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SINGULAR REGULARIZATION OF OPERATOR EQUATIONS IN L_1 SPACES VIA FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. An abstract causal operator equation $y = Ay$ defined on a space of the form $L_1([0,\tau], X)$, with X a Banach space, is regularized by the fractional differential equation

$$\varepsilon(D_0^\alpha y_\varepsilon)(t) = -y_\varepsilon(t) + (Ay_\varepsilon)(t), \quad t \in [0,\tau],$$

where D_0^α denotes the (left) Riemann-Liouville derivative of order $\alpha \in (0,1)$. The main procedure lies on properties of the Mittag-Leffler function combined with some facts from convolution theory. Our results complete relative ones that have appeared in the literature; see, e.g. [5] in which regularization via ordinary differential equations is used.

1. Introduction

Regularization employs several techniques in order to approximate solutions of ill-posed problems such as

$$My = f, \tag{1.1}$$

where M is an operator acting on a space X and taking values in another space Y. Basically, the problem is characterized as an ill-posed problem, if either solutions do not exist for some f, or uniqueness of solutions is not guaranteed, or continuous dependence on data does not hold. The latter is equivalent to saying that there is no continuous inverse of M. In order to solve an ill- posed problem (approximately), we should regularize it, namely, replace this problem by a suitable family of well-posed problems whose solutions approximate (in some sense) the solution of the ill-posed problem which we look for.

However, it is not true that such a process may produce an approximation of the solutions of the original equation for all situations. To see it, we borrow an example from the literature (e.g., [17, 18]) adopted to our situation, as follows: Consider the 2×2 matrix-operator M and the function f given by

$$M := \begin{bmatrix} \frac{d}{dt} & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad f(t) := \begin{bmatrix} 0 \\ p(t) \end{bmatrix},$$

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where p is a differentiable function on $[0,1]$, say. The exact solution of the operator equation (1.1) in the space $C^1([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$ is given by

$$x(t)=p(t),\quad y(t)=p'(t),\quad t\in[0,1].$$

Take a small number ε and let

$$f_\varepsilon(t):=f(t)+\begin{bmatrix}0\\ \varepsilon\sin(t/\varepsilon^2)\end{bmatrix}$$

be a small perturbation of f. Then we obtain the exact solution

$$x_\varepsilon(t)=p(t)+\varepsilon\sin(t/\varepsilon^2),\quad y_\varepsilon(t)=p'(t)+\frac{1}{\varepsilon}\cos(t/\varepsilon^2).$$

Hence the quantity

$$\begin{bmatrix}x_\varepsilon(t)\\ y_\varepsilon(t)\end{bmatrix}-\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}\varepsilon\sin(t/\varepsilon^2)\\ \frac{1}{\varepsilon}\cos(t/\varepsilon^2)\end{bmatrix}$$

becomes large enough if the number ε tends to 0. This means that the solution changes a lot after a small change in the right side of equation.

In case that M is a compact linear operator between two Hilbert spaces, a regularizing form should consist of the equation

$$(M^*M+\varepsilon)x_\varepsilon=M^*f, \tag{1.2}$$

where M^* is the adjoint of M, see [10]. In [7] the regularization (1.2) has its right side M^*f_δ, where f_δ is a (noisy) approximation of f. The works [21, 22] refer to *Tikhonov-regularization*, i.e. regularization of minimazing problems. According to such problems, an equation of the form

$$\int_a^b k(t,s)x(s)ds=f(t) \tag{1.3}$$

is replaced by the equation

$$\int_a^b k(t,s)x_\varepsilon(s)ds+\varepsilon x_\varepsilon(t)=f(t),$$

or the equation

$$\int_a^b k(t,s)x_\varepsilon(s)ds+\varepsilon x_\varepsilon(t)=f_\delta(t),$$

and then one looks for the convergence of the net x_ε. Here a noisy f_δ replaces f, for small δ; see, e.g., the interesting survey presented in [16]. Approximation of the kernel k of (1.3) is used by other authors, see, e.g., [19]. Approximation of both the perturbation and the operator applies elsewhere, [9]. Some authors, as, e.g. [3], dealing with the Volterra equation

$$\int_0^t k(t,s)x(s)ds=f(t)\,, \tag{1.4}$$

apply the so called method of *the simplified (or Lavrentiev) regularization*, consisting of an approximation of the perturbation f and the *local regularization*, realized by an approximate equation of the form

$$\int_t^{t+\varepsilon} k(t+\varepsilon,s)x(s)ds+\int_0^t k(t+\varepsilon,s)x(s)ds=f(t+\varepsilon),$$

where ε is a parameter tending to 0.

In [27] another approach is applied to (1.3) by taking an approximation of both the kernel k and the output f. For a more general setting see, also, [28].

Regularization of abstract equations of the form (1.1) can be realized by approximating the output f, as, e.g. in [8] and for Fredholm integral equations, as, e.g., in [30]. Regularization of the Hammerstein's type equation $x+BAx=f$, is achieved, (see, e.g., [26]) by replacing it with the equation $x_\varepsilon+(B+\varepsilon J)(A+\varepsilon J)x_\varepsilon=f_\delta$, where ε,δ are positive reals tending to 0 and the functions f,f_δ are such that $\|f-f_\delta\|\leq\delta$. Here A and B are operators, and x,f are elements in a given Banach space X, with x being the unknown element in X.

In case that the operator M has the form $My=Ay-y+f$, the problem (1.1) leads to the fixed point problem

$$y=Ay. \tag{1.5}$$

It is known (see, e.g., [5, p. 89]) that a continuous compact operator A (in the sense of Krasnoselskii) defined on a locally convex Hausdorff space has a fixed point. Regularization theory of such an equation (especially), when A is a monotone or a non-expansive operator defined in a Hilbert or (even in a) Banach space, forms a large field, and most of the authors make use of variation techniques, see, e.g. [1, 2, 4, 14, 29] and the references therein.

