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EXISTENCE OF MULTIPLE SOLUTIONS FOR QUASILINEAR DIAGONAL ELLIPTIC SYSTEMS

MARCO SQUASSINA

Abstract. Nonsmooth-critical-point theory is applied in proving multiplicity results for the quasilinear symmetric elliptic system

$$-\sum_{i,j=1}^{n} D_j(a_{ij}^k(x,u)D_iu_k)+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{h=1}^{N} D_{s_k}a_{ij}^h(x,u)D_iu_hD_ju_h=g_k(x,u)\,,$$

for $k=1,..,N$.

1. Introduction

Many papers have been published on the study of multiplicity of solutions for quasilinear elliptic equations via nonsmooth-critical-point theory; see e.g. [2, 3, 4, 7, 8, 9, 10, 18, 20]. However, for the vectorial case only a few multiplicity results have been proven: [20, Theorem 3.2] and recently [4, Theorem 3.2], where systems with multiple identity coefficients are treated. In this paper, we consider the following diagonal quasilinear elliptic system, in an open bounded set $\Omega\subset\mathbb{R}^n$ with $n\geq 3$,

$$\begin{aligned}-\sum_{i,j=1}^{n} D_j(a_{ij}^k(x,u)D_iu_k)+\frac{1}{2}\sum_{i,j=1}^{n}\sum_{h=1}^{N} D_{s_k}a_{ij}^h(x,u)D_iu_hD_ju_h=\\ =D_{s_k}G(x,u)\quad\text{in }\Omega\,,\end{aligned}\tag{1}$$

for $k=1,..,N$, where $u:\Omega\to\mathbb{R}^N$ and $u=0$ on $\partial\Omega$. To prove the existence of weak solutions, we look for critical points of the functional $f:H_0^1(\Omega,\mathbb{R}^N)\to\mathbb{R}$,

$$f(u)=\frac{1}{2}\int_\Omega\sum_{i,j=1}^{n}\sum_{h=1}^{N} a_{ij}^h(x,u)D_iu_hD_ju_h\,dx-\int_\Omega G(x,u)\,dx\,.\tag{2}$$

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This functional is not locally Lipschitz if the coefficients a_{ij}^h depend on u; however, as pointed out in [2, 7], it is possible to evaluate f',

$$\begin{aligned} f'(u)(v) &= \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j v_h \, dx + \\ &\quad + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u) \cdot v D_i u_h D_j u_h \, dx - \int_\Omega D_s G(x,u) \cdot v \, dx \end{aligned}$$

for all $v \in H_0^1(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$. We shall apply the nonsmooth-critical-point theory developed in [11, 13, 15, 16]. For notation and related results, the reader is referred to [9]. To prove our main result and to provide some regularity of solutions, we consider the following assumptions.

(A1) The matrix $\left(a_{ij}^h(\cdot, s)\right)$ is measurable in x for every $s \in \mathbb{R}^N$, and of class C^1 in s for a.e. $x \in \Omega$ with

$$a_{ij}^h = a_{ji}^h \,.$$

Furthermore, we assume that there exist $\nu > 0$ and $C > 0$ such that for a.e. $x \in \Omega$, all $s \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$

$$\sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,s) \xi_i^h \xi_j^h \geq \nu |\xi|^2, \quad \left|a_{ij}^h(x,s)\right| \leq C, \quad \left|D_s a_{ij}^h(x,s)\right| \leq C \tag{3}$$

and

$$\sum_{i,j=1}^n \sum_{h=1}^N s \cdot D_s a_{ij}^h(x,s) \xi_i^h \xi_j^h \geq 0. \tag{4}$$

(A2) There exists a bounded Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$, such that for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^{nN}$, $\sigma \in \{-1, 1\}^N$ and $r, s \in \mathbb{R}^N$

$$\sum_{i,j=1}^n \sum_{h=1}^N \left(\frac{1}{2} D_s a_{ij}^h(x,s) \cdot \exp_\sigma(r,s) + a_{ij}^h(x,s) D_{s_h} (\exp_\sigma(r,s))_h \right) \xi_i^h \xi_j^h \leq 0, \tag{5}$$

where $(\exp_\sigma(r,s))_i := \sigma_i \exp[\sigma_i(\psi(r_i) - \psi(s_i))]$ for each $i = 1, .., N$.

(G1) The function $G(x,s)$ is measurable in x for all $s \in \mathbb{R}^N$ and of class C^1 in s for a.e. $x \in \Omega$, with $G(x,0) = 0$. Moreover for a.e. $x \in \Omega$ we will denote with $g(x, \cdot)$ the gradient of G with respect to s.

