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High-order mixed-type differential equations with weighted integral boundary conditions *

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Abstract

In this paper, we prove the existence and uniqueness of strong solutions for high-order mixed-type problems with weighted integral boundary conditions. The proof uses energy inequalities and the density of the range of the operator generated.

1 Introduction

Let α be a positive integer and Q be the set $(0,1)\times(0,T)$. We consider the equation

$$\mathcal{L}u := \frac{\partial^2 u}{\partial t^2} + (-1)^\alpha a(t)\frac{\partial^{2\alpha+1}u}{\partial x^{2\alpha}\partial t} = f(x,t), \tag{1}$$

where the function $a(t)$ and its derivative are bounded on the interval $[0,T]$:

$$0 < a_0 \le a(t) \le a_1\,, \tag{2}$$

$$\frac{da(t)}{dt} \le a_2\,. \tag{3}$$

To equation (1) we attach the initial conditions

$$l_1 u = u(x,0) = \varphi(x), \quad l_2 u = \frac{\partial u}{\partial t}(x,0) = \psi(x) \quad x \in (0,1), \tag{4}$$

the boundary conditions

$$\frac{\partial^i}{\partial x^i}u(0,t) = \frac{\partial^i}{\partial x^i}u(1,t) = 0, \quad \text{for } \ 0 \le i \le \alpha-k-1,\ t \in (0,T), \tag{5}$$

and integral conditions

$$\int_0^1 x^i u(x,t)dx = 0, \ \text{for } \ 0 \le i \le 2k-1, \quad 1 \le k \le \alpha \quad t \in (0,T). \tag{6}$$

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where φ and ψ are known functions which satisfy the compatibility conditions given in (5)-(6).

Various problems arising in heat conduction [3, 4, 8, 9], chemical engineering [6], underground water flow [7], thermo-elasticity [14], and plasma physics [12] can be reduced to the nonlocal problems with integral boundary conditions. This type of boundary value problems has been investigated in [1, 2, 3, 4, 5, 6, 8, 9, 10, 13, 16] for parabolic equations and in [11, 15] for hyperbolic equations. The basic tool in [2, 10, 16] is the energy inequality method which, of course, requires appropriate multipliers and functional spaces. In this paper, we extend this method to the study of a high-order mixed-type partial differential equations.

2 Preliminaries

In this paper, we prove existence and uniqueness of a strong solution of problem (1)-(6). For this, we consider the problem (1)-(6) as a solution of the operator equation

$$Lu = \mathcal{F}, \tag{7}$$

where $L = (\mathcal{L}, l_1, l_2)$, with domain of definition $D(L)$ consisting of functions $u \in W_2^{2\alpha,2}(Q)$ such that $\frac{\partial^i}{\partial x^i}(\frac{\partial^{\alpha-k+1}u}{\partial x^{\alpha-k}\partial t}) \in L_2(Q)$, $i = \overline{0, \alpha+k-1}$ and u satisfies conditions (5)-(6); the operator L is considered from E to F, where E is the Banach space consisting of functions $u \in L_2(Q)$, satisfying (5)-(6), with the finite norm

$$\begin{aligned} \|u\|_E^2 &= \int_Q \big|J^k \frac{\partial^2 u}{\partial t^2}\big|^2 + \sum_{i=0}^{\alpha-k} \big|\frac{\partial^{i+1}u}{\partial x^i \partial t}\big|^2 \,dx\,dt \\ &\quad + \sup_{0\le t\le T} \int_0^1 \sum_{i=0}^{\alpha-k} \big|\frac{\partial^i u}{\partial x^i}\big|^2 + \big|\frac{\partial^{\alpha-k+1}u}{\partial x^{\alpha-k}\partial t}\big|^2 \,dx\,, \end{aligned} \tag{8}$$

where $J^k u = \int_0^x \int_0^{\xi_1} \dots \int_0^{\xi_{k-1}} u(\xi,t)d\xi$. Here F is the Hilbert space of vector-valued functions $\mathcal{F} = (f, \varphi, \psi)$ obtained by completing of the space $L_2(Q) \times W_2^{2\alpha}(0,1) \times W_2^{2\alpha}(0,1)$ with respect to the norm

$$\|\mathcal{F}\|_F^2 = \int_Q \big|f\big|^2 \,dx\,dt + \int_0^1 \sum_{i=0}^{\alpha-k} \big|\frac{\partial^i \varphi}{\partial x^i}\big|^2 + \big|\frac{\partial^{\alpha-k}\psi}{\partial x^{\alpha-k}}\big|^2 \,dx\,. \tag{9}$$

