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EXISTENCE, CHARACTERIZATION AND NUMBER OF GROUND STATES FOR COUPLED EQUATIONS

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Abstract. This article concerns the existence, characterization and number of ground states for the system consisting of m coupled semilinear equations

$$-\Delta u_i + \lambda u_i = \sum_{j=1}^{m} k_{ij} \frac{q_{ij}}{p+1} |u_j|^{p_{ij}} |u_i|^{q_{ij}-2} u_i, \quad x \in \Omega,$$
$$u_i \in H_0^1(\Omega), \quad i = 1, 2, \dots, m.$$

We extend the characterization results obtained by Correia [5, 6] to the above problem. Also we give a new characterization of the ground states, which provides a more convenient way for finding or checking ground states. This study may be the first result not only positive ground states but also for semi-trivial ground states, and it shows that the positive ground state is unique for some special cases.

1. Introduction and statement of main results

In this article, we study the existence, characterization and number of ground states of the system consisting of m semilinear equations

$$\begin{gathered} -\Delta u_i + \lambda u_i = \sum_{j=1}^{m} k_{ij} \frac{q_{ij}}{p+1} |u_j|^{p_{ij}} |u_i|^{q_{ij}-2} u_i, \quad x \in \Omega, \\ u_i \in H_0^1(\Omega), \quad i = 1, 2, \dots, m, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ may be bounded or $\mathbb{R}^N$ and $\{p_{ij}\}, \{q_{ij}\}, \{k_{ij}\} \subset \mathbb{R}$ satisfy the following assumptions:

(1) $1 < q_{ij} = p_{ji}$, $p_{ij} + q_{ij} = 2p + 2$, where $0 < p < 2/(N-2)$ if $N \geq 3$ and $0 < p < +\infty$ if $N = 1, 2$;
(2) $k_{ij} = k_{ji}$ and for any fixed $i \in \{1, 2, \dots, m\}$, there exists at least one $j \in \{1, 2, \dots, m\}$ such that $k_{ij} > 0$, which implies that the problem (1.1) may contain attraction and repulsion.

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Problem (1.1) arises when one looks for standing waves $\Psi(t,x) = e^{-\mathbf{i}\lambda t}U$ with $U = (u_1, u_2, \dots, u_m) \in \big(H_0^1(\Omega)\big)^m$ to the equations

$$\begin{gathered}\mathbf{i}(v_i)_t + \Delta v_i + \sum_{j=1}^m k_{ij}\frac{q_{ij}}{p+1}|v_j|^{p_{ij}}|v_i|^{q_{ij}-2}v_i = 0,\\ (t,x) \in \mathbb{R}^+ \times \Omega, \quad i = 1, 2, \dots, m,\end{gathered} \tag{1.2}$$

where $\mathbf{i}$ denotes the imaginary unit. This coupled system of nonlinear Schrödinger equations with power-type nonlinearities comes from physical problems, such as nonlinear optics and Bose-Einstein condensates. It models a physical system in which the field has more than one component. According to the results in [1], one can see that u_j denotes the jth component of the beam in Kerr-like photo-refractive media and the coupling constant k_{ij} acts to the interaction between the ith and the jth components of the beam. System (1.2) also stems from the Hartree-Fock theory for a m-component Bose-Einstein condensate. Readers can learn more about the derivation and applications of this system in[7, 18].

Because of both physical and mathematical reasons, the ground states are the most important solutions. At the same time, the existence, uniqueness and multiplicity of solutions are important characteristic. Therefore, we pay attention to the existence, uniqueness or multiplicity of the ground states. Researchers studied the existence, non-existence, and uniqueness of ground states to the scalar equation in [2, 9, 14, 16, 17] and the references therein. Results about ground states for 2 and 3 coupled systems can be found in [3, 4, 10, 13, 15]. Recently, Correia [5, 6] studied the system of m coupled equations

$$-\Delta u_i + \lambda u_i = \sum_{j=1}^m k_{ij}|u_j|^{p+1}|u_i|^{p-1}u_i, \quad x \in \Omega, i = 1, 2, \dots, m, \tag{1.3}$$

and not only presented sufficient conditions for the existence of nontrivial ground states, but also gave a characterization of the ground states. We want to point out that the characterization given by Correia depends heavily on the maximum point of a function constraint on the unit spherical surface, which makes difficult to find or check a ground state. As far as we know, there are no results on the uniqueness or multiplicity of the ground states of (1.1) and (1.3). So we want to study the existence, the form, and the number of the ground states of (1.1). Also we give sufficient conditions for the existence of nontrivial ground states, a new characterization which is easy to find or to check a ground state, and an estimate on the number of the ground states.

Before stating our results, we introduce some definitions and notation. We say a solution U of (1.1) is called positive if $U_i > 0$ for any $i \in \{1, 2, \dots, m\}$ and a solution U of (1.1) is nontrivial if $U \in H := \big(H_0^1(\Omega)\big)^m \setminus \{\vec{0}\}$. We denote by A_m the set consisting of all nontrivial solutions and call $U \in A_m$ a ground state of (1.1) if

$$S_m(U) := \frac{1}{2}I_m(U) - \frac{1}{2p+2}J_m(U) \le S_m(V), \quad \forall\, V \in A_m,$$

where

$$I_m(v) := \sum_{i=1}^m \int_\Omega (|Dv_i|^2 + \lambda|v_i|^2),$$

$$J_m(v) := \sum_{i,j=1}^{m} k_{ij} \int_\Omega |v_j|^{p_{ij}} |v_i|^{q_{ij}}.$$

We denote by G_m the set of ground states of (1.1). Let

$$G_m^+ := \{U \in G_m : u_i > 0,\ i = 1, 2, \dots, m\},$$

and for $n \in \{1, \dots, m\}$,

$$T_n := \{U \in G_m : U \text{ has exactly } n \text{ nontrivial components}\}$$

and

$$T_n^+ := \{U \in T_n : U \ \text{ has exactly } n \text{ positive components}\}.$$

For $\gamma > 0$, we define

$$I_m^\gamma := \inf_{J_m(u)=\gamma} I_m(u), \quad S_m^\gamma := \inf_{J_m(u)\geq\gamma} I_m(u),$$
$$\gamma_G := \big(\inf_{J_m(u)=1} I_m(u)\big)^{\frac{p+1}{p}}.$$

It is easy to check that $I_m^1 = \gamma^{-\frac{1}{p+1}} I_m^\gamma$ and $I_m^1 = \gamma_G^{\frac{p}{p+1}}$.

Let G be the set of the ground states to the equation

$$\begin{gathered} -\Delta u + \lambda u = u^{2p+1}, \quad u > 0,\ x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.4}$$

Let $|G|$ be the number of the elements in G, and $\lambda_1(\Omega)$ be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, if Ω is bounded and 0 if $\Omega = \mathbb{R}^N$. According to the results of [2, 9, 11, 12, 17], we can see that if $\lambda > -\lambda_1(\Omega)$, then $G \neq \emptyset$. In particular, when $\Omega = \mathbb{R}^N$ and $\lambda > 0$, $|G| = 1$. Moreover, ones can check that if $w \in G$, then

$$G = \{u : I_1(u) = \min_{J(Q)=J(w)} I_1(Q),\ J(u) = J(w)\},$$

where $J(v) = \int_\Omega |v|^{2p+2}$.

The above facts are very important for our results since the existence of the ground states of (1.1) depends heavily on the existence of the ground states of the scalar equation (1.4). If (1.4) has no ground states, so does (1.1), which will be presented in Theorems 1.2 and 1.5. Our result about the existence of the ground states of (1.1) can be stated as follows.

Theorem 1.1. *Assume that there exists $u \in H$ such that*

$$\sum_{i,j=1}^{m} k_{ij} \int_\Omega |u_j|^{p_{ij}} |u_i|^{q_{ij}} > 0.$$

If $\lambda > -\lambda_1(\Omega)$, then $G_m \neq \emptyset$.