(G2) For every $\varepsilon > 0$ there exists $a_\varepsilon \in L^{2n/(n+2)}(\Omega)$ such that

$$|g(x,s)| \leq a_\varepsilon(x) + \varepsilon |s|^{(n+2)/(n-2)} \tag{6}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^N$ and that there exist $q > 2$, $R > 0$ such that for all $s \in \mathbb{R}^N$ and for a.e. $x \in \Omega$

$$|s| \geq R \Longrightarrow 0 < qG(x,s) \leq s \cdot g(x,s). \tag{7}$$

(AG) There exists $\gamma \in (0, q-2)$ such that for all $\xi \in \mathbb{R}^{nN}$, $s \in \mathbb{R}^N$ and a.e. in Ω

$$\sum_{i,j=1}^n \sum_{h=1}^N s \cdot D_s a_{ij}^h(x,s) \xi_i^h \xi_j^h \leq \gamma \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,s) \xi_i^h \xi_j^h \,. \tag{8}$$

Under these assumptions we will prove the following.

Theorem 1. *Assume that for a.e.* $x \in \Omega$ *and for each* $s \in \mathbb{R}^N$

$$a_{ij}^h(x,-s) = a_{ij}^h(x,s), \quad g(x,-s) = -g(x,s)\,.$$

Then there exists a sequence $(u^m) \subseteq H_0^1(\Omega,\mathbb{R}^N)$ *of weak solutions to* (1) *such that*

$$\lim_m f(u^m) = +\infty\,.$$

The above result is well known for the semilinear scalar problem

$$-\sum_{i,j=1}^n D_j(a_{ij}(x)D_iu) = g(x,u) \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega\,.$$

A. Ambrosetti and P. H. Rabinowitz in [1, 19] studied this problem using techniques of classical critical point theory. The quasilinear scalar problem

$$-\sum_{i,j=1}^n D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u)D_iuD_ju = g(x,u) \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega\,,$$

was studied in [7, 8, 9] and in [18] in a more general setting. In this case the functional

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u)D_iuD_ju\,dx - \int_\Omega G(x,u)\,dx$$

is continuous under appropriate conditions, but it is not locally Lipschitz. Consequently, techniques of nonsmooth-critical-point theory have to be applied. In the vectorial case, to my knowledge, problem (1) has only been considered in [20, Theorem 3.2] and recently in [4, Theorem 3.2] for coefficients of the type $a_{ij}^{hk}(x,s) = \delta^{hk}\alpha_{ij}(x,s)$. In [4] a new technical condition is introduced to be compared with our (5). They assume that there exist $K > 0$ and an increasing bounded Lipschitz function $\psi : [0,+\infty[\to [0,+\infty[$, with $\psi(0) = 0$, ψ' nonincreasing, $\psi(t) \to K$ as $t \to +\infty$ and such that for all $\xi \in \mathbb{R}^n$, for a.e. $x \in \Omega$ and for all $r, s \in \mathbb{R}^N$

$$\sum_{i,j=1}^n \sum_{k=1}^N |D_{s_k} a_{ij}(x,s)\xi_i\xi_j| \le 2e^{-4K}\psi'(|s|)\sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j\,. \tag{9}$$

The proof itself of [4, Lemma 6.1] shows that condition (9) implies our assumption **(A2)**. On the other hand, if $N \ge 2$, the two conditions look quite similar. However, condition **(A2)** seems to be preferable, because when $N = 1$ it reduces to the inequality

$$\left|\sum_{i,j=1}^n D_s a_{ij}(x,s)\xi_i\xi_j\right| \le 2\psi'(s)\sum_{i,j=1}^n a_{ij}(x,s)\xi_i\xi_j,$$

which is not so restrictive in view of (3), while (9) is in this case much stronger.

2. Boundedness of concrete Palais-Smale sequences

Definition 2. *Let $c \in \mathbb{R}$. A sequence $(u^m) \subseteq H_0^1(\Omega;\mathbb{R}^N)$ is said to be* a concrete Palais-Smale sequence at level c *($(CPS)_c-$*sequence*, in short) for f, if $f(u^m) \to c$,*

$$\sum_{i,j=1}^{n} \sum_{h=1}^{N} D_{s_k} a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \in H^{-1}(\Omega;\mathbb{R}^N)$$

eventually as $m \to \infty$, and

$$-\sum_{i,j=1}^{n} D_j(a_{ij}^k(x,u^m) D_i u_k^m) + \frac{1}{2} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_{s_k} a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m - g_k(x,u^m)$$

converges to zero strongly in $H^{-1}(\Omega;\mathbb{R}^N)$. We say that f satisfies the concrete Palais-Smale condition at level c *($(CPS)_c$ in short), if every $(CPS)_c-$sequence for f admits a strongly convergent subsequence in $H_0^1(\Omega;\mathbb{R}^N)$.*

Next we state and prove a vectorial version of Brezis-Browder's Theorem [6].