Then we establish an energy inequality

$$\|u\|_E \le C_1 \|Lu\|_F\,, \tag{10}$$

and we show that the operator L has the closure $\overline{L}$.

Definition A solution of the operator equation $\overline{L}u = \mathcal{F}$ is called a strong solution of the problem (1)-(6).

Inequality (10) can be extended to $u \in D(\overline{L})$, i.e.,

$$\|u\|_E \le C_1 \left\|\overline{L}u\right\|_F , \quad \forall u \in D(\overline{L}).$$

From this inequality we obtain the uniqueness of a strong solution if it exists, and the equality of sets $R(\overline{L})$ and $\overline{R(L)}$. Thus, to prove the existence of a strong solution of the problem (1)-(6) for any $\mathcal{F} \in F$, it remains to prove that the set $R(L)$ is dense in F.

Lemma 2.1 *For any function $u \in E$, we have*

$$\int_0^1 \left|J^{2k}\frac{\partial^2 u}{\partial t^2}\right|^2 dx \le 4^k \int_0^1 \left|J^k\frac{\partial^2 u}{\partial t^2}\right|^2 dx\,. \tag{11}$$

Proof Integrating $-\int_0^1 xJ^k\frac{\partial^2 u}{\partial t^2}J^{k+1}\frac{\partial^2 \overline{u}}{\partial t^2}dx$ by parts, and using elementary inequalities we obtain (11). $\diamondsuit$

Lemma 2.2 *For $u \in E$ and $0 \le i \le \alpha - k$, we have*

$$\int_0^1 \left|\frac{\partial^{i+1}u}{\partial x^i \partial t}\right|^2 dx \le 4^{(\alpha-k)-i}\int_0^1 \left|\frac{\partial^\alpha}{\partial x^\alpha}(J^k\frac{\partial u}{\partial t})\right|^2 dx\,. \tag{12}$$

Proof Integrating by parts $-\int_0^1 x\frac{\partial^{\alpha-i}}{\partial x^{\alpha-i}}(J^k\frac{\partial u}{\partial t})\frac{\partial^{\alpha-i-1}}{\partial x^{\alpha-i-1}}(J^k\frac{\partial \overline{u}}{\partial t})\,dx$ and using elementary inequalities yield (12). $\diamondsuit$

Lemma 2.3 *For $u \in E$ satisfying the condition (4) we have*

$$\begin{aligned}
&\sum_{i=0}^{\alpha-k}\int_0^1 \exp(-c\tau)\left|\frac{\partial^i u(x,\tau)}{\partial x^i}\right|^2 dx \\
&\le \sum_{i=0}^{\alpha-k}\int_0^1 (1-x)\left|\frac{\partial^i \varphi}{\partial x^i}\right|^2 dx \\
&\quad +\frac{1}{3}(4^{\alpha-k+1}-1)\int_0^\tau\int_0^1 \exp(-ct)\left|\frac{\partial^\alpha}{\partial x^\alpha}(J^k\frac{\partial u}{\partial t})\right|^2 dx\,dt\,,
\end{aligned} \tag{13}$$

where $c \ge 1$ and $0 \le \tau \le T$.

Proof Integrating by parts $\int_0^\tau \exp(-ct)\frac{\partial^{i+1}u}{\partial x^i \partial t}\frac{\partial^i u}{\partial x^i}dt$ for $i = \overline{0, \alpha-k-1}$, using elementary inequalities, and lemma 2.2, we obtain (13).