In case (1.5) refers to a space of functions $y:[0,1]\to\mathbb{R}$, say, namely we have

$$y(t)=(Ay)(t),\quad t\in[0,1], \tag{1.6}$$

regularization is achieved by a differential equation of the form

$$\varepsilon\frac{d}{dt}y(t)+y(t)-(Ay)(t)=0. \tag{1.7}$$

This is done elsewhere (see, e.g., the book [5, p. 140], and the references therein), when y has to be a continuous function, say, $y\in C([0,T],\mathbb{R})$. Similar things occur for a neutral differential equation discussed in [11]. An immediate consequence of this approach is that, in this case, a solution of (1.6) is approximated by a sequence (y_{ε_n}) of real-valued functions having continuous first order derivatives.

For fractional differential equations a few results, analogous to above, are known. We should refer to the problem

$$D_0^\alpha(x-x(0)-\varepsilon)=f(t,x)+\varepsilon,\quad x(0)=x_0+\varepsilon,$$

discussed in [15], where conditions are given so that, as ε tends to 0, the maximal solution $\eta(t;\varepsilon)$ tends to the maximal solution $\eta(t)$ of the problem

$$D_0^\alpha(x-x(0))=f(t,x),\quad x(0)=x_0,$$

uniformly on any compact interval $[0,t_1]$ of the domain of η. In this work we assume that A is defined on an L_1-space of X-valued functions, where X is a Banach space, and we regularize (1.6) by an equation involving continuous functions with Lebesgue-integrable first order derivatives. To succeed in such an approach we work in L_1-spaces and use the fractional equation

$$\varepsilon(D_0^\alpha y_\varepsilon)(t)=-y_\varepsilon(t)+(Ay_\varepsilon)(t),\quad \text{a.a.}\quad t\in[0,\tau]:=I_\tau, \tag{1.8}$$

for ε tending to 0. Here, $D_0^\alpha y_\varepsilon$ is the (left) Riemann-Liouville derivative of f of order α.

A central role to our approach is played by some facts from convolution theory, as well as the Mittag-Leffler function. It is known that the relation of the latter with the fractional calculus, is analogous of that of the exponential function with standard calculus. See, for instance, [12, subsection 3.2].

We investigate when, for some $\tau \in (0, T]$, there is a sequence of solutions of the fractional differential equation (1.8) converging in the sense of L_1-norm on $[0, \tau]$ to solutions of equation (1.6), when the parameter ε approaches 0.

2. Preliminaries

2.1. **Fractional calculus.** Throughout this paper we shall work on a real Banach space X endowed with a norm $\|\cdot\|_X$, and on the space $L_1^T := L_1([0, T], X)$, for some $T > 0$ fixed, with norm

$$\|y\|_1^\tau := \int_0^\tau \|y(s)\|_X ds.$$

Several books in the literature present surveys on the classical fractional calculus. Two exhaustive such books are the ones by Podlubny [24] and Miller and Ross [20]. We recall some basic definitions and results adopted for our purposes, namely we consider the meaning of fractional derivative and integral on an X-valued function defined on the interval $[0, T]$.

Let Γ be the Euler Gamma function. It is well known (see, e.g., [31]) that on the positive real axis the function Γ admits a local minimum 0.885603... at $x_{\min} = 1.461632144...$ and it is increasing for $x > x_{\min}$. Later on we shall use the monotonicity of Γ on the interval $[2, +\infty)$.

For $u \in L_1^T$ and $\alpha \in (0, 1)$, the (left) fractional Riemann-Liouville derivative of f of order α, is defined by

$$(D_0^\alpha u)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s) ds,$$

where the integral is in the Bohner sense.

As in [24], [pp. 59-73, and relation (2.122)], we can see that the first composition formula with integer order n derivative holds[1]:

$$D_0^\alpha(u^{(n)})(t) = D_0^{\alpha+n} u(t) - \sum_{j=0}^{n-1} \frac{u^{(j)}(0) t^j}{\Gamma(j+1)}. \tag{2.1}$$

Now consider the problem

$$(D_0^\alpha u)(t) = f(t), \quad \text{a.a. } t \in [0, T], \quad (D_0^{\alpha-1} u)(t)\Big|_{t=0} = b, \tag{2.2}$$

where $b \in X$.

Although the following result can be implied from arguments borrowed from the literature (see, e.g., [24] Theorem 3.1, p. 122 and relation (3.7) in p. 123), we shall give our proof for two reasons: First we want this work to be complete. Second, the functions used here take values in the abstract Banach space X and not in $\mathbb{R}$, as it is used elsewhere (and in [24, Theorem 3.1]).

Let B be the (real) Betta function, namely the function defined for $\rho, \sigma > 0$ by

$$B(\rho, \sigma) = \int_0^1 (1-\theta)^{\rho-1} \theta^{\sigma-1} d\theta$$

[1] The relation holds even for $\alpha < 0$.

This is connected with the Gamma function by the relation

$$B(\rho,\sigma)=\frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)}.$$

Lemma 2.1. *The function y defined by*

$$y(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}b+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds,\quad a.a.\ t\in[0,T],$$

is the only solution of the problem (2.2).