Set $(\mathbb{R}_0^+)^m := \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0,\ \forall i = 1, \dots, m\}$, define

$$f(x) := \sum_{i,j=1}^{m} k_{ij} |x_j|^{p_{ij}} |x_i|^{q_{ij}}, \quad x \in (\mathbb{R}_0^+)^m \tag{1.5}$$

and let $X \subset (\mathbb{R}_0^+)^m$ be the set of solutions of the following maximization problem

$$f_{\max} := \max_{|x|=1} f(x) \tag{1.6}$$

and $\mathbb{C}$ be the set of complex numbers.

Here are our results about the characterization of the ground states of (1.1).

Theorem 1.2. (1) *If $u \in G_m$, then there exist $a_i \in \mathbb{C}, i = 1, 2, \dots, m$, and $w \in G$ such that $u = (a_1 w, a_2 w, \dots, a_m w)$ and $f_{\max}^{\frac{1}{2p}}\big(|a_1|, |a_2|, \dots, |a_m|\big) \in X$;*

(2) *For any $w \in G$, if $b_i \in \mathbb{C}, i = 1, 2, \dots, m$, satisfy $f_{\max}^{\frac{1}{2p}}\big(|b_1|, |b_2|, \dots, |b_m|\big) \in X$, we have $(b_1 w, b_2 w, \dots, b_m w) \in G_m$;*

(3) *If $w \in G$ and $\big(c_1 w, c_2 w, \dots, c_m w\big)$ is a ground state of (1.1), then*

$$\sum_{i=1}^m c_i^2 = f_{\max}^{-\frac{1}{p}}, \quad f(c) = f_{\max}^{-\frac{1}{p}}, \quad f_{\max} = \big(\frac{I_1(w)}{\gamma_G}\big)^p$$

and for any fixed $i \in \{1, 2, \dots, m\}$,

$$\sum_{j=1}^m k_{ij} \frac{q_{ij}}{p+1} |c_j|^{p_{ij}} |c_i|^{q_{ij}-2} = \begin{cases} 1, & \text{if } |c_i| > 0, \\ 0, & \text{if } |c_i| = 0, \end{cases}$$

where $c = (c_1, \dots, c_m)$, f and $f_{\max}$ have been defined in (1.5) and (1.6) respectively, and we need a special definition: $0^q = 0$, for any fixed $q \in \mathbb{R}$.

In particular, $G_m^+ \neq \emptyset$ if and only if there is an $x \in X$ such that $x_i \neq 0, i = 1, 2, \dots, m$, and $G_m = T_m$ if and only if all the elements of X have no zero components.

From the above results, we can obtain the following corollary easily.

Corollary 1.3. *$\lambda > -\lambda_1(\Omega)$ is a necessary condition to the existence of the ground states of (1.1).*

Using Theorem 1.2, we can show the following proposition.

Proposition 1.4. *Suppose that $\Omega = \mathbb{R}^N$ and there exists a partition $\{Y_k\}_{1 \le k \le K}$ of $\{1, 2, \dots, m\}$ such that for any given i, j with $i \neq j$,*

$$k_{ij} \ge 0 \text{ if and only if there exists } k \text{ such that } i, j \in Y_k.$$

Furthermore if $u = (u_1, u_2, \dots, u_m) \in G_m$, then there exists $k_0 \in \{1, 2, \dots, K\}$ such that $u_l \neq 0$ for some $l \in Y_{k_0}$ and $u_s = 0$ for any $s \notin Y_{k_0}$.

In Theorem 1.2, X is the set consisting of the solutions of a maximization problem constraint on the unit spherical surface, which causes a big difficulty to find a point $x \in X$ or check whether a point x belongs to X. So we want to find a new characterization of the ground states of (1.1), which can give a more convenient way to find or check a ground state of (1.1). To get this goal, we consider the maximization problem

$$\hat{f}_{\max} := \max_{|x| \neq 0} \frac{f(x)}{|x|^{2p+2}} \tag{1.7}$$

and let X^0 be the set of maximizers in $(\mathbb{R}_0^+)^m$. By the homogeneity of $f(x)|x|^{-2p-2}$, we can check that if $x \in X$ and $t > 0$, then $tx \in X^0$, which implies that $X^0 \neq \emptyset$.

A new characterization of the ground states of (1.1) can be summarized as follows.

Theorem 1.5. (1) *If $u \in G_m$, then there exist $a_i \in \mathbb{C}, i = 1, 2, \dots, m$, and $w \in G$ such that*

$$u = \frac{a}{|a|} \hat{f}_{\max}^{-1/(2p)} w \quad \text{and} \quad (|a_1|, |a_2|, \dots, |a_m|) \in X^0,$$

where $a=(a_1,\dots,a_m)$. In particular, if $u\in T_n^+$, then there exist $x_0\in X^0$ and $w\in G$ such that $u=\frac{x_0}{|x_0|}\hat f_{\max}^{-1/(2p)}w$.

(2) *$\frac{b}{|b|}\hat f_{\max}^{-1/(2p)}w\in G_m$ for any $w\in G$ and any $b:=(b_1,b_2,\dots,b_m)$ with $(|b_1|,|b_2|,\dots,|b_m|)\in X^0$.*

Next we discuss the number of the ground states of (1.1).

Theorem 1.6. *Assume that $u^1,u^2\in G_m$ with $u_i^1,u_i^2\geq 0$, $i\in\{1,2,\dots,m\}$. Then for any fixed $i\in\{1,2,\dots,m\}$, one of the following two conclusions hold:*

(1) *there exists a positive constant c_0 such that $u_i^1=c_0w_1, u_i^2=c_0w_2$, where $w_1,w_2\in G$;*
(2) *$u_i^1\equiv 0$ or $u_i^2\equiv 0$.*

The following corollary is a direct consequence of Theorems 1.2 and 1.6.

Corollary 1.7. *T_n^+ has at most $|G|C_m^n$ elements, where C_m^n is the combinatorial number. In particular, if $G_m^+\neq\emptyset$, then $|G_m^+|=|T_m^+|=|G|$.*

Before we end this section, we outline the main ideas and the approaches in the proofs of our main results. We will introduce a constraint minimization problem and show that if the constraint minimization problem can be obtained, then G_m exactly consists of all the reached function of the constraint minimization problem. So to prove Theorem 1.1, we show that the constraint minimization problem can be obtained by the concentration-compactness lemma.

For Theorems 1.2 and 1.5, we firstly prove that if $u\in G_m$, then

$$\Big(\sum_{i=1}^m|u_i|^2\Big)^{1/2}=f_{\max}^{-1/(2p)}w$$

for some $w\in G$. Secondly, we show that

$$(|u_1|,|u_2|,\dots,|u_m|)=x\Big(\sum_{i=1}^m u_i^2\Big)^{1/2}$$

for some $x\in X$. Finally, using complex analysis and the integration, we conclude that

$$u_i=x_ie^{\mathbf{i}\theta_i}\Big(\sum_{i=1}^m u_i^2\Big)^{1/2}=x_ie^{\mathbf{i}\theta_i}f_{\max}^{-1/(2p)}w.$$

The proof of Theorem 1.6 is inspired by [3, 8]. But we encounter three main difficulties. Firstly, we can not consider a perturbation problem of (1.1) as [3, 8] since k_{ii} may be zero; Secondly, when Ω is bounded, we have no results on the uniqueness of the ground states of (1.4). The last difficulty is that the extreme points, corresponding to semi-trivial ground state, cannot be interior points. Thus we can not determine that the first derivative at the extreme point is zero and the second derivative is not zero, which play a key role in using the Implicit Function Theorem. Therefore, we have to make some changes. Under our careful observation, we find that the purpose of studying a perturbation problem is to obtain a perturbation least energy and get an equivalence by using the derivative of the perturbation least energy. So we introduce a new perturbation system which is different from that in [3, 8].

This article is organized as follows: We will show the existence of the ground states of (1.1) in part 2. The proofs of the characterization of the ground states of (1.1) would be put into part 3. The last part contributes to the proof of the number of the ground states of (1.1).