3 An energy inequality and its applications

Theorem 3.1 *For any function $u \in D(L)$, we have*

$$\|u\|_E \le C_1 \|Lu\|_F\,, \tag{14}$$

where $C_1 = \exp(cT)\max(8.4^k, \frac{a_1}{2})/\min(\frac{a_0}{2}, \frac{7}{8})$, with the constant c satisfying

$$c \ge 1 \quad \textit{and} \quad 3(ca_0 - a_2) \ge 2(4^{\alpha-k-1}-1). \tag{15}$$

Proof Let

$$Mu = (-1)^k J^{2k} \frac{\partial^2 u}{\partial t^2}\,.$$

For a constant c satisfying (15), we consider the quadratic form

$$\operatorname{Re} \int_0^\tau \int_0^1 \exp(-ct) \mathcal{L}u \overline{Mu}\, dx\, dt, \tag{16}$$

which is obtained by multiplying (1) by $\exp(-ct)\overline{Mu}$, then integrating over Q^τ, with $Q^\tau = (0,1) \times (0,\tau)$, $0 \le \tau \le T$, and then taking the real part. Integrating by parts in (16) with the use of boundary conditions (5) and (6), we obtain

$$\begin{aligned} &\operatorname{Re} \int_0^\tau \int_0^1 \exp(-ct) \mathcal{L}u \overline{Mu}\, dx\, dt \\ &= \int_0^\tau \int_0^1 \exp(-ct) \big| J^k \frac{\partial^2 u}{\partial t^2}\big|^2\, dx\, dt \\ &\quad + \operatorname{Re} \int_0^\tau \int_0^1 \exp(-ct) a(t) \frac{\partial^{\alpha-k+1} u}{\partial x^{\alpha-k} \partial t} \frac{\partial^{\alpha-k+2} \overline{u}}{\partial x^{\alpha-k} \partial t^2}\, dx\, dt\,. \end{aligned} \tag{17}$$

By substituting the expression of Mu in (16), using elementary inequalities and Lemma 2.1 we obtain

$$\begin{aligned} \operatorname{Re} \int_0^\tau \int_0^1 \exp(-ct) \mathcal{L}u \overline{Mu}\, dx\, dt \quad &\le \quad 8.4^k \int_0^\tau \int_0^1 \exp(-ct) \big|\mathcal{L}u\big|^2\, dx\, dt \\ &\quad + \frac{1}{8} \int_0^\tau \int_0^1 \exp(-ct) \big| J^k \frac{\partial^2 u}{\partial t^2}\big|^2\, dx\, dt\,. \end{aligned} \tag{18}$$

By integrating the last term on the right-hand side of (17) and combining the obtained results with the inequalities (15), (18) and lemmas 2.2, 2.3 we obtain

$$\begin{aligned} &8.4^k \int_0^\tau \int_0^1 \exp(-ct) \big|\mathcal{L}u\big|^2\, dx\, dt + \frac{1}{2} \int_0^1 a(0) \big| \frac{\partial^{\alpha-k} \psi}{\partial x^{\alpha-k}}\big|^2 dx + \sum_{i=0}^{\alpha-k} \int_0^1 \big| \frac{\partial^i \varphi}{\partial x^i}\big|^2 dx \\ &\ge \frac{7}{8} \int_0^\tau \int_0^1 \exp(-ct) \big| J^k \frac{\partial^2 u}{\partial t^2}\big|^2\, dx\, dt + \sum_{i=0}^{\alpha-k} \int_0^1 \exp(-c\tau) \big| \frac{\partial^i u(x,\tau)}{\partial x^i}\big|^2 dx \\ &\quad + \sum_{i=0}^{\alpha-k} \int_0^\tau \int_0^1 \exp(-ct) \big| \frac{\partial^{i+1} u}{\partial x^i \partial t}\big|^2\, dx\, dt \\ &\quad + \frac{1}{2} \int_0^1 \exp(-c\tau) a(\tau) \big| \frac{\partial^{\alpha-k+1} u(x,\tau)}{\partial x^{\alpha-k} \partial t}\big|^2 dx \end{aligned} \tag{19}$$