Proof. We show that y satisfies the problem (2.2). We have

$$\begin{aligned}
(D_0^\alpha y)(t)&=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-s)^{-\alpha}s^{\alpha-1}dsb\\
&\quad+\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-s)^{-\alpha}\int_0^s(s-r)^{\alpha-1}f(r)\,dr\,ds\\
&=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}B(1-\alpha,\alpha)b\\
&\quad+\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-s)^{-\alpha}\int_r^t(s-r)^{\alpha-1}f(r)dsdr\\
&=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t f(r)drB(1-\alpha,\alpha)\\
&=\frac{d}{dt}\int_0^t f(r)dr=f(t),\quad \text{a.e.},
\end{aligned}\tag{2.3}$$

where, in the integration, we used the substitution $s=:(1-\theta)r+\theta t,\quad \theta\in[0,1]$. Similarly we obtain

$$(D_0^{\alpha-1}y)(t)\Big|_{t=0}=\frac{d}{dt}(t)\Big|_{t=0}b+\frac{d}{dt}\int_0^t(t-r)f(r)dr\Big|_{t=0}=b.$$

The inverse is implied by an application [24, Theorem 3.1, p.122]. □

2.2. **The Mittag-Leffler function.** The Mittag-Leffler function of order $\alpha(>0)$ is defined on the complex plane by

$$E_\alpha(z):=\sum_0^\infty\frac{z^j}{\Gamma(ja+1)}\,.$$

From a result of Feller referred by Pollard [25], we know that there is a nondecreasing and bounded function F_α such that

$$E_\alpha(-x)=\int_0^{+\infty}e^{-xs}dF_\alpha(s),\quad x\geq 0.\tag{2.4}$$

It follows that this function is positive, non-increasing, it tends to 0 as $x\to+\infty$ and since $E_\alpha(0)=1$, the quantity $E_\alpha(-x)$ is not greater than 1. More properties of this function and of some generalizations of it can be found in [24].

3. Main results

Let $A : L_1^T \to L_1^T$ be a causal operator, namely, it satisfies $(Ax)(t) = (Ay)(t)$, whenever $x(s) = y(s)$, for a.a. $s \in [0,t]$, (for the continuous case see, e.g., [13], [23] and the references therein). This characteristic guarantees that, for any $\tau \in (0,T]$, the operator A maps the ball

$$B_\tau^r := \{y \in L_1^\tau : \|y\|_1^\tau < r\},$$

into the space L_1^τ. Suppose, also, that A is continuous and compact in the sense that, it maps bounded sets into relatively compact sets. Hence, in case that for some $\tau > 0$ it holds

$$A(\overline{B_\tau^r}) \subseteq \overline{B_\tau^r},$$

the following Schauder's fixed point theorem applies and ensures the existence of a fixed point of A in $\overline{B_\tau^r}$.

Theorem 3.1 ([5, p. 89])**.** *Let E be a real Banach space and $K \subset E$ a closed, bounded and convex set. If $C : K \to K$ is a continuous compact operator, then C has at least one fixed point.*

Now, for any fixed $\varepsilon > 0$ and small enough, say $\varepsilon < 1$, consider the fractional differential equation

$$\varepsilon(D_0^\alpha y)(t) = -y(t) + (Ay)(t), \quad \text{a.a. } \ t \in [0,T], \tag{3.1}$$

where the derivative $D_0^\alpha y$ is in the sense of Riemann-Liouville and $\alpha \in (0,1)$.

Let b be a (nonzero) real number and consider the initial value problem

$$(D_0^\alpha y)(t) = -\frac{1}{\varepsilon} y(t) + \frac{1}{\varepsilon}(Ay)(t), \quad (D_0^{\alpha-1}y)(t)\Big|_{t=0} = b.$$

According to Lemma 2.1, a function y is a solution of the problem, if and only if it satisfies the equation

$$y(t) = \frac{b}{\Gamma(\alpha)} t^{\alpha-1} - \frac{1}{\varepsilon\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{\varepsilon\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (Ay)(s) ds. \tag{3.2}$$

Our main result in this work is given in the following theorem:

Theorem 3.2. *If A is a causal, compact and continuous operator on L_1^T, then, there exists a certain $\tau \in (0,T]$, such that, for any sequence (ε_n) converging to 0, there is a sequence of solutions (y_n) of equation* (3.2) *converging in the L_1^τ-sense to a solution y of equation*

$$y(t) = (Ay)(t), \quad a.a\ t \in [0,\tau].$$

The proof of the above theorem will be given in the last section. It is noteworthy that the theorem has several interesting consequences, as the following one.

Corollary 3.3. *Let k be a positive integer, W a continuous and causal operator defined on the $C^k([0,T],X)$-space and let $\alpha \in (0,1)$. Then, there exists a certain $\tau \in (0,T]$ such that, for any sequence (ε_n) converging to 0, there is a sequence of solutions (x_n) of the problem*

$$\varepsilon(D_0^{k+\alpha} x)(t) = -x^{(k)}(t) + (Wx)(t), \quad a.a.\ t \in [0,\tau], \tag{3.3}$$

$$x^{(j)}(0) = 0, \quad j = 0,1,\dots,k-1, \quad (D_0^{k+\alpha-1}x)(t)\Big|_{t=0} = b,$$

converging, in the sup-norm $\|\cdot\|_\infty^\tau$ *sense, to a solution of the problem*

$$\begin{gathered} x^{(k)}(t) = (Wx)(t) \\ x^{(j)}(0) = 0, \quad j = 0, 1, \dots, k-1. \end{gathered}$$

Proof. Set $y = x^{(k)}$. Then, due to (2.1), we have

$$(D_0^\alpha y)(t) = (D_0^{k+\alpha} x)(t) \quad \text{and} \quad (D_0^{\alpha-1} y)(t)\big|_{t=0} = (D_0^{k+\alpha-1} x)(t)\Big|_{t=0} = b$$

and, moreover,

$$x(t) = \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} y(s) ds =: (Uy)(t).$$

Thus problem (3.3) is transformed into problem (1.6), where $Au := W \circ U(u)$, with A continuous, compact and, obviously, causal.