2. Existence of ground states

Lemma 2.1. *Let*

$$\begin{aligned} E_1 &:= \{U \in H : I_m^{\gamma_G} \text{ is achieved by } U\}, \\ E_2 &:= \{U \in H : S_m^{\gamma_G} \text{ is achieved by } U\}. \end{aligned}$$

Then $E_1 = E_2$ *and* $I_m^{\gamma_G} = S_m^{\gamma_G}$.

A similar proof can be found in [5]. But for the readers' convenience and the completeness, we would give a detailed proofs.

Proof. **Step 1:** We prove that $E_1 \subset E_2$. For any $u^0 \in E_1$, we assume that there exists Q with $J_m(Q) \geq \gamma_G, I_m(Q) < I_m(u^0)$. Then we can choose some constant $0 < C \leq 1$ such that $J_m(CQ) = \gamma_G$. From the minimality of u^0, $I_m(u_0) \leq I_m(CQ) \leq I_m(Q)$, which is impossible. So for any $Q \in H$ with $J_m(Q) \geq \gamma_G$, we have $I_m(Q) \geq I_m(u^0)$, which, combining $J_m(u^0) = \gamma_G$, implies that $u^0 \in E_2$. Thus, $E_1 \subset E_2$.

Step 2: We show that $E_2 \subset E_1$. For any $v^0 \in E_2$, we have $I_m(v^0) = S_m^{\gamma_G}$ and $J_m(v^0) \geq \gamma_G$. If $J_m(v^0) > \gamma_G$, then there is a constant $0 < c < 1$ such that $J_m(cv^0) = \gamma_G$ and $I_m(cv^0) = c^2 I_m(v^0) < I_m(v^0)$, which contradicts to the minimality of v^0. So $J_m(v^0) = \gamma_G$. Hence

$$I_m(v^0) \geq \min_{J_m(v)=\gamma_G} I_m(v) = I_m^{\gamma_G}. \tag{2.1}$$

By the minimality of v^0, we have

$$I_m(v^0) = \min_{J_m(v)\geq\gamma_G} I_m(v) \leq \min_{J_m(v)=\gamma_G} I_m(v) = I_m^{\gamma_G}. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$I_m(v^0) = I_m^{\gamma_G} \quad \text{and} \quad J_m(v^0) = \gamma_G.$$

Therefore $v^0 \in E_1$, and hence $E_2 \subset E_1$.

So $E_1 = E_2$, which also implies that $I_m^{\gamma_G} = S_m^{\gamma_G}$. □

The next Lemma will give a relation of G_m and E_1.

Lemma 2.2. *If* $E_1 \neq \emptyset$, *then* $G_m = E_1$, *where* E_1 *is defined in Lemma 2.1.*

Proof. **Step 1:** We prove that $E_1 \subset G_m$. If $u^0 \in E_1$, then we can find some $\mu \in \mathbb{R}$ such that for any $h := (h_1, h_2, \dots, h_m) \in H$ and any $i \in \{1, 2, \dots, m\}$, we have

$$\int_\Omega (Du_i^0 Dh_i + \lambda u_i^0 h_i) = \mu(p+1) \sum_{j=1}^m k_{ij} \frac{q_{ij}}{p+1} \int_\Omega |u_j^0|^{p_{ij}} |u_i^0|^{q_{ij}-2} u_i^0 h_i. \tag{2.3}$$

Taking $h = u^0$, we obtain

$$\begin{aligned}
\gamma_G = \gamma_G^{\frac{1}{p+1}} I_m^1 = I_m^{\gamma_G} &= I_m(u^0)\\
&= \mu(p+1)\sum_{i=1}^m\sum_{j=1}^m k_{ij}\frac{q_{ij}}{p+1}\int_\Omega |u_j^0|^{p_{ij}}|u_i^0|^{q_{ij}}\\
&= \mu(p+1)\sum_{i,j=1}^m k_{ij}\int_\Omega |u_j^0|^{p_{ij}}|u_i^0|^{q_{ij}}\\
&= \mu(p+1)J_m(u^0)\\
&= \mu(p+1)\gamma_G,
\end{aligned}\tag{2.4}$$

which implies

$$\mu(p+1) = 1. \tag{2.5}$$

From (2.3)–(2.5), we see that

$$I_m(u^0) = \gamma_G \quad \text{and} \quad u^0 \in A_m. \tag{2.6}$$

Let v^0 be a solution of (1.1). Then $I_m(v^0) = J_m(v^0) > 0$. From the definition of I_m^γ, we have

$$\gamma_G^{\frac{p}{p+1}} = I_m^1 = \frac{I_m^\gamma}{\gamma^{\frac{1}{p+1}}} = \frac{I_m^{J_m(v^0)}}{J_m^{\frac{1}{p+1}}(v^0)} \le \frac{I_m(v^0)}{J_m^{\frac{1}{p+1}}(v^0)} = I_m^{\frac{p}{p+1}}(v^0),$$

which implies that $I_m(v^0) \ge \gamma_G$. Therefore, $u^0 \in G_m$ and $E_1 \subset G_m$.

Step 2: We proof that $G_m \subset E_1$. Let $V^1 \in G_m$ and $V^2 \in E_1$. Then as above we have $I_m(V^1) = J_m(V^1) \ge \gamma_G$ and $V^2 \in A_m, I_m(V^2) = J_m(V^2) = \gamma_G$. Thus

$$S_m(V^1) = \frac{p}{2p+2} I_m(V^1) \ge \frac{p}{2p+2}\gamma_G = \frac{p}{2p+2} I_m(V^2) = S_m(V^2).$$

Since V^1 is a ground state, we have $I_m(V^1) = J_m(V^1) = I_m(V^2) = J_m(V^2) = \gamma_G$, which means that $V^1 \in E_1$. So $G_m \subset E_1$.

Therefore $E_1 = G_m$. □

Proof of Theorem 1.1. According to Lemma 2.2, it suffices to prove that $E_1 \neq \emptyset$. Let $\{U_n\}$ be a minimizing sequence of $I_m^{\gamma_G}$. We divide the proof into two cases:

Case I: Ω is bounded. It is easy to check that $\{U_n\}$ is bounded in H. So using the compactness of $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ $(1 \le q < 2^*)$, we can have, up to a subsequence, for $i \in \{1, 2, \dots, m\}$,

$$\begin{aligned}
U_{ni} &\rightharpoonup U_i \quad \text{weakly in } H_0^1(\Omega),\\
U_{ni} &\to U_i \quad \text{strongly in } L^q(\Omega),
\end{aligned}$$

which implies that

$$J_m(U) = \lim_{n\to+\infty} J_m(U_n) = \gamma_G, \quad I_m(U) \le \lim_{n\to+\infty} I_m(U_n) = I_m^{\gamma_G}. \tag{2.7}$$

Since $J_m(U) = \gamma_G$,

$$I_m^{\gamma_G} = \min_{J_m(v)=\gamma_G} I_m(v) \le I_m(U) \tag{2.8}$$

It follows from (2.7) and (2.8) that $U \in E_1$. So $E_1 \neq \emptyset$.