Using elementary inequalities and (2) we obtain

$$8.4^k \int_Q \big|\mathcal{L}u\big|^2\, dx\, dt + \frac{a_1}{2} \int_0^1 \big| \frac{\partial^{\alpha-k} \psi}{\partial x^{\alpha-k}}\big|^2 dx + \sum_{i=0}^{\alpha-k} \int_0^1 \big| \frac{\partial^i \varphi}{\partial x^i}\big|^2 dx$$

$$\geq \exp(-cT)\Big[\frac{7}{8}\int_0^\tau\int_0^1 \big|J^k\frac{\partial^2 u}{\partial t^2}\big|^2\,dx\,dt + \sum_{i=0}^{\alpha-k}\int_0^1 \big|\frac{\partial^i u(x,\tau)}{\partial x^i}\big|^2\,dx \tag{20}$$
$$+\sum_{i=0}^{\alpha-k}\int_0^\tau\int_0^1 \big|\frac{\partial^{i+1}u}{\partial x^i\partial t}\big|^2\,dx\,dt + \frac{a_0}{2}\int_0^1 \big|\frac{\partial^{\alpha-k+1}u(x,\tau)}{\partial x^{\alpha-k}\partial t}\big|^2 dx\Big]$$

As the left hand side of (20) is independent of τ, by replacing the right hand side by its upper bound with respect to τ in the interval $[0,T]$, we obtain the desired inequality. $\diamondsuit$

Lemma 3.2 *The operator L from E to F admits a closure.*

Proof Suppose that $(u_n)\in D(L)$ is a sequence such that

$$u_n \to 0 \quad \text{in } E \tag{21}$$

and

$$Lu_n \to \mathcal{F} \quad \text{in } F\,. \tag{22}$$

We need to show that $\mathcal{F}=(f,\varphi_1,\varphi_2)=0$. The fact that $\varphi_i=0$; $i=1,2$; results directly from the continuity of the trace operators l_i.

Introduce the operator

$$\mathcal{L}_0 v = \frac{\partial^2((1-x)^{\alpha+k}J^k v)}{\partial t^2} + (-1)^{\alpha+1}\frac{\partial^{\alpha+1}}{\partial x^\alpha \partial t}\big(a(t)\frac{\partial^\alpha((1-x)^{\alpha+k}J^k v)}{\partial x^\alpha}\big),$$

defined on the domain $D(\mathcal{L}_0)$ of functions $v\in W_2^{2\alpha,2}(Q)$ satisfying

$$v|_{t=T}=0 \quad \frac{\partial v}{\partial t}|_{t=T}=0 \quad \frac{\partial^i v}{\partial x^i}|_{x=0}=\frac{\partial^i v}{\partial x^i}|_{x=1}=0\,,\quad i=\overline{0,\alpha-1}\,.$$

we note that $D(\mathcal{L}_0)$ is dense in the Hilbert space obtained by completing $L_2(Q)$ with respect to the norm

$$\|v\|^2=\int_Q (1-x)^{2(\alpha+k)}\big|J^k v\big|^2\,dx\,dt.$$

Since

$$\begin{aligned}\int_Q f\overline{(1-x)^{\alpha+k}J^k v}\,dx\,dt &= \lim_{n\to\infty}\int_Q \mathcal{L}u_n\overline{(1-x)^{\alpha+k}J^k v}\,dx\,dt\\ &= \lim_{n\to\infty}\int_Q u_n\mathcal{L}_0\overline{v}\,dx\,dt = 0\,,\end{aligned}$$

holds for every function $v\in D(\mathcal{L}_0)$, it follows that $f=0$. $\diamondsuit$

Theorem 3.1 is valid for strong solutions, i.e., we have the inequality

$$\|u\|_E \le C_1\left\|\overline{L}u\right\|_F\,,\ \forall\, u\in D(\overline{L}),$$

hence we obtain the following.

Corollary 3.3 *A strong solution of (1)-(6) is unique if it exists, and depends continuously on* $\mathcal{F} = (f, \varphi, \psi) \in F$.