Take any sequence (ε_n) converging to 0. Then applying the results above, we obtain the existence of a sequence of solutions y_n of (3.1) satisfying $(D_0^{\alpha-1} y_n)(t)\big|_{t=0} = b$ and converging in the L_1^τ-sense to a solution of equation $y = Ay$. We set

$$x_n := Uy_n \quad \text{and} \quad x := Uy.$$

Then, evidently, x_n satisfies the problem (3.3) and

$$x^{(k)}(t) = y(t) = (Ay)(t) = W(Uy)(t) = Wx(t),$$

for a.a. $t \in [0, \tau]$ and $x^{(j)}(0) = 0$, $j = 0, 1, \dots, k-1$. Finally, we observe that

$$\|x_n - x\|_\infty^\tau = \sup_{t \in [0,\tau]} \Big\| \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} [y_n(s) - y(s)] ds \Big\|_X \leq \frac{\tau^{k-1}}{(k-1)!} \|y_n - y\|_1^\tau.$$

The right-hand side tends to zero. The proof is complete. □

4. Auxiliary Lemmas

Before giving the proof of Theorem 3.2, we need some auxiliary facts concerning the series

$$\sum_{j=1}^\infty \frac{(-1)^{j-1} s^{j\alpha-1}}{\varepsilon^j \Gamma(j\alpha)}, \quad s > 0. \tag{4.1}$$

Lemma 4.1. *The series* (4.1) *converges absolutely and uniformly on compact subsets of* $[0, +\infty)$ *to a function* $k(s;\varepsilon)$, $\quad s > 0$*, which is continuous and positive.*

Proof. Define the sets

$$Q_1 := \{j \in \mathbb{Z} : \alpha \leq j\alpha < 1\}, \quad Q_k := \{j \in \mathbb{Z} : k \leq j\alpha < k+1\}, \quad k = 2, 3, \dots$$

Obviously, for $k \geq 2$ the set Q_k has at most $\mu := [\frac{1}{\alpha}] + 1$ elements. Absolutely, the series can be written as

$$\sum_{j=1}^\infty \frac{s^{j\alpha-1}}{\varepsilon^j \Gamma(j\alpha)} = \Lambda(s) + \sum_{k=3}^\infty \sum_{j \in Q_k} \frac{s^{j\alpha-1}}{\varepsilon^j \Gamma(j\alpha)},$$

where

$$\Lambda(s) := \sum_{k=1}^2 \sum_{j \in Q_k} \frac{s^{j\alpha-1}}{\varepsilon^j \Gamma(j\alpha)}, \quad s > 0$$

is an L_1^T function, for any $T > 0$.

Now, by using the fact that $(s+1)^{\alpha} > 1 > \varepsilon$ and the monotonicity of the function Γ on the interval $[2,+\infty)$, we obtain

$$\begin{aligned}
\sum_{j=1}^{\infty}\frac{(s+1)^{j\alpha-1}}{\varepsilon^{j}\Gamma(j\alpha)} &\leq \Lambda(s)+\sum_{k=3}^{\infty}\sum_{j\in Q_k}\frac{1}{s+1}\frac{(\frac{(s+1)^{\alpha}}{\varepsilon})^{j}}{\Gamma(k)} \\
&\leq \Lambda(s)+\sum_{k=3}^{\infty}\sum_{j\in Q_k}\frac{1}{s+1}\frac{(\frac{(s+1)^{\alpha}}{\varepsilon})^{\frac{k+1}{\alpha}}}{\Gamma(k)} \\
&= \Lambda(s)+\sum_{k=3}^{\infty}\sum_{j\in Q_k}\varepsilon^{\frac{1}{\alpha}}\frac{(\frac{(s+1)}{\varepsilon^{1/\alpha}})^{k}}{\Gamma(k)} \\
&\leq \Lambda(s)+\mu\sum_{k=3}^{\infty}\varepsilon^{\frac{1}{\alpha}}\frac{(\frac{(s+1)}{\varepsilon^{1/\alpha}})^{k}}{(k-1)!} \\
&= \Lambda(s)+\mu(s+1)\sum_{k=3}^{\infty}\frac{(\frac{(s+1)}{\varepsilon^{1/\alpha}})^{k-1}}{(k-1)!} \\
&= \Lambda(s)-\mu(s+1)(1+\frac{(s+1)}{\varepsilon^{1/\alpha}})+\mu(s+1)\exp(\frac{(s+1)}{\varepsilon^{1/\alpha}}).
\end{aligned}$$

The right-hand side defines an L_1^T function, for any $T>0$. Obviously, this proves the first part of the lemma.

It remains to show that the function $k(\cdot;\varepsilon)$ is positive. Indeed, by the previous arguments, we can apply the Lebesgue Dominated Convergence Theorem and get, for fixed $\theta\in[0,t]$, that

$$\begin{aligned}
\int_{t-\theta}^{t}k(s;\varepsilon)ds = \int_0^{\theta}k(t-s;\varepsilon)ds &= \int_0^{\theta}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}(t-s)^{j\alpha-1}}{\varepsilon^{j}\Gamma(j\alpha)}ds \\
&= \sum_{j=1}^{\infty}\frac{(-1)^{j}(t-\theta)^{j\alpha}}{\varepsilon^{j}\Gamma(j\alpha+1)}-\sum_{j=1}^{\infty}\frac{(-1)^{j}t^{j\alpha}}{\varepsilon^{j}\Gamma(j\alpha+1)} \\
&= \sum_{j=0}^{\infty}\frac{(-1)^{j}(t-\theta)^{j\alpha}}{\varepsilon^{j}\Gamma(j\alpha+1)}-\sum_{j=0}^{\infty}\frac{(-1)^{j}t^{j\alpha}}{\varepsilon^{j}\Gamma(j\alpha+1)} \\
&= E_{\alpha}(\frac{-(t-\theta)^{\alpha}}{\varepsilon})-E_{\alpha}(\frac{-t^{\alpha}}{\varepsilon}).
\end{aligned}\tag{4.2}$$