Case II: $\Omega = \mathbb{R}^N$. For any fixed small $\varepsilon > 0$, we let $\delta(\varepsilon) := C_0\varepsilon$, where C_0 is a very large positive constant. By the concentration-compactness principle [11, 12], up to a subsequence, it is possible to associate to each $U_{ni}, 1 \leq i \leq M$, a set of functions $\{U_{ni}^l, W_{ni}\}_{1\leq l\leq L_i} \subset H$ (a set of bubbles plus a remainder), such that

(1) each U_{ni}^l has support in a ball of radius R and the distance between the supports of U_{ni}^l and $U_{ni}^j, (j \neq l)$, goes to $+\infty$ as $n \to +\infty$;

(2) one has

$$\int_{\mathbb{R}^N} \Big| |U_{ni}|^{2p+2} - \sum_{l=1}^{L_i} |U_{ni}^l|^{2p+2} \Big| < \delta(\varepsilon), \tag{2.9}$$

$$\|DU_{ni}\|_2^2 \geq \sum_{l=1}^{L_i} \|DU_{ni}^l\|_2^2 - \delta(\varepsilon), \quad \|U_{ni}\|_2^2 \geq \sum_{l=1}^{L_i} \|U_{ni}^l\|_2^2 - \delta(\varepsilon). \tag{2.10}$$

Essentially, one applies successively the concentration-compactness principle to U_{ni} to obtain the various bubbles. This process will end in finite steps since the total L^2 norm is finite and one always picks up the bubble whose L^2 norm is larger than a positive constant uniformly, which implies that, after L_i steps, the remainder W_{ni} has L^{2p+2} norm smaller than ε. So it is easy to see that the bubbles $\{U_{ni}^l\}_{1\leq l\leq L_i}$ satisfy the above condition (1) and inequality (2.10). We give a proof of the inequality (2.9) as below. Since the supports of U_{ni}^l and $U_{ni}^j (j \neq l)$ have no intersection,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big| |U_{ni}|^{2p+2} - \sum_{l=1}^{L_i} |U_{ni}^l|^{2p+2} \Big| \\
&= \int_{\mathbb{R}^N} \Big| |\sum_{l=1}^{L_i} U_{ni}^l + W_{ni}|^{2p+2} - |\sum_{l=1}^{L_i} U_{ni}^l|^{2p+2} \Big| \\
&\leq C \int_{\mathbb{R}^N} \Big(|\sum_{l=1}^{L_i} U_{ni}^l|^{2p+1} |W_{ni}| + |W_{ni}|^{2p+2} \Big) \\
&\leq C \Big(\int_{\mathbb{R}^N} |W_{ni}|^{2p+2} \Big)^{\frac{1}{2p+2}} + C \int_{\mathbb{R}^N} |W_{ni}|^{2p+2} \\
&< C\varepsilon < \delta(\varepsilon).
\end{aligned}$$

Setting $L = \max_{1\leq i\leq M}\{L_i\}$, we define, for each i, $U_{ni}^l = 0$ if $L_i < l \leq L$. Up to a subsequence, it is possible to group the bubbles into several clusters in such a way that:

(3) each cluster has one and only one bubble from $U_{ni} (1 \leq i \leq M)$;

(4) the supports of two bubbles $U_{ni}^l, U_{nj}^s (1 \leq i \neq j \leq M, 1 \leq l, s \leq L)$ have a nonempty intersection if and only if U_{ni}^l and U_{nj}^s belong to the same cluster. Obviously, we shall end up with L clusters. Define U_n^l as the vector of bubbles from the cluster l. Then, by the definition of U_n^l and the fact that $\big||a|^p - |b|^p\big|^q \leq C\big||a|^{pq} - |b|^{pq}\big|$ for any $q > 1$ and some positive constant C, dependent of q, we have

$$U_n^l \cdot U_n^j = 0, \quad x \in \mathbb{R}^N \text{ for any } l \neq j$$

and

$$\Big| J_m(U_n) - \sum_{l=1}^{L} J_m(U_n^l) \Big|$$

$$
\begin{aligned}
&= \Big| \sum_{i,j=1}^{m} k_{ij} \int_{\mathbb{R}^N} |U_{nj}|^{p_{ij}} |U_{ni}|^{q_{ij}} - \sum_{l=1}^{L} \sum_{i,j=1}^{m} k_{ij} \int_{\mathbb{R}^N} |U_{nj}^l|^{p_{ij}} |U_{ni}^l|^{q_{ij}} \Big| \\
&\le \sum_{i,j=1}^{m} |k_{ij}| \Big| \int_{\mathbb{R}^N} |U_{nj}|^{p_{ij}} |U_{ni}|^{q_{ij}} - \int_{\mathbb{R}^N} (\sum_{l=1}^{L} |U_{nj}^l|)^{p_{ij}} (\sum_{l=1}^{L} |U_{ni}^l|)^{q_{ij}} \Big| \\
&\le C \Big| \int_{\mathbb{R}^N} |U_{nj}|^{p_{ij}} \Big[|U_{ni}|^{q_{ij}} - (\sum_{l=1}^{L} |U_{ni}^l|)^{q_{ij}} \Big] \Big| \\
&\quad + C \Big| \int_{\mathbb{R}^N} \Big[|U_{nj}|^{p_{ij}} - (\sum_{l=1}^{L} |U_{nj}^l|)^{p_{ij}} \Big] \Big(\sum_{l=1}^{L} |U_{ni}^l| \Big)^{q_{ij}} \Big| \\
&\le C \Big(\int_{\mathbb{R}^N} |U_{ni}|^{q_{ij}} - \Big(\sum_{l=1}^{L} |U_{ni}^l| \Big)^{q_{ij}} \Big|^{\frac{2p+2}{q_{ij}}} \Big)^{\frac{q_{ij}}{2p+2}} \\
&\quad + C \Big(\int_{\mathbb{R}^N} \Big| U_{nj}|^{p_{ij}} - (\sum_{l=1}^{L} |U_{nj}^l|)^{p_{ij}} \Big|^{\frac{2p+2}{p_{ij}}} \Big)^{\frac{p_{ij}}{2p+2}} \\
&\le C \Big(\int_{\mathbb{R}^N} |U_{ni}|^{2p+2} - \Big(\sum_{l=1}^{L} |U_{ni}^l| \Big)^{2p+2} \Big| \Big)^{\frac{q_{ij}}{2p+2}} \\
&\quad + C \Big(\int_{\mathbb{R}^N} \Big| U_{nj}|^{2p+2} - \Big(\sum_{l=1}^{L} |U_{nj}^l| \Big)^{2p+2} \Big| \Big)^{\frac{p_{ij}}{2p+2}} \\
&= C \Big(\int_{\mathbb{R}^N} \Big| U_{ni}|^{2p+2} - \sum_{l=1}^{L} |U_{ni}^l|^{2p+2} \Big| \Big)^{\frac{q_{ij}}{2p+2}} \\
&\quad + C \Big(\int_{\mathbb{R}^N} \Big| U_{nj}|^{2p+2} - \sum_{l=1}^{L} |U_{nj}^l|^{2p+2} \Big| \Big)^{\frac{p_{ij}}{2p+2}} \\
&\le C\delta(\varepsilon)^{\frac{q_{ij}}{2p+2}} + C\delta(\varepsilon)^{\frac{p_{ij}}{2p+2}}.
\end{aligned}
$$

It follows form (2.10) that

$$
\begin{gathered}
\sum_{i=1}^{m} \|U_{ni}\|_2^2 \ge \sum_{i=1}^{m} \sum_{l=1}^{L} \|U_{ni}^l\|_2^2 - M\delta(\varepsilon), \\
\sum_{i=1}^{m} \|DU_{ni}\|_2^2 \ge \sum_{i=1}^{m} \sum_{l=1}^{L} \|DU_{ni}^l\|_2^2 - M\delta(\varepsilon).
\end{gathered}
$$

Up to a subsequence, we can define $\gamma_l := \lim_{n\to+\infty} J_m(U_n^l)$, $1 \le l \le L$. Using a diagonalization process, we obtain, for each n, a decomposition of $\{U_n\}$ in L_n bubbles (where $L_n \to \hat{L} \in N \cup \{\infty\}$) such that

$$
\sum_{i=1}^{m} \|U_{ni}\|_2^2 \ge \sum_{i=1}^{m} \sum_{l=1}^{L_n} \|U_{ni}^l\|_2^2 - M\delta(\frac{1}{n}), \tag{2.11}
$$

$$
\sum_{i=1}^{m} \|DU_{ni}\|_2^2 \ge \sum_{i=1}^{m} \sum_{l=1}^{L_n} \|DU_{ni}^l\|_2^2 - M\delta(\frac{1}{n}), \tag{2.12}
$$

$$|J_m(U_n)-\sum_{l=1}^{L_n}J_m(U_n^l)|\le \delta^\sigma(\frac{1}{n}), \tag{2.13}$$

$$\gamma_G=\sum_{l=1}^{\hat{L}}\gamma_l, \tag{2.14}$$

where $\sigma:=\min\{\frac{q_{ij}}{2p+2},\frac{p_{ij}}{2p+2}\}$.