Corollary 3.4 *The range* $R(L)$ *of the operator* $\overline{L}$ *is closed in* F, *and* $R(\overline{L}) = \overline{R(L)}$.

4 Solvability of Problem (1)-(6)

To proof solvability of (1)-(6), it is sufficient to show that $R(L)$ is dense in F. The proof is based on the following lemma

Lemma 4.1 *Let* $D_0(L) = \{u \in D(L) : l_1 u = 0, l_2 u = 0\}$. *If for* $u \in D_0(L)$ *and some* $\omega \in L_2(Q)$, *we have*

$$\int_Q (1-x)^{2k} \mathcal{L}u\varpi \, dx\, dt = 0\,, \tag{23}$$

then $\omega = 0$.

Proof The equality (23) can be written as follows

$$-\int_Q (1-x)^{2k} \frac{\partial^2 u}{\partial t^2} \varpi \, dx\, dt = (-1)^\alpha \int_Q (1-x)^{2k} \frac{\partial^\alpha}{\partial x^\alpha} (a \frac{\partial^{\alpha+1} u}{\partial x^\alpha \partial t}) \varpi \, dx\, dt. \tag{24}$$

For $\omega(x,t)$ given, we introduce the function $v(x,t) = (-1)^k \partial^{2k}((1-x)^{2k}\omega)/\partial x^{2k}$, then we have $\int_0^1 x^i v(x,t) dx = 0$ for $i = \overline{0, 2k-1}$. Then from equality (24) we have

$$-\int_Q \frac{\partial^2 u}{\partial t^2} J^{2k} \overline{v} \, dx\, dt = (-1)^\alpha \int_Q \frac{\partial^\alpha}{\partial x^\alpha} (a \frac{\partial^{\alpha+1} u}{\partial x^\alpha \partial t}) J^{2k} \overline{v} \, dx\, dt. \tag{25}$$

Integrating by parts the right hand side of (25) $2k$ times, we get

$$-\int_Q \frac{\partial^2 u}{\partial t^2} J^{2k} \overline{v} \, dx\, dt = \int_Q A(t) \frac{\partial u}{\partial t} \overline{v} \, dx\, dt, \tag{26}$$

where $A(t)u = (-1)^\alpha \frac{\partial^{\alpha-k}}{\partial x^{\alpha-k}} (a \frac{\partial^{\alpha-k} u}{\partial x^{\alpha-k}})$.

When we introduce the smoothing operators $J_\varepsilon^{-1} = (I + \varepsilon \frac{\partial}{\partial t})^{-1}$ and $(J_\varepsilon^{-1})^*$ with respect to t [16], then these operators provide solutions of the problems

$$\begin{aligned} \varepsilon \frac{d g_\varepsilon(t)}{dt} + g_\varepsilon(t) &= g(t), \\ g_\varepsilon(t)|_{t=0} &= 0, \end{aligned} \tag{27}$$

and

$$\begin{aligned} -\varepsilon \frac{d g_\varepsilon^*(t)}{dt} + g_\varepsilon^*(t) &= g(t), \\ g_\varepsilon^*(t)|_{t=T} &= 0\,. \end{aligned} \tag{28}$$

The solutions have the following properties: for any $g \in L_2(0,T)$, the functions $g_\varepsilon = (J_\varepsilon^{-1})g$ and $g_\varepsilon^* = (J_\varepsilon^{-1})^* g$ are in $W_2^1(0,T)$ such that $g_\varepsilon\mid_{t=0}= 0$ and $g_\varepsilon^*\mid_{t=T}= 0$. Moreover, J_ε^{-1} commutes with $\frac{\partial}{\partial t}$, so $\int_0^T \left|g_\varepsilon - g\right|^2 dt \to 0$ and $\int_0^T \left|g_\varepsilon^* - g\right|^2 dt \to 0$, for $\varepsilon \to 0$.