By using (2.4), relation (4.2) gives

$$\int_0^{\theta}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}(t-s)^{j\alpha-1}}{\varepsilon^{j}\Gamma(j\alpha)}ds = \int_0^{+\infty}(e^{-(t-\theta)s}-e^{-ts})dF_{\alpha}(s)\geq 0.$$

From the properties of E_{α} which we mentioned in Subsection 2.2, it follows that the quantity $E_{\alpha}(\frac{-t^{\alpha}}{\varepsilon})$ is positive and less than 1 and it tends to zero monotonically when t tends to $+\infty$. The latter implies that

$$\lim_{x\to+\infty}E_{\alpha}(-x)=0, \tag{4.3}$$

namely,

$$0<E_{\alpha}(\frac{-t^{\alpha}}{\varepsilon})\leq 1, \tag{4.4}$$

$$\lim_{t\to+\infty} E_\alpha(\frac{-t^\alpha}{\varepsilon}) = 0. \tag{4.5}$$

Obviously, (4.4) implies that

$$0 \le \int_0^t k(s;\varepsilon)ds < 1.$$

Finally, since the function

$$t \to \int_0^t k(s;\varepsilon)ds = 1 - E_\alpha(\frac{-t^\alpha}{\varepsilon}), \quad t \ge 0 \tag{4.6}$$

is increasing, its derivative, namely the function $k(t;\varepsilon)$, is positive. □

Lemma 4.2. *The following properties*[2] *hold:*

$$\lim_{\varepsilon\to 0} \int_0^t k(s;\varepsilon)ds = 1, \tag{4.7}$$

uniformly for t in intervals of the form $[r,T]$, for all $r \in (0,T]$ and

$$\lim_{\varepsilon\to 0} \int_\delta^t k(s;\varepsilon)ds = 0, \tag{4.8}$$

for all $t \in (0,T]$ and $\delta \in (0,t)$. For each $u \in L_1^T$ it holds

$$\lim_{\varepsilon\to 0} \int_0^t k(t-s;\varepsilon)u(s)ds = u(t). \tag{4.9}$$

Proof. Property (4.7) is easily implied from (4.3) and (4.2), while (4.8) follows from (4.2) and the fact that $\int_\delta^t k(s;\varepsilon)ds = E_\alpha(\frac{-\delta^\alpha}{\varepsilon}) - E_\alpha(\frac{-t^\alpha}{\varepsilon})$.

Next, let $u \in L_1^T$ and $\eta > 0$. Extend u from $[0,T]$ to $\mathbb{R}$ by setting $\bar{u}(s) = 0$, if $s \notin [0,\tau]$ and $\bar{u}(s) = u(s)$, $s \in [0,T]$. Then $\bar{u}$ is an element of $L_1(\mathbb{R},X)$ and, so it satisfies $\lim_{s\to 0} \|\bar{u}(\cdot - s) - \bar{u}(\cdot)\|_1^T = 0$, (see. e.g. [6, Thm 1.4.2 p. 298]). This means that there is an $s_0 > 0$ such that

$$\|\bar{u}(\cdot - s) - \bar{u}(\cdot)\|_1^T \le \eta, \quad 0 \le s \le s_0.$$

Take any $\delta \in (0,s_0]$. By (4.7), there is some $\varepsilon_\delta > 0$, such that for all $\varepsilon \in (0,\varepsilon_\delta]$ it holds

$$\Big|\int_0^t k(t-s;\varepsilon)ds - 1\Big| < \eta, \quad t \in [\delta,T].$$

Hence, we have

$$\Big\|\int_0^t k(t-s;\varepsilon)u(t)ds - u(t)\Big\|_X \le \eta\|u(t)\|_X, \quad t \in [\delta,T],$$

or

$$\Big\|\int_0^t \big[k(s;\varepsilon)u(s) - \frac{1}{t}u(t)\big]ds\Big\|_X \le \eta\|u(t)\|_X, \quad t \in [\delta,T]. \tag{4.10}$$

[2]These properties are enough to characterize the function k as an approximate identity of the convolution, which resembles to the well known Dirac sequences in the convolutions theory.

Taking into account Lemma 4.1 (i.e. that k is positive), we observe that

$$\begin{aligned}
&\int_\delta^T \|\int_0^t \big[k(t-s;\varepsilon)u(s)ds-u(t)\big]\|_X dt\\
&=\int_\delta^T \|\int_0^t \big[k(s;\varepsilon)\bar u(t-s)-\frac{1}{t}\bar u(t)\big]ds\|_X dt\\
&\le \int_\delta^T \|\int_0^t \big[k(s;\varepsilon)\bar u(t-s)ds-\int_0^t k(s;\varepsilon)\bar u(t)ds\big]\|_X dt\\
&\quad+\int_\delta^T \|\int_0^t \Big(k(s;\varepsilon)\bar u(t)-\frac{1}{t}\bar u(t)\Big)ds\|_X dt\\
&\le \int_\delta^T \|\int_0^t k(s;\varepsilon)[\bar u(t-s)-\bar u(t)]ds\|_X dt+\eta\int_\delta^T \|\bar u(t)\|_X dt\\
&\le \int_\delta^T\int_0^\delta k(s;\varepsilon)\|\bar u(t-s)-\bar u(t)\|_X\,ds\,dt\\
&\quad+\int_\delta^T\int_\delta^t k(s;\varepsilon)\|\bar u(t-s)-\bar u(t)\|_X\,ds\,dt+\eta\|u\|_1^T.
\end{aligned}\tag{4.11}$$