Lemma 3. *Let $T \in L^1_{\rm loc}(\Omega,\mathbb{R}^N) \cap H^{-1}(\Omega,\mathbb{R}^N)$, $v \in H_0^1(\Omega,\mathbb{R}^N)$ and $\eta \in L^1(\Omega)$ with $T \cdot v \geq \eta$. Then $T \cdot v \in L^1(\Omega)$ and*

$$\langle T, v\rangle = \int_\Omega T \cdot v \, dx$$

Proof. Let $(v_h) \subseteq C_c^\infty(\Omega,\mathbb{R}^N)$ with $v_h \to v$. Define $\Theta_h(v) \in H_0^1 \cap L^\infty$ with compact support in Ω by setting

$$\Theta_h(v) = \min\{|v|,|v_h|\} \frac{v}{\sqrt{|v|^2 + \frac{1}{h}}}.$$

Since

$$\min\{|v|,|v_h|\} \frac{T \cdot v}{\sqrt{|v|^2 + \frac{1}{h}}} \geq -\eta^- \in L^1(\Omega),$$

and

$$\left\langle T, \min\{|v|,|v_h|\} \frac{v}{\sqrt{|v|^2 + \frac{1}{h}}} \right\rangle = \int_\Omega \min\{|v|,|v_h|\} \frac{T \cdot v}{\sqrt{|v|^2 + \frac{1}{h}}} \, dx,$$

a variant of Fatou's Lemma implies $\int_\Omega T \cdot v \, dx \leq \langle T, v\rangle$, so that $T \cdot v \in L^1(\Omega)$. Finally, since

$$\left| \min\{|v|,|v_h|\} \frac{T \cdot v}{\sqrt{|v|^2 + \frac{1}{h}}} \right| \leq |T \cdot v|,$$

Lebesgue's Theorem yields

$$\langle T, v\rangle = \int_\Omega T \cdot v \, dx,$$

and the proof is complete. □

The first step for the $(CPS)_c$ to hold is the boundedness of $(CPS)_c$ sequences.

Lemma 4. *Assume* **(A1)**, **(G1)**, **(G2)** *and* **(AG)**. *Then for all $c \in \mathbb{R}$ each $(CPS)_c$ sequence of f is bounded in $H_0^1(\Omega,\mathbb{R}^N)$.*

Proof. Let $a_0 \in L^1(\Omega)$ be such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^N$

$$qG(x,s) \leq s \cdot g(x,s) + a_0(x).$$

Now let (u^m) be a $(CPS)_c$ sequence for f and let $w^m \to 0$ in $H^{-1}(\Omega,\mathbb{R}^N)$ such that for all $v \in C_c^\infty(\Omega,\mathbb{R}^N)$,

$$\begin{aligned}\langle w^m, v\rangle \;=\; & \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j v_h \, dx + \\ & + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u^m)\cdot v D_i u_h^m D_j u_h^m \, dx - \int_\Omega g(x,u^m)\cdot v \,.\end{aligned}$$

Taking into account the previous Lemma, for every $m \in \mathbb{N}$ we obtain

$$\begin{aligned}& -\|w^m\|_{H^{-1}(\Omega,\mathbb{R}^N)}\|u^m\|_{H_0^1(\Omega,\mathbb{R}^N)} \leq \\ & \leq \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \, dx + \\ & \quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u^m)\cdot u^m D_i u_h^m D_j u_h^m \, dx - \int_\Omega g(x,u^m)\cdot u^m \, dx \leq \\ & \leq \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \, dx + \\ & \quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u^m)\cdot u^m D_i u_h^m D_j u_h^m \, dx + \\ & \quad - q\int_\Omega G(x,u^m)\, dx + \int_\Omega a_0 \, dx \,.\end{aligned}$$

Taking into account the expression of f and assumption **(AG)**, we have that for each $m \in \mathbb{N}$,

$$\begin{aligned}& -\|w^m\|_{H^{-1}(\Omega,\mathbb{R}^N)}\|u^m\|_{H_0^1(\Omega,\mathbb{R}^N)} \leq \\ & \leq -\Big(\frac{q}{2}-1\Big)\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \, dx + \\ & \quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u^m)\cdot u^m D_i u_h^m D_j u_h^m \, dx + qf(u^m) + \int_\Omega a_0 \, dx \leq \\ & \leq -\Big(\frac{q}{2}-1-\frac{\gamma}{2}\Big)\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \, dx + \\ & \quad + qf(u^