Replacing in (26), $\frac{\partial u}{\partial t}$ by the smoothed function $J_\varepsilon^{-1}\frac{\partial u}{\partial t}$, using the relation $A(t)J_\varepsilon^{-1} = J_\varepsilon^{-1}A(\tau) + \varepsilon J_\varepsilon^{-1}\frac{\partial A(\tau)}{\partial \tau}J_\varepsilon^{-1}$, and using properties of the smoothing operators we obtain

$$\int_Q \frac{\partial u}{\partial t}\overline{J^{2k}(\frac{\partial v_\varepsilon^*}{\partial t})}\,dx\,dt = \int_Q A(t)\frac{\partial u}{\partial t}\overline{v_\varepsilon^*}\,dx\,dt + \varepsilon\int_Q \frac{\partial A}{\partial t}(\frac{\partial u}{\partial t})_\varepsilon \overline{v_\varepsilon^*}\,dx\,dt. \tag{29}$$

Passing to the limit,(29) is satisfied for all functions satisfying the conditions (4)-(6) such that $\frac{\partial^i}{\partial x^i}(a\frac{\partial^{\alpha+1}u}{\partial x^\alpha \partial t}) \in L_2(Q)$ for $0 \le i \le \alpha$.

The operator $A(t)$ has a continuous inverse on $L_2(0,1)$ defined by

$$\begin{aligned} &A^{-1}(t)g \qquad (30)\\ &= (-1)^\alpha \int_0^x \int_0^{\eta_{\alpha-k-1}} \dots \int_0^{\eta_1} \Big[\int_0^\eta \int_0^{\xi_{\alpha-k-1}} \dots \int_0^{\xi_1} \frac{1}{a} g(\xi)d\xi d\xi_1 \dots d\xi_{\alpha-k-1} \\ &\quad + \sum_{i=1}^{\alpha-k} C_i(t)\frac{\eta^{i-1}}{(i-1)!}\Big]\,d\eta d\eta_1 \dots d\eta_{\alpha-k-1}\,. \end{aligned}$$

Then we have

$$\int_0^1 A^{-1}(t)g\,dx = 0\,. \tag{31}$$

Hence the function $(\frac{\partial u}{\partial t})_\varepsilon$ can be represented in the form $(\frac{\partial u}{\partial t})_\varepsilon = J_\varepsilon^{-1}A^{-1}A\frac{\partial u}{\partial t}$. Then $\frac{\partial A}{\partial t}(\frac{\partial u}{\partial t})_\varepsilon = A_\varepsilon(t)A(t)\frac{\partial u}{\partial t}$, where

$$A_\varepsilon(t)g = (-1)^\alpha\big[a'(t)J_\varepsilon^{-1}\frac{g}{a} + \sum_{i=1}^{\alpha-k}\frac{\partial^{\alpha-k}}{\partial x^{\alpha-k}}\big\{\frac{x^{i-1}}{(i-1)!}J_\varepsilon^{-1}C_i\big\}\big]. \tag{32}$$

Consequently, (29) becomes

$$\int_Q \frac{\partial u}{\partial t}\overline{J^{2k}(\frac{\partial v_\varepsilon^*}{\partial t})}\,dx\,dt = \int_Q A(t)\frac{\partial u}{\partial t}(\overline{v_\varepsilon^* + \varepsilon A_\varepsilon^* v_\varepsilon^*})\,dx\,dt, \tag{33}$$

in which $A_\varepsilon^*(t)$ is the adjoint of the operator $A_\varepsilon(t)$. The left-hand side of (33) is a continuous linear functional of $\frac{\partial u}{\partial t}$. Hence the function $h_\varepsilon = v_\varepsilon^* + \varepsilon A_\varepsilon^* v_\varepsilon^*$ has the derivatives $\frac{\partial^i h_\varepsilon}{\partial x^i} \in L_2(Q)$, $\frac{\partial^i}{\partial x^i}(a\frac{\partial^{\alpha-k}h_\varepsilon}{\partial x^{\alpha-k}}) \in L_2(Q)$, $i = \overline{0,\alpha-k}$, and the following conditions are satisfied

$$\frac{\partial^i h_\varepsilon}{\partial x^i}\big|_{x=0} = \frac{\partial^i h_\varepsilon}{\partial x^i}\big|_{x=1} = 0\,,\ i = \overline{0,\alpha-k-1}. \tag{34}$$