We estimate the right-hand side of relation (4.11). We have

$$\begin{aligned}
&\int_\delta^T\int_0^\delta k(s;\varepsilon)\|\bar u(t-s)-\bar u(t)\|_X\,ds\,dt\\
&=\int_0^\delta k(s;\varepsilon)\int_\delta^T \|\bar u(t-s)-\bar u(t)\|_X\,dt\,ds\\
&\le \int_0^\delta k(s;\varepsilon)\|\bar u(\cdot-s)-\bar u(\cdot)\|_1^T ds\\
&\le \eta\int_0^\delta k(s;\varepsilon)ds.
\end{aligned}$$

Also

$$\begin{aligned}
&\int_\delta^T\int_\delta^t k(s;\varepsilon)\|\bar u(t-s)-\bar u(t)\|_X\,ds\,dt\\
&=\int_\delta^T k(s;\varepsilon)\int_s^T \|\bar u(t-s)-\bar u(t)\|_X\,dt\,ds\\
&\le \int_\delta^T\int_0^T \big(k(s;\varepsilon)(\|\bar u(t-s)\|_X+\|\bar u(t)\|_X\big)\,dt\,ds\\
&\le 2\|u\|_1^T\int_\delta^T k(s;\varepsilon)ds.
\end{aligned}$$

Hence, (4.6) becomes

$$\begin{aligned}
&\int_\delta^T \|\int_0^t \big[k(t-s;\varepsilon)u(s)-\frac{1}{t}u(t)\big]ds\|_X dt\\
&\le \eta\int_0^\delta k(s;\varepsilon)ds+2\|u\|_1^T\int_\delta^T k(s;\varepsilon)ds+\eta\|u\|_1^T.
\end{aligned}$$

Now, in view of (4.7) and (4.8) as ε tends to 0, the right-hand side tends to $\eta(1+\|u\|_1^T)$. Since δ is arbitrary and small, we obtain

$$\int_0^T \Big\| \int_0^t \big[k(t-s;\varepsilon)u(s) - \frac{1}{t}u(t)\big]ds\Big\|_X dt \le \eta(1+\|u\|_1^T).$$

The fact that η is arbitrary completes the proof of relation (4.9). $\square$

5. Proof of theorem 3.2

To simplify notation, we set

$$\phi(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}b, \quad t \in (0,T]$$

and observe that ϕ is an element of L_1^T, for all $T>0$. Also, consider the operator

$$(L_\varepsilon u)(t) := \frac{1}{\varepsilon\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds, \quad u \in L_1^T.$$

Then relation (3.2) takes the form

$$y(t) = \phi(t) - (L_\varepsilon y)(t) + (L_\varepsilon Ay)(t)$$

which, by iteration, for each $n = 1,2,\dots$, gives

$$y(t) = \sum_{j=0}^{n-1}(-1)^j(L_\varepsilon^{(j)}\phi)(t) + (-1)^n(L_\varepsilon^n y)(t) + \sum_{j=1}^{n}(-1)^{j-1}(L_\varepsilon^{(j)}Ay)(t). \tag{5.1}$$

Let $u \in L_1^T$. We observe that

$$(L_\varepsilon^{(2)}u)(t) = \frac{1}{\varepsilon^2\Gamma(2\alpha)}\int_0^t (t-s)^{2\alpha-1}u(s)ds.$$

By induction we obtain

$$(L_\varepsilon^{(j)}u)(t) = \frac{1}{\varepsilon^j\Gamma(j\alpha)}\int_0^t (t-s)^{j\alpha-1}u(s)ds, \quad j = 1,2,\dots.$$

Then we have

$$\begin{aligned}
\|L_\varepsilon^{(j)}u\|_1^T &= \int_0^T \Big\|\frac{1}{\varepsilon^j\Gamma(j\alpha)}\int_0^t (t-s)^{j\alpha-1}u(s)ds\Big\|_X dt \\
&\le \int_0^T \frac{1}{\varepsilon^j\Gamma(j\alpha)}\int_s^t (t-s)^{j\alpha-1}\|u(s)\|_X\,dt\,ds \\
&\le \frac{T^{j\alpha}}{\varepsilon^j\Gamma(j\alpha+1)}\|u\|_1^T.
\end{aligned}$$

Since by definition

$$\sum_0^{+\infty}\frac{T^{j\alpha}}{\varepsilon^j\Gamma(j\alpha+1)} = E_\alpha(\frac{T^\alpha}{\varepsilon}),$$

where E_α is the Mittag-Leffler function, it follows that both series in (5.1) converge, yet

$$\lim_j L_\varepsilon^{(j)}u = 0.$$

So the right side of (5.1) converges to

$$\sum_{j=0}^{\infty}(-1)^j(L_\varepsilon^{(j)}\phi)(t)+\sum_{j=1}^{\infty}(-1)^{j-1}(L_\varepsilon^{(j)}Au)(t)=:Su(t)$$

and, therefore, we obtain

$$\begin{aligned}
Su(t)-\phi(t)&=\sum_{j=1}^{\infty}(-1)^{j-1}\big(L_\varepsilon^{(j)}(Au-\phi)\big)(t)dt\\
&=\sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{\varepsilon^j\Gamma(j\alpha)}\int_0^t(t-s)^{j\alpha-1}(Au(s)-\phi(s))ds\\
&=\int_0^t\sum_{j=1}^{\infty}\frac{(-1)^{j-1}(t-s)^{j\alpha-1}}{\varepsilon^j\Gamma(j\alpha)}(Au(s)-\phi(s))ds\\
&=\int_0^t k(t-s;\varepsilon)(Au(s)-\phi(s))ds,
\end{aligned}\tag{5.2}$$

where

$$k(s;\varepsilon):=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}s^{j\alpha-1}}{\varepsilon^j\Gamma(j\alpha)}.$$

