\lambda^2}d\lambda\\
&+\frac{1}{2\pi i}\oint_{C_n}\frac{(\cot\alpha-\cot\beta+A_1(\lambda)+D_1(\lambda))(A_2(\lambda)+D_2(\lambda))\cot\lambda\pi}{\lambda^2}d\lambda\\
&+O(\frac{1}{n^3}).
\end{aligned}
$$

which implies, using residue calculation, that

$$
\begin{aligned}
\lambda_n &= n+\frac{\cot\beta-\cot\alpha-A_1(n)-D_1(n)}{n\pi}\\
&\quad+\Big(2D_2(n)\cot\alpha+(A_1(n)+D_1(n))(A_2(n)+D_2(n))-C_2(n)\\
&\quad+B_2(n)-E_2(n)+H_2(n)\Big)/(n^2\pi)\\
&\quad-\frac{(\cot\alpha-\cot\beta+A_1(n)+D_1(n))^2}{n^3}+O(\frac{1}{n^3}).
\end{aligned}
\tag{2.26}
$$

Thus we have proven the following theorem.

Theorem 2.1. *The spectrum of problem* (1.1)–(1.7) *has the* (2.26) *asymptotic distribution for sufficiently large* $|n|$.

Finally, we will get regularized trace formula for problem (1.1)–(1.7). The asymptotic formula (2.25) for the eigenvalues implies that for all sufficiently large N_0, the numbers λ_n with $|n|\le N_0$ are inside Γ_{N_0}, and the numbers λ_n with $|n|>N_0$ are outside Γ_{N_0}. It follows that

$$
\begin{aligned}
&\lambda_{-0}^2+\lambda_0^2+\sum_{0\neq n=-N_0}^{N_0}(\lambda_n^2-n^2)\\
&=-\frac{1}{2\pi i}\oint_{\Gamma_n}2\lambda\ln\frac{W(\lambda)}{W_0(\lambda)}d\lambda\\
&=-\frac{1}{2\pi i}\oint_{\Gamma_n}2(\cot\alpha-\cot\beta+A_1(\lambda)+D_1(\lambda))\cot\lambda\pi d\lambda\\
&-\frac{1}{2\pi i}\oint_{\Gamma_n}2(A_2(\lambda)+D_2(\lambda))d\lambda\\
&-\frac{1}{2\pi i}\oint_{\Gamma_n}2\Big[-2D_2(\lambda)\cot\alpha-(A_1(\lambda)+D_1(\lambda))(A_2(\lambda)+D_2(\lambda))\\
&+C_2(\lambda)-B_2(\lambda)+E_2(\lambda)-H_2(\lambda)\Big]\frac{\cot\lambda\pi}{\lambda}d\lambda\\
&-\frac{1}{2\pi i}\oint_{\Gamma_n}2\Big[\cot\alpha\cot\beta+(\cot\beta-\cot\alpha)A_1(\lambda)+(\cot\alpha+\cot\beta)D_1(\lambda)
\end{aligned}
$$

$$
\begin{aligned}
&- C_1(\lambda) + B_1(\lambda) - E_1(\lambda) + H_1(\lambda)\Big] \frac{d\lambda}{\lambda} \\
&+ \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{(\cot\alpha - \cot\beta + A_1(\lambda) + D_1(\lambda))^2 \cot^2 \lambda\pi}{\lambda} d\lambda \\
&+ \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{(A_2(\lambda) + D_2(\lambda))^2}{\lambda} d\lambda + O(\frac{1}{N_0})
\end{aligned}
$$

by calculations, which implies

$$
\begin{aligned}
&\lambda_{-0}^2 + \lambda_0^2 + \sum_{0\neq n=-N_0}^{N_0} (\lambda_n^2 - n^2) \\
&= -\frac{2}{\pi} \sum_{0\neq n=-N_0}^{N_0} (\cot\alpha - \cot\beta + A_1(n) + D_1(n)) \\
&\quad - \frac{2}{\pi}(\cot\alpha - \cot\beta + A_1(0) + D_1(0)) \\
&\quad - \sum_{0\neq n=-N_0}^{N_0} 2\big[- 2D_2(n)\cot\alpha - (A_1(n) + D_1(n))(A_2(n) + D_2(n)) \\
&\quad + C_2(n) - B_2(n) + E_2(n) - H_2(n)\Big] \frac{1}{n\pi} + T \\
&\quad - 2\Big[\cot\alpha \cot\beta + (\cot\beta - \cot\alpha)A_1(0) + (\cot\alpha + \cot\beta)D_1(0) \\
&\quad - C_1(0) + B_1(0) - E_1(0) + H_1(0)\Big] \\
&\quad - (\cot\alpha - \cot\beta + A_1(0) + D_1(0))^2 + (A_2(0) + D_2(0))^2 + O(\frac{1}{N_0}),
\end{aligned} \tag{2.27}
$$

where

$$
\begin{aligned}
T = \operatorname{Res}_{\lambda=0} \Big\{ 2\Big[&2D_2(\lambda)\cot\alpha + (A_1(\lambda) + D_1(\lambda))(A_2(\lambda) + D_2(\lambda)) \\
&- C_2(\lambda) + B_2(\lambda) - E_2(\lambda) + H_2(\lambda)\Big] \frac{\cot\lambda\pi}{\lambda} \Big\}.
\end{aligned}
$$

Passing to the limit as $N_0 \to \infty$ in (32), we have

$$
\begin{aligned}
&\lambda_{-0}^2 + \lambda_0^2 + \sum_{0\neq n=-\infty}^{\infty} \Big\{ \lambda_n^2 - n^2 - \frac{2}{\pi}[\cot\beta - \cot\alpha - A_1(n) - D_1(n)] \\
&+ \Big[- 2D_2(n)\cot\alpha + (A_1(n) + D_1(n))(A_2(n) + D_2(n)) \\
&+ C_2(n) - B_2(n) + E_2(n) + H_2(n)\Big] \frac{1}{n\pi} \Big\} \\
&= \frac{2}{\pi}(\cot\beta - \cot\alpha - A_1(0) - D_1(0)) - \cot^2\alpha - \cot^2\beta \\
&\quad - 2(\cot\alpha - \cot\beta)(A_1(0) + D_1(0)) - (A_1(0) + D_1(0))^2 \\
&\quad + (A_2(0) + D_2(0))^2 - 2\Big[\cot\alpha\cot\beta + (\cot\beta - \cot\alpha)A_1(0) \\
&\quad + (\cot\alpha + \cot\beta)D_1(0) - C_1(0) + B_1(0) - E_1(0) + H_1(0)\Big] + T.
\end{aligned} \tag{2.28}
$$

The series on the left side of this equality is called the regularized trace of the problem (1.1)–(1.7). The main result of this article is given by the following theorem.

Theorem 2.2. *Formula* (2.28) *holds for the first regularized trace of the problem* (1.1)*–*(1.7)*.*

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