The operators $A_\varepsilon^*(t)$ are bounded in $L_2(Q)$, for ε sufficiently small we have $\|\varepsilon A_\varepsilon^*(t)\|_{L_2(Q)} < 1$; hence the operator $I + \varepsilon A_\varepsilon^*(t)$ has a bounded inverse in

$L_2(Q)$. In addition, the operators $\frac{\partial^i A_\varepsilon^*(t)}{\partial x^i}$, $i=\overline{0,\alpha-k}$ are bounded in $L_2(Q)$. From the equality

$$\frac{\partial^i h_\varepsilon}{\partial x^i} = (I+\varepsilon A_\varepsilon^*(t))\frac{\partial^i v_\varepsilon^*}{\partial x^i} + \varepsilon\sum_{k=1}^{i} \mathcal{C}_i^k \frac{\partial^k A_\varepsilon^*(t)}{\partial x^k}\frac{\partial^{i-k} v_\varepsilon^*}{\partial x^{i-k}}, \quad i=\overline{0,\alpha-k-1} \tag{35}$$

we conclude that v_ε^* has derivatives $\frac{\partial^i v_\varepsilon^*}{\partial x^i}$ in $L_2(Q)$, $i=\overline{0,\alpha-k-1}$. Taking into account (34) and (35), for $i=\overline{0,\alpha-k-1}$, we have

$$\big[(I+\varepsilon A_\varepsilon^*(t))\frac{\partial^i v_\varepsilon^*}{\partial x^i} + \varepsilon\sum_{k=1}^{i} \mathcal{C}_i^k \frac{\partial^k A_\varepsilon^*(t)}{\partial x^k}\frac{\partial^{i-k} v_\varepsilon^*}{\partial x^{i-k}}\big]_{x=0} = 0\,, \tag{36}$$

$$\big[(I+\varepsilon A_\varepsilon^*(t))\frac{\partial^i v_\varepsilon^*}{\partial x^i} + \varepsilon\sum_{k=1}^{i} \mathcal{C}_i^k \frac{\partial^k A_\varepsilon^*(t)}{\partial x^k}\frac{\partial^{i-k} v_\varepsilon^*}{\partial x^{i-k}}\big]_{x=1} = 0\,. \tag{37}$$

Similarly, for ε sufficiently small, and each fixed $x\in[0,1]$ the operators $\frac{\partial^i A_\varepsilon^*(t)}{\partial x^i}$, $i=\overline{0,\alpha-k}$ are bounded in $L_2(Q)$ and the operator $I+\varepsilon A_\varepsilon^*(t)$ is continuously invertible in $L_2(Q)$. From (36) and (37) result that v_ε^* satisfies the conditions

$$\frac{\partial^i v_\varepsilon^*}{\partial x^i}\Big|_{x=0} = \frac{\partial^i v_\varepsilon^*}{\partial x^i}\Big|_{x=1} = 0, \quad i=\overline{0,\alpha-k-1},$$

So, for ε sufficiently small, the function v_ε^* has the same properties as h_ε. In addition v_ε^* satisfies the integral conditions (6).

Putting $u=\int_0^t\int_0^\tau \exp(c\eta)v_\varepsilon^*(\eta,\tau)d\eta d\tau$ in (26), with the constant c satisfying $ca_0-a_2-\frac{a_2^2}{a_0}\geq 0$, and using (28), we obtain

$$\begin{aligned}\int_Q(-1)^k\exp(ct)v_\varepsilon^*\overline{J^{2k}v}\,dx\,dt &= -\int_Q(-1)^kA(t)\frac{\partial u}{\partial t}\exp(-ct)\frac{\partial^2\overline{u}}{\partial t^2}\,dx\,dt\\ &\quad+\varepsilon\int_Q(-1)^kA(t)\frac{\partial u}{\partial t}\frac{\partial\overline{v_\varepsilon^*}}{\partial t}\,dx\,dt\,.\end{aligned} \tag{38}$$