The interchange of integration and summation is permitted because of Lemma 4.1. From (5.2) and the fact that k is positive, we obtain

$$\begin{aligned}
\|Su-\phi\|_1^T&=\int_0^T\|Su(t)-\phi(t)\|_X dt\\
&\le\int_0^T\int_0^t k(t-s;\varepsilon)\|Au(s)-\phi(s)\|_X\,ds\,dt\\
&=\int_0^T\int_s^T k(t-s;\varepsilon)\|Au(s)-\phi(s)\|_X\,dt\,ds\\
&=\int_0^T\Big[1-E_\alpha\Big(\frac{-(T-s)^\alpha}{\varepsilon}\Big)\Big]\|Au(s)-\phi(s)\|_X ds\\
&\le\|Au-\phi\|_1^T.
\end{aligned}\tag{5.3}$$

We claim that, for any $R>0$, there exists $\tau\in(0,T]$, such that in the space L_1^τ, it holds

$$S(\overline{B(\phi,R)})\subseteq\overline{B(\phi,R)}.$$

By (5.3), to show this fact, it is sufficient to prove that there is a $\tau\in(0,T]$, such that in the space L_1^τ, it holds

$$A(\overline{B(\phi,R)})\subseteq\overline{B(\phi,R)}.\tag{5.4}$$

Let $\overline{B(\phi,R)}$ be the closed ball $\{u\in L_1^\tau:\quad\|u-\phi\|_1^T\le R\}$. Fix any $\zeta\in(0,\frac{R}{2}]$. Since the set $A(\overline{B(\phi,R)})$ has compact closure, there is a finite ζ-dense subset of it, say, $Au_1,Au_2,\dots,Au_k\in A(\overline{B(\phi,R)})$. Also, we can find $\tau\in(0,T]$ such that

$$\|Au_j-\phi\|_1^\tau=\int_0^\tau\|(Au_j)(t)-\phi(t)\|_X dt\le\zeta,\quad j=1,2,\dots,k.$$

Take any $u \in \overline{B(\phi, R)}$. Then $Au \in A(\overline{B(\phi, R)})$ and, thus, $\|Au - Au_j\|_1^\tau \le \zeta$, for some j. Hence,

$$\|Au - \phi\|_1^\tau \le \|Au - Au_j\|_1^\tau + \|Au_j - \phi\|_1^\tau \le 2\zeta \le R.$$

Therefore (5.4) is true.

Because of the previous facts, the fixed point Theorem 3.1 applies and we conclude that there is $y_\varepsilon \in \overline{B([0,\tau], R)}$, such that

$$y_\varepsilon(t) = (Sy_\varepsilon)(t) = \sum_{j=0}^{\infty}(-1)^j (L_\varepsilon^{(j)}\phi)(t) + \sum_{j=1}^{\infty}(-1)^{j-1}(L_\varepsilon^{(j)}Ay_\varepsilon)(t), \quad t \in [0, \tau],$$

or, by (5.2),

$$y_\varepsilon(t) - \phi(t) = \int_0^t k(t-s;\varepsilon)(Ay_\varepsilon(s) - \phi(s))ds, \quad t \in [0,\tau].$$

Next, we take any sequence ε_n tending to 0, and denote by y_n the solution y_{ε_n}. Hence we have

$$y_n(t) - \phi(t) = \int_0^t k(t-s;\varepsilon_n)(Ay_n(s) - \phi(s))ds, \quad t \in [0,\tau]. \tag{5.5}$$

By the relative compactness of the set $A(\overline{(B(\phi, R)})$, we can assume that the sequence (Ay_n) converges to some $y \in L_1^\tau$. Then, for almost all $t \in [0,\tau]$, from (5.5) we obtain

$$y_n(t) - y(t) = \int_0^t k(t-s;\varepsilon_n)(Ay_n(s) - \phi(s))ds - (y(t) - \phi(t))$$

and, therefore, it follows that

$$\begin{aligned}
\|y_n - y\|_1^\tau &= \int_0^\tau \Big\|\Big(\int_0^t k(t-s;\varepsilon_n)\big[Ay_n(s) - \phi(s)\big]ds\Big) - (y(t) - \phi(t))\Big\|_X dt \\
&\le \int_0^\tau \int_0^t k(t-s;\varepsilon_n)\|Ay_n(s) - y(s)\|_X \,ds\,dt \\
&\quad + \int_0^\tau \Big\| \int_0^t k(t-s;\varepsilon_n)(y(s) - \phi(s))ds - (y(t) - \phi(t))\Big\|_X dt.
\end{aligned}$$

For the first integral on the right side we have

$$\begin{aligned}
&\int_0^\tau \int_0^t k(s;\varepsilon_n)\big\|(Ay_n)(t-s) - y(t-s)\big\|_X \,ds\,dt \\
&= \int_0^\tau \int_s^\tau k(s;\varepsilon_n)\big\|(Ay_n)(t-s) - y(t-s)\big\|_X \,dt\,ds \\
&\le \int_0^\tau k(s;\varepsilon_n)\int_s^\tau \big\|(Ay_n)(t-s) - y(t-s)\big\|_X \,dt\,ds \\
&= \int_0^\tau k(s;\varepsilon_n)\int_0^{\tau-s} \big\|(Ay_n)(\xi) - y(\xi)\big\|_X d\xi ds \\
&\le \int_0^\tau k(s;\varepsilon_n)ds\|Ay_n - y\|_1^\tau,
\end{aligned}$$

which tends to 0. Also, the sequence

$$\int_0^\tau \Big\| \int_0^t k(t-s;\varepsilon_n)(y(s) - \phi(s))ds - (y(t) - \phi(t))\Big\|_X dt$$

tends to 0, because of (4.9). Hence, we have $\lim y_n = y$ and, by the continuity of A, it follows that $y = \lim Ay_n = Ay$. The proof is complete.