Case 1: If $\gamma_l\ge 0$ for any l, one has

$$J_m\Big((\frac{\gamma_l}{J_m(U_n^l)})^{\frac{1}{2p+2}}U_n^l\Big)=\gamma_l,$$

which, combining (2.11) and (2.12), implies that

$$\begin{aligned} I_m^{\gamma_G}&=\lim_{n\to\infty}I_m(U_n)\\ &\ge\limsup_{n\to\infty}\sum_{l=1}^{L_n}(\frac{J_m(U_n^l)}{\gamma_l})^{\frac{1}{p+1}}I_m\Big((\frac{\gamma_l}{J_m(U_n^l)})^{\frac{1}{2p+2}}U_n^l\Big)\\ &\ge\limsup_{n\to\infty}\sum_{l=1}^{L_n}I_m^{\gamma_l}=\sum_{l=1}^{\hat{L}}I_m^{\gamma_l}. \end{aligned} \tag{2.15}$$

However, the function

$$\gamma\to I_m^\gamma=\gamma^{\frac{1}{p+1}}I_m^1$$

is strictly concave in $\mathbb{R}^+$, which implies that there exists l_0 such that, for any $l\ne l_0$, $\gamma_l=0$. By (2.14), we see that $\gamma_{l_0}=\gamma_G$. Therefore, defining

$$V_n:=\Big(\frac{\gamma_G}{J_m(U_n^{l_0})}\Big)^{\frac{1}{2p+2}}U_n^{l_0},$$

it follows from (2.15) that $\liminf_{n\to+\infty}I_m(V_n)=I_m^{\gamma_G}$, $J_m(V_n)=\gamma_G$ and so $\{V_n\}$ is a minimizing sequence for $I_m^{\gamma_G}$, for which the compactness alternative from the concentration-compactness principle is verified (recall that V_n is, up to a multiplicative factor, the vector of a group of bubbles of U_n). Since $\{V_n\}$ is bounded in H, there exists $W\in H$ such that $V_n\rightharpoonup W$ weakly in H and $V_n\to W$ strongly in $(L^2(\mathbb{R}^N)\cap L^{2p+2}(\mathbb{R}^N))^m$. In particular, it holds that

$$I_m(W)\le\lim I_m(V_n)=I_m^{\gamma_G},\quad J_m(W)=\lim J_m(V_n)=\gamma_G.$$

Therefore W is a minimizer of $I_m^{\gamma_G}$.

Case 2: Now suppose that

$$L_n^-:=\big\{l:\gamma_l<0\big\}\ne\emptyset.$$

Let L_n^+ be the complementary set of L_n^-, set $L^+:=\lim_{n\to+\infty}L_n^+$ and

$$\eta_l:=\frac{\sum_{j=1}^{\hat{L}}\gamma_j}{\sum_{i\in L_n^+}\gamma_i}\gamma_l.$$

Notice that (2.14) implies $L^+\ne\emptyset$. Furthermore,

$$\gamma_G=\sum_{l\in L^+}\eta_l. \tag{2.16}$$

Since

$$J_m\Big((\frac{\eta_l}{J_m(U_n^l)})^{\frac{1}{2p+2}}U_n^l\Big)=\eta_l>0,\quad l\in L_n^+,$$

using (2.11) and (2.12), one has

$$
\begin{aligned}
I_m^{\gamma_G} &= \lim_{n\to+\infty} I_m(U_n)\\
&\geq \limsup_{n\to+\infty} \sum_{l\in L_n^+} \big(\frac{J_m(U_n^l)}{\eta_l}\big)^{\frac{1}{p+1}} I_m\Big(\big(\frac{\eta_l}{J_m(U_n^l)}\big)^{\frac{1}{2p+2}} U_n^l\Big)\\
&\geq \limsup_{n\to+\infty} \sum_{l\in L_n^+} I_m^{\eta_l}\\
&= \sum_{l\in L^+} I_m^{\eta_l}.
\end{aligned}
\tag{2.17}
$$

Similarly, since

$$\gamma \to I_m^\gamma = \gamma^{\frac{1}{p+1}} I_m^1$$

is strictly concave in $\mathbb{R}^+$, which, combining (2.16), implies that there exists l_0 such that $\gamma_{l_0} = \gamma_G$. Proceeding as the previous case, we can complete the proof. □

3. Form of the ground states

Proof of Theorem 1.2. **Step 1:** We show that if $u \in G_m$, then $(\sum_{i=1}^m |u_i|^2)^{1/2} = f_{\max}^{-1/(2p)} w$ for some $w \in G$. Let $u \in G_m$, $|u| := (|u_1|, |u_2|, \dots, |u_m|)$ and $\|u(x)\|_{\mathbb{R}^1}^2 := \sum_{i=1}^m |u_i(x)|^2$. Since $J_m(u) = J_m(|u|)$ and $I_m(u) \geq I_m(|u|)$, we have, by Lemma 2.2, that $|u| \in G_m$. Fixing $x_0 \in X$, we can conclude that

$$
\begin{aligned}
J_m(|u|) &= J_m(u)\\
&= \int_\Omega f(u) = \int_\Omega f\big(\frac{u}{\|u(x)\|_{\mathbb{R}^1}}\big)\|u(x)\|_{\mathbb{R}^1}^{2p+2}\\
&\leq \int_\Omega f(x_0)\|u(x)\|_{\mathbb{R}^1}^{2p+2}\\
&= \int_\Omega f\big(x_0\|u(x)\|_{\mathbb{R}^1}\big)\\
&= J_m\big(x_0\|u(x)\|_{\mathbb{R}^1}\big)
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
I_m\big(x_0\|u(x)\|_{\mathbb{R}^1}\big) &= \int_\Omega \Big(\lambda\|u(x)\|_{\mathbb{R}^1}^2|x_0|^2 + \big|D\|u(x)\|_{\mathbb{R}^1}\big|^2|x_0|^2\Big)\\
&= \int_\Omega \Big(\sum_{i=1}^m \lambda|u_i|^2 + \Big|\frac{\sum_{i=1}^m |u_i|\big|D|u_i|\big|}{(\sum_{i=1}^m |u_i|^2)^{\frac12}}\Big|^2\Big)\\
&\leq \sum_{i=1}^m \int_\Omega \Big(\lambda|u_i|^2 + \big|D|u_i|\big|^2\Big)\\
&= I_m(|u|),
\end{aligned}
\tag{3.2}
$$

where we have used the Cauchy-Schwarz inequality. Let $0 < c \leq 1$ be such that $J_m\big(cx_0\|u(x)\|_{\mathbb{R}^1}\big) = J_m(|u|)$. Then by the minimality of $|u|$ and (3.2), we can see that

$$I_m(|u|) \leq I_m\big(cx_0\|u(x)\|_{\mathbb{R}^1}\big) = c^2 I_m\big(x_0\|u(x)\|_{\mathbb{R}^1}\big) \leq I_m\big(x_0\|u(x)\|_{\mathbb{R}^1}\big) \leq I_m(|u|),$$

which implies that $c = 1$. So $J_m\big(x_0\|u(x)\|_{\mathbb{R}^1}\big) = J_m(|u|)$ and $I_m\big(x_0\|u(x)\|_{\mathbb{R}^1}\big) = I_m(|u|)$. Therefore, $x_0\|u(x)\|_{\mathbb{R}^1}$ is also a ground state of (1.1).

Since $x_0\|u(x)\|_{\mathbb{R}^1}$ is a ground state of (1.1), it is easy to check that

$$-\Delta\|u(x)\|_{\mathbb{R}^1}+\lambda\|u(x)\|_{\mathbb{R}^1}=f_{\max}\|u(x)\|_{\mathbb{R}^1}^{2p+1}. \tag{3.3}$$

Let $c_0:=f_{\max}^{\frac{1}{2p}}$. Then $c_0\|u(x)\|_{\mathbb{R}^1}$ is a solution of (1.4). By the maximum principle and the fact that $J_m(|u|)>0$, we obtain $\|u(x)\|_{\mathbb{R}^1}>0$ in Ω. In fact, $x_0\|u(x)\|_{\mathbb{R}^1}$ is a ground state of (1.1) implies that $c_0\|u(x)\|_{\mathbb{R}^1}$ is a ground state of (1.4). Hence $\|u(x)\|_{\mathbb{R}^1}=c_0^{-1}(c_0\|u(x)\|_{\mathbb{R}^1})=:c_0^{-1}w$, where w is a ground state of (1.4).