Integrating by parts each term in the right-hand side of (38), we have

$$\begin{aligned}&\mathrm{Re}\int_Q(-1)^kA(t)\frac{\partial u}{\partial t}\exp(-ct)\frac{\partial^2\overline{u}}{\partial t^2}\,dx\,dt\\ &\geq \frac{c}{2}\int_Q a(t)e^{-ct}\Big|\frac{\partial^{\alpha-k+1}u}{\partial x^{\alpha-k}\partial t}\Big|^2\,dx\,dt-\frac{1}{2}\int_Q\frac{\partial a}{\partial t}e^{-ct}\Big|\frac{\partial^{\alpha-k+1}u}{\partial x^{\alpha-k}\partial t}\Big|^2\,dx\,dt\,.\end{aligned} \tag{39}$$

$$\mathrm{Re}(-\varepsilon\int_Q(-1)^kA(t)\frac{\partial u}{\partial t}\frac{\partial\overline{v_\varepsilon^*}}{\partial t}\,dx\,dt)\geq\frac{-\varepsilon a_2^2}{2a_0}\int_Q\exp(-ct)\Big|\frac{\partial^{\alpha-k+1}u}{\partial x^{\alpha-k}\partial t}\Big|^2\,dx\,dt. \tag{40}$$

Now, using (39) and (40) in (38), with the choice of c indicated above, we have $2\,\mathrm{Re}\int_Q\exp(ct)v_\varepsilon^*\overline{j^{2k}v}\,dx\,dt\leq 0$, then $2\,\mathrm{Re}\int_Q\exp(ct)v\overline{J^{2k}v}\,dx\,dt\leq 0$ as ε approaches zero. Since $\mathrm{Re}\int_Q\exp(ct)\big|J^k\overline{v}\big|^2\,dx\,dt=0$, we conclude that $J^{2k}v=0$, hence $\omega=0$, which completes the present proof. $\diamondsuit$

Theorem 4.2 *The range $R(L)$ of L coincides with F.*

Proof Since F is a Hilbert space, we have $R(L) = F$ if and only if the following implication is satisfied:

$$\int_Q \mathcal{L}u\overline{f}\,dx\,dt + \int_0^1 (\sum_{i=0}^{\alpha-k} \frac{\partial^i l_1 u}{\partial x^i}\frac{\partial^i \overline{\varphi}}{\partial x^i} + \frac{\partial^{\alpha-k} l_2 u}{\partial x^{\alpha-k}}\frac{\partial^{\alpha-k}\overline{\psi}}{\partial x^{\alpha-k}})dx = 0, \tag{41}$$

for arbitrary $u \in E$ and $\mathcal{F} = (f, \varphi, \psi) \in F$, implies that f, φ, and ψ are zero.

Putting $u \in D_0(L)$ in (41), we obtain $\int_Q \mathcal{L}u\overline{f}\,dx\,dt = 0$. Taking $\omega = f/(1-x)^{2k}$, and using lemma 4.1 we obtain that $f/(1-x)^{2k} = 0$, then $f = 0$. Consequently, $\forall u \in D(L)$ we have

$$\int_0^1 \sum_{i=0}^{\alpha-k} \frac{\partial^i l_1 u}{\partial x^i}\frac{\partial^i \overline{\varphi}}{\partial x^i} + \frac{\partial^{\alpha-k} l_2 u}{\partial x^{\alpha-k}}\frac{\partial^{\alpha-k}\overline{\psi}}{\partial x^{\alpha-k}}\,dx = 0\,. \tag{42}$$

The range of the trace operator (l_1, l_2) is everywhere dense in a Hilbert space with norm

$$\Big[\int_0^1 \sum_{i=0}^{\alpha-k} \Big|\frac{\partial^i \varphi}{\partial x^i}\Big|^2 + \Big|\frac{\partial^{\alpha-k}\psi}{\partial x^{\alpha-k}}\Big|^2 dx\Big]^{1/2}.$$

Therefore, $(\varphi, \psi) = (0, 0)$ and the present proof is complete.

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