References

[1] Ya. I. Alber; Recurrence relations and variational inequalities, *Soviet Math. Dokl.*, **27** (1983), 511-517.

[2] Yakov Alber, Simeon Reich, David Shoikhet; Iterative approximations of null points of uniformly accretive operators with estimates of the convergence rate, *Commun. Appl. Nonlinear Anal.* **3** (8-9) (2002), 1107–1124.

[3] Cara D. Brooks, Patricia K. Lamm, Xiaoyue Luo; Local regularization of nonlinear Volterra equations of Hammerstein type, *Integral Equations Appl.*, 09/2010; 22(2010). DOI: 10.1216/JIE-2010-22-3-393.

[4] F. E. Browder, W. V. Petryshyn; Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.*, **20** (1967), 197-228.

[5] C. Corduneanu; *Integral Equations and Applications*, Cambridge Univ. Press, New York, 1991.

[6] Misha Cotlar, Roberto Cignoli; *An Introduction to Functional Analysis*, American Elsevier Publ. Co. New York, 1974.

[7] Heinz W. Engl; On the choice of the regularization parameter for iterated Tikhonov regularization of ill-posed problems, *Journal of Approx. Theory* 49 (1987), 55-63.

[8] Markus Haltmeier, Antonio Leitão, Otmar Scherzer; Kaczmarz methods for regularizing nonlinear ill-posed equations I: Convergence analysis, *Inverse Problems and Imaging* 1(2007), 289-298.

[9] A. L. Gaponenko, Yu L. Gaponenko; A method of regularization for operator equations of the first kind, *Zh. vỹchisl. Mat. mat. Fiz.*, 16 (1976), 577-584.

[10] C. W. Groetsch; Integral equations of the first kind, inverse problems and regularization: a crash course, *Journal of Physics: Conference Series* 73 (2007) 1-32.

[11] Nicola Guglielmi, Ernst Hairer; Regularization of neutral delay differential equations with several delays, *J. Dynam. Differential Equations* 7, (2012), 1-26.

[12] O. Jumarie; Modified Riemann-Liouville Derivative and Fractional Taylor Series of Nondifferentiable Functions Further Results, *Comput. Math. Appl.* 51 (2006) 1367-1376.

[13] George L. Karakostas; Causal operators and Topological Dynamics, *Ann. Matematica Pura ed Appl.* Vol. CXXXI, 1982, 1-27.

[14] George L. Karakostas; Strong approximation of the solutions of a system of operator equations in Hilbert spaces, *J. Difference Equ. Appl.* 12 (2006), 619-632.

[15] V. Lakshmikantham, A. S. Vatsala; General uniqueness and monotone iterative technique for fractional differential equations, *Appl. Math. Lett.* 21 (2008), 828-834.

[16] Patricia K. Lamm; A Survey of Regularization Methods for First-Kind Volterra Equations, Mathematics Dept., Michigan State University, E. Lansing, MI 48824-1027 USA, http://www.mth.msu.edu/lamm (May 19, 2015).

[17] Ping Lin; Regularization methods for differential equations and their numerical solution, Ph. D. Thesis, The University of British Columbia, 1995.

[18] R. März; *Numerical methods for differential-algebraic equations., Part I: Characterizing DAEe*, Preprint No. 91-32/I, Humboldt Universität zu Berlin, 1991.

[19] Abdelaziz Mennouni; A regularization procedure for solving some singular integral equations of the second kind, *Internat. J. Difference Equations* 8 (2013), 71-76.

[20] Kenneth S. Miller, Bertram Ross; *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc. New York, 1993.

[21] A. Neubauer; Tikhonov-Regularization of ill-Posed Linear Operator Equations on Closed Convex Sets, *J. Approx. Theory* 53(1988), 304-320.

[22] Abdul-Majid Wazwaz; Solving Schlömilch's integral equation by the regularization-Adomian method, *Rom. Journ. Phys.*, 60 (2015), 56-72.

[23] L. W. Neustadt; On the solutions of certain integral like operator equations. Existence, uniqueness and dependence theorem, *Arch. Rat. Mech. Anal.*, 38 (1970), 131-160.

[24] Igor Podlubny; *Fractional Differential Equation*, Mathematics in Science and Engineering, Vol. 118, Acad. Press, 1999.

[25] Harry Pollard; The completely monotonic character of the Mittag-Leffler function $E_\alpha(x)$, *Bull. Amer. Math. Soc.* Vol. 54, (12), (1948), 1115-1116.
[26] E. Prempeh, I. Owusu-Mensay, K. Piesie-Frimbong; On the regularization of Hammerstein's type operator equations, *Aust. J. Math. Anal. Appl.*, 11 (2014), 1-10.
[27] T. I. Savelova; Optimal regularization of equations of the convolution type with random noise in the kernel and right-hand side, *U.S.U.R. Comput. Math. Phys.* 18(1978), 1-7.
[28] T. I. Savelova; Regularization of non-linear integral equations of the convolution type, *U.S.U.R. Comput. Math. Phys.* 19(1979), 20-27.
[29] Ishikawa Shiro; Fixed points by a new iteration method, *Proc. Amer. Math. Soc.*, **44**(1) (1974), 147-150.
[30] Jin Wen, Ting Wei; Regularized solution to the Fredholm integral equation of the first kind with noisy data, *J. Appl. Math. and Informatics* 29(2011), 23-37.
[31] Wikipedia, http://en.wikipedia.org/wiki/Particular_values_of_the_Gamma_function #Other_constants (May 26, 2015).

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