Step 2: We show that $(|u_1|,|u_2|,\dots,|u_m|)=x(\sum_{i=1}^m u_i^2)^{1/2}$ for some $x\in X$. Since $\big|\frac{|u|}{\|u(x)\|_{\mathbb{R}^1}}\big|=1$, by the definition of X, we have $f(x_0)\ge f\big(\frac{|u|}{\|u(x)\|_{\mathbb{R}^1}}\big)$. It follows from $J_m(|u|)=J_m\big(x_0\|u(x)\|_{\mathbb{R}^1}\big)$ that

$$\int_\Omega f\big(\frac{|u|}{\|u(x)\|_{\mathbb{R}^1}}\big)\|u(x)\|_{\mathbb{R}^1}^{2p+2}=\int_\Omega f(x_0)\|u(x)\|_{\mathbb{R}^1}^{2p+2}.$$

Combining $\|u(x)\|_{\mathbb{R}^1}>0$ in Ω and $f(x_0)\ge f\big(\frac{|u|}{\|u(x)\|_{\mathbb{R}^1}}\big)$, implies that $f\big(\frac{|u|}{\|u(x)\|_{\mathbb{R}^1}}\big)=f(x_0)$ a.e $x\in\Omega$. That is, $\frac{|u|}{\|u(x)\|_{\mathbb{R}^1}}=X(x)$ a.e $x\in\Omega$, where $X(x)$ satisfies $f(X(x))=f(x_0)$ i.e $X(x)\in X$. So $|u|=X(x)\|u(x)\|_{\mathbb{R}^1}$ a.e $x\in\Omega$.

Since $|u|=X(x)\|u(x)\|_{\mathbb{R}^1}$ is a solution of (1.1), inserting this expression into (1.1) and using (3.3), we can have

$$2D\|u(x)\|_{\mathbb{R}^1}DX_i(x)+\|u(x)\|_{\mathbb{R}^1}\Delta X_i(x)=0, i=1,2,\dots,m. \tag{3.4}$$

Using integration by parts and (3.4), we obtain

$$\begin{aligned}
&-\int_\Omega \|u(x)\|_{\mathbb{R}^1}^2X_i(x)\Delta X_i(x)\\
&=\int_\Omega DX_i(x)D\big(\|u(x)\|_{\mathbb{R}^1}^2X_i(x)\big)\\
&=\int_\Omega DX_i(x)\Big(\|u(x)\|_{\mathbb{R}^1}^2DX_i(x)+2X_i(x)\|u(x)\|_{\mathbb{R}^1}D\|u(x)\|_{\mathbb{R}^1}\Big)\\
&=\int_\Omega\Big(|DX_i(x)|^2\|u(x)\|_{\mathbb{R}^1}^2+2X_i(x)DX_i(x)\|u(x)\|_{\mathbb{R}^1}D\|u(x)\|_{\mathbb{R}^1}\Big)\\
&=\int_\Omega |DX_i(x)|^2\|u(x)\|_{\mathbb{R}^1}^2-\int_\Omega\|u(x)\|_{\mathbb{R}^1}^2X_i(x)\Delta X_i(x),
\end{aligned}\tag{3.5}$$

which implies that

$$\int_\Omega |DX_i(x)|^2\|u(x)\|_{\mathbb{R}^1}^2=0,\quad i=1,2,\dots,m.$$

So we know that $X_i(x)$ is a constant. Therefore, there exists an $\hat{X}\in X$ such that $|u|=\hat{X}\|u(x)\|_{\mathbb{R}^1}$.

Step 3: We show that $u_i=x_ie^{\mathbf{i}\theta_i}\big(\sum_{i=1}^m u_i^2\big)^{1/2}$. Since $\|u(x)\|_{\mathbb{R}^1}>0$ in Ω, one may assume that $u_i(x)=|u_i|e^{\mathbf{i}\theta_i(x)}=X_i\|u(x)\|_{\mathbb{R}^1}e^{\mathbf{i}\theta_i(x)}$. Then it follows from $I_m(u)=I_m(|u|)$ that

$$\begin{aligned}
&\int_\Omega\big(\big|D\|u(x)\|_{\mathbb{R}^1}\big|^2+\lambda\|u(x)\|_{\mathbb{R}^1}^2\big)\\
&=\sum_{i=1}^m\int_\Omega\big(|X_i|^2\big|D\|u(x)\|_{\mathbb{R}^1}\big|^2+\lambda|X_i|^2\|u(x)\|_{\mathbb{R}^1}^2\big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \int_{\Omega} \big(\big|D|u_i|\big|^2 + \lambda \big|u_i\big|^2 \big) = I_m(|u|) = I_m(u) \\
&= \sum_{i=1}^{m} \int_{\Omega} \big(|Du_i|^2 + \lambda |u_i|^2 \big) \\
&= \sum_{i=1}^{m} \int_{\Omega} \Big(|X_i|^2 \big|D\|u(x)\|_{\mathbb{R}^1}\big|^2 + \lambda |X_i|^2 \|u(x)\|_{\mathbb{R}^1}^2 + |X_i|^2 \|u(x)\|_{\mathbb{R}^1}^2 \big|D\theta_i(x)\big|^2 \Big) \\
&= \int_{\Omega} \big(\big|D\|u(x)\|_{\mathbb{R}^1}\big|^2 + \lambda \|u(x)\|_{\mathbb{R}^1}^2 + \sum_{i=1}^{m} |X_i|^2 \|u(x)\|_{\mathbb{R}^1}^2 \big|D\theta_i(x)\big|^2 \big),
\end{aligned}
$$

where the first, second and sixth equalities have used $\sum_{i=1}^m |X_i|^2 = 1, |u_i| = X_i \|u(x)\|_{R^1}, u_i = X_i \|u(x)\|_{\mathbb{R}^1} e^{\mathbf{i}\theta_i(x)}$ respectively. The above equality implies that

$$
\int_{\Omega} |X_i|^2 \|u(x)\|_{\mathbb{R}^1}^2 \big|D\theta_i(x)\big|^2 = 0, \quad i = 1, 2, \dots, m.
$$

So we can conclude that for any $i \in \{1, 2, \dots, m\}$, $X_i = 0$ or θ_i is a constant. So $u_i = 0$ or $u_i = X_i \|u(x)\|_{\mathbb{R}^1} e^{\mathbf{i}\theta_i}$. In a word, $u_i = X_i \|u(x)\|_{\mathbb{R}^1} e^{\mathbf{i}\theta_i}$.

From to Steps 1, 2 and 3, we can get the conclusion (1) directly.

Step 4: We show that for any $w \in G$, if $b_i \in \mathbb{C}, i = 1, 2, \dots, m$, satisfy $f_{\max}^{\frac{1}{2p}}\big(|b_1|, |b_2|, \dots, |b_m|\big) \in X$, then we have $(b_1 w, b_2 w, \dots, b_m w) \in G_m$. For any $w \in G$, let $b_i \in \mathbb{C}, 1 \le i \le M$, be such that $f_{\max}^{\frac{1}{2p}}(|b_1|, |b_2|, \dots, |b_m|) \in X$ and set $U := (b_1 w, b_2 w, \dots, b_m w)$. Then

$$
f(b) = f_{\max}^{-\frac{1}{p}}, \quad \sum_{i=1}^{m} b_i^2 = f_{\max}^{-\frac{1}{p}}, \tag{3.6}
$$

$$
J_m(U) = f(b) \int_{\Omega} |w|^{2p+2} = f_{\max}^{-\frac{1}{p}} \int_{\Omega} |w|^{2p+2} = f_{\max}^{-\frac{1}{p}} \int_{\Omega} \big(|Dw|^2 + \lambda |w|^2 \big), \tag{3.7}
$$

$$
I_m(U) = \sum_{i=1}^{m} b_i^2 \int_{\Omega} \big(|Dw|^2 + \lambda |w|^2 \big) = f_{\max}^{-\frac{1}{p}} \int_{\Omega} \big(|Dw|^2 + \lambda |w|^2 \big). \tag{3.8}
$$

Since $x_0 \|u(x)\|_{\mathbb{R}^1}$ is a ground state of (1.1), it follows that

$$
\begin{aligned}
\gamma_G &= I_m\big(x_0 \|u(x)\|_{\mathbb{R}^1}\big) = I_1\big(\|u(x)\|_{\mathbb{R}^1}\big) \\
&= I_1(c_0^{-1} w) = f_{\max}^{-\frac{1}{p}} \int_{\Omega} \big(|Dw|^2 + \lambda |w|^2 \big).
\end{aligned} \tag{3.9}
$$

From (3.7)–(3.9), we can see that U is a minimizer of $I_m^{\gamma_G}$. By Lemma 2.2, $U \in G_m$, which implies that our conclusion (2) is true.

Following from the process of the proof of our conclusion (2), it is easy to get our conclusion (3). □

Proof of Proposition 1.4. We define an equivalence relation in $\{Y_k\}_{1 \le k \le K}$,

$$
i \preceq j \text{ if and only if exists } k \text{ such that } i, j \in Y_k. \tag{3.10}
$$

Let

$$
k_{ij}^{-\infty} := \begin{cases} k_{ij}, & \text{if } i \preceq j, \\ -\infty, & \text{if } i \not\preceq j, \end{cases}
$$

$$B := \big\{x \in (\mathbb{R}_0^+)^m : f^{-\infty}(x) = f_{\max}, |x| = 1\big\},$$

where

$$f^{-\infty}(x) := \sum_{i,j=1}^{m} k_{ij}^{-\infty}|x_j|^{p_{ij}}|x_i|^{q_{ij}}.$$

We firstly prove that $X = B$.

(1) For all $x^0 \in B$, we have $|x^0| = 1, f^{-\infty}(x^0) = f_{\max}$, which implies that $x_i^0 \cdot x_j^0 = 0$ if $i \not\preceq j$. So $f(x^0) = f^{-\infty}(x^0) = f_{\max}$. Therefore, $x^0 \in X$ and then $B \subset X$.

(2) If $x^1 \in X$ and $x_i^1 \cdot x_j^1 = 0, \forall\ i \not\preceq j$, then $x^1 \in B$ and $X \subset B$. We may assume that $\exists\ i_0 \not\preceq j_0$ such that $x_{i_0}^1 \cdot x_{j_0}^1 \neq 0$. Since $x^1 \in X$, by Theorem 1.2, $Q := f_{\max}^{-1/(2p)} x^1 w$ is a ground state of (1.1), where $w \in G$. That is,

$$I_m(Q) = J_m(Q) = \gamma_G.$$

Note that $Q_{i_0} \cdot Q_{j_0} \not\equiv 0$, since $x_{i_0}^1 \cdot x_{j_0}^1 \neq 0$. Let $Q_i^R = Q_i$ if $i \not\preceq j_0$, and $Q_i^R = Q_i(\cdot + Re_1)$ if $i \preceq j_0$. Then for large R,

$$\begin{gathered}
\int_\Omega |Q_j^R|^{p_{ij}}|Q_i^R|^{q_{ij}} \le \int_\Omega |Q_j|^{p_{ij}}|Q_i|^{q_{ij}}, \quad \text{if } i \not\preceq j,\\
\int_\Omega |Q_{j_0}^R|^{p_{i_0j_0}}|Q_{i_0}^R|^{q_{i_0j_0}} < \int_\Omega |Q_{j_0}|^{p_{i_0j_0}}|Q_{i_0}|^{q_{i_0j_0}}\\
\int_\Omega |Q_j^R|^{p_{ij}}|Q_i^R|^{q_{ij}} = \int_\Omega |Q_j|^{p_{ij}}|Q_i|^{q_{ij}}, \quad \text{if } i \preceq j.
\end{gathered}$$

Combining the fact that $k_{ij} \ge 0$ if and only if $i \preceq j$, implies that $J_m(Q^R) > J_m(Q)$. From

$$J_m\Big(\big(\frac{J_m(Q)}{J_m(Q^R)}\big)^{\frac{1}{2p+2}} Q^R\Big) = J_m(Q) = \gamma_G$$

and the minimality of Q,

$$I_m(Q) \le I_m\Big(\big(\frac{J_m(Q)}{J_m(Q^R)}\big)^{\frac{1}{2p+2}} Q^R\Big) = \big(\frac{J_m(Q)}{J_m(Q^R)}\big)^{\frac{1}{p+1}} I_m(Q^R) < I_m(Q^R) = I_m(Q),$$

which is impossible. So $x_i^1 \cdot x_j^1 = 0, \forall\ i \not\preceq j$. Therefore, $X \subset B$.

From the above discussions, we can get that $X = B$. If $u \in G_m$, then $|u| := \big(|u_1|, |u_2|, \dots, |u_m|\big) \in G_m$ and so, by Theorem 1.2, there exists an $x_0 \in X = B$ such that $|u| = f_{\max}^{-1/(2p)} x_0 w$. Since $x_0 \in B$, it is easy to check that there exists a $k_0 \in \{1, 2, \dots, K\}$ such that $u_l \neq 0$ for some $l \in Y_{k_0}$ and $u_s = 0$ for any $s \notin Y_{k_0}$. This completes the proof. $\square$

Proof of Theorem 1.5. Let $\tilde{X^0} := \Big\{\frac{x}{|x|} : x \in X^0\Big\}$. We would prove that $X = \tilde{X^0}$.

(1) For any $x^1 \in X$, we have $|x^1| = 1$ and

$$\frac{f(x^1)}{|x^1|^{2p+2}} = f(x^1) = \max_{|y|=1} f(y) = \max_{|y|\neq 0} \frac{f(y)}{|y|^{2p+2}}.$$

So $x^1 \in X^0$ and $x^1 = \frac{x^1}{|x^1|} \in \tilde{X^0}$, which implies that $X \subset \tilde{X^0}$.

(2) For any $y^0 \in \tilde{X^0}$, there exists an $x^1 \in X^0$ such that $y^0 = \frac{x^1}{|x^1|}$. Therefore,

$$f(y^0) = f\big(\frac{x^1}{|x^1|}\big) = \frac{f(x^1)}{|x^1|^{2p+2}} = \max_{|y|\neq 0} \frac{f(y)}{|y|^{2p+2}} = \max_{|y|=1} f(y),$$

which implies that $y^0 \in X$. So $\tilde{X^0} \subset X$.

According to the above discussions, we obtain that $X = \tilde{X^0}$, which combined with Theorem 1.2, implies that our conclusions of the theorem 1.5 and completes the proof. $\square$

4. Number of ground states

In this part, we study the number of the ground states and show Theorem 1.6.

Proof of Theorem 1.6. We argue by contradiction. Suppose to the contrary that there exist an $i \in \{1, 2, \dots, m\}$ and $c_1 \neq c_2 \in \mathbb{R}^+$ such that $u_i^1 > 0, u_i^2 > 0$, $u_i^1 = c_1 w_1$ and $u_i^2 = c_2 w_2$, where $w_1, w_2 \in G$. Without loss of generality, we may assume that $i = 1$. Consider the perturbed system

$$
\begin{gathered}
\mu(-\Delta u_1 + \lambda u_1) = \sum_{j=1}^m k_{1j} \frac{q_{1j}}{p+1} |u_j|^{p_{1j}} |u_1|^{q_{1j}-2} u_1, \quad x \in \Omega, \\
-\Delta u_i + \lambda u_i = \sum_{j=1}^m k_{ij} \frac{q_{ij}}{p+1} |u_j|^{p_{ij}} |u_i|^{q_{ij}-2} u_i, \quad x \in \Omega, i = 2, 3, \dots, m, \\
u_l = 0, \quad x \in \partial\Omega,\ l = 1, 2, \dots, m.
\end{gathered} \tag{4.1}
$$

For $\mu > 0$, we set

$$
F(x, \mu) := \frac{f^{\frac{1}{p+1}}(x)}{\sum_{i=2}^m x_i^2 + \mu x_1^2}.
$$

Since $u^1, u^2 \in G_m$ with $u_i^1, u_i^2 \geq 0, i = 1, 2, \dots, m$, it follows from Theorem 1.5 that there exist $y^1, y^2 \in X^0$ and $w_1, w_2 \in G$ such that

$$
u^1 = \frac{y^1}{|y^1| \hat{f}_{\max}^{\frac{1}{2p}}} w_1, \quad u^2 = \frac{y^2}{|y^2| \hat{f}_{\max}^{\frac{1}{2p}}} w_2, \tag{4.2}
$$

$$
F(y^j, 1) = \max_{|y| \neq 0, y_i \geq 0} F(y, 1), \quad j = 1, 2. \tag{4.3}
$$

According to the assumption and (4.2), it is easy to see that $y_1^1 > 0, y_1^2 > 0$ and $\frac{y_1^1}{|y^1|} \neq \frac{y_1^2}{|y^2|}$. We set

$$
\hat{F}(y_1, \mu) := F(x, \mu)\Big|_{x = (y_1, y_2^j, y_3^j, \dots, y_m^j)}.
$$

Then y_1^j is an interior maximum point of $\hat{F}(y_1, 1)$ in $\{y_1 > 0\}$. In fact, using (4.3), we see that

$$
\max_{|y| \neq 0, y_i \geq 0} F(y, 1) = F(y^j, 1) = \hat{F}(y_1^j, 1) \leq \max_{y_1 > 0} \hat{F}(y_1, 1) \leq \max_{|y| \neq 0, y_i \geq 0} F(y, 1),
$$

which implies that $\hat{F}(y_1^j, 1) = \max_{y_1 > 0} \hat{F}(y_1, 1)$.

Since y_1^j is an interior maximum point of $\hat{F}(y_1, 1)$ in $\{y_1 > 0\}$, we have

$$
\frac{\partial \hat{F}(y_1, 1)}{\partial y_1}\Big|_{y_1 = y_1^j} = 0, \quad \frac{\partial^2 \hat{F}(y_1, 1)}{(\partial y_1)^2}\Big|_{y_1 = y_1^j} < 0.
$$

Let $G(y_1, \mu) := \frac{\partial \hat{F}(y_1, \mu)}{\partial y_1}$. Then

$$
G(y_1, \mu)\Big|_{(y_1, \mu) = (y_1^j, 1)} = \frac{\partial \hat{F}(y_1, \mu)}{\partial y_1}\Big|_{(y_1, \mu) = (y_1^j, 1)} = 0,
$$

$$\frac{\partial G(y_1,\mu)}{\partial y_1}\Big|_{(y_1,\mu)=(y_1^j,1)}=\frac{\partial^2\hat{F}(y_1,\mu)}{(\partial y_1)^2}\Big|_{(y_1,\mu)=(y_1^j,1)}<0.$$

By the Implicit Function Theorem, we obtain that there exist a small constant $\varepsilon_1>0$ and a constant $y_1^j(\mu)\in C^1\big((1-\varepsilon_1,1+\varepsilon_1),\mathbb{R}^+\big)$ such that $y_1^j(\mu)$ is an interior maximum point of $\hat{F}(y_1,\mu)$ in $\{y_1>0\}$ for $\mu\in(1-\varepsilon_1,1+\varepsilon_1)$ and the least energy of (4.1) is $E_m(\mu)=\frac{p}{2p+2}\hat{F}^{-\frac{p+1}{p}}(y_1^j(\mu),\mu)B_1\in C^1\big((1-\varepsilon_1,1+\varepsilon_1),\mathbb{R}^+\big)$, where $B_1:=I_1(w)$ for $w\in G$.

By direct computation, we have

$$E_m(\mu)=\inf_{u\in H\setminus\{\vec{0}\}}\max_{t>0}I_m^\mu(tu),$$

where

$$I_m^\mu(u):=\frac{1}{2}\Big(\mu I_1(u_1)+\sum_{i=2}^m I_1(u_i)\Big)-\frac{1}{2p+2}J_m(u).$$

We denote

$$C^j:=\sum_{i=2}^m I_1(u_i^j),\quad D^j:=I_1(u_1^j),\quad G^j:=J_m(u^j).$$

We can check that there exists a $t^j(\mu)$ such that

$$\max_{t>0}I_m^\mu(tu^j)=I_m^\mu\big(t^j(\mu)u^j\big)$$

with $t^j(\mu)>0$ satisfying $H\big(\mu,t^j(\mu)\big)=0$, where

$$H(\mu,t):=(C^j+\mu D^j)-G^jt^{2p}.$$

Note that $H(1,1)=0$ and $\frac{\partial H}{\partial t}(1,1)<0$. By the Implicit Function Theorem, there is a small constant $0<\varepsilon_2<\varepsilon_1$ such that $t^j(\mu)\in C^1\big((1-\varepsilon_2,1+\varepsilon_2),\mathbb{R}^+\big)$ and $(t^j)'(1)=\frac{D^j}{2pG^j}$. By the Taylor expansion, we see that

$$\begin{gathered}t^j(\mu)=1+(t^j)'(1)(\mu-1)+O(|\mu-1|^2),\\ \big(t^j(\mu)\big)^2=1+2(t^j)'(1)(\mu-1)+O(|\mu-1|^2).\end{gathered}$$

Since u^j is a ground state of (1.1), we conclude that

$$G^j=C^j+D^j=\frac{2p+2}{p}E_m(1)$$

and hence

$$\begin{aligned}E_m(\mu)&\le I_m^\mu\big(t^j(\mu)u^j\big)\\ &=\frac{p}{2p+2}\big(t^j(\mu)\big)^2\big(C^j+\mu D^j\big)\\ &=\frac{p}{2p+2}\big(t^j(\mu)\big)^2\big(C^j+D^j\big)+\frac{p}{2p+2}\big(t^j(\mu)\big)^2\big(\mu-1\big)D^j\\ &=\big(t^j(\mu)\big)^2E_m(1)+\frac{p}{2p+2}\big(t^j(\mu)\big)^2\big(\mu-1\big)D^j\\ &=E_m(1)+\big(\mu-1\big)\frac{D^j}{2p+2}+\frac{p}{2p+2}\big(t^j(\mu)\big)^2(\mu-1)D^j+O(|\mu-1|^2).\end{aligned}\tag{4.4}$$

From (4.4), we can see that, for any $\mu>1$,

$$\frac{E_m(\mu)-E_m(1)}{\mu-1}\le\frac{D^j}{2p+2}+\frac{p}{2p+2}\big(t^j(\mu)\big)^2D^j+O(|\mu-1|),\tag{4.5}$$

which implies that

$$E'_m(1) \le \frac{1}{2} D^j \quad \text{as } \mu \searrow 1. \tag{4.6}$$

Similarly, we have

$$E'_m(1) \ge \frac{1}{2} D^j \quad \text{as } \mu \nearrow 1. \tag{4.7}$$

It follows from (4.6) and (4.7) that

$$E'_m(1) = \frac{1}{2} D^j,$$

which implies that

$$\frac{|y_1^1|^2}{2|y^1|^2 \hat{f}_{\max}^{\frac{1}{p}}} I_1(w_1) = \frac{1}{2} D^1 = E'_m(1) = \frac{1}{2} D^2 = \frac{|y_1^2|^2}{2|y^2|^2 \hat{f}_{\max}^{\frac{1}{p}}} I_1(w_2).$$

Since $y_1^j > 0, \hat{f}_{\max} > 0$ and $w_1, w_2 \in G$, we can conclude that $\frac{y_1^1}{|y^1|} = \frac{y_1^2}{|y^2|}$, which contradicts to $\frac{y_1^1}{|y^1|} \neq \frac{y_1^2}{|y^2|}$. The proof is complete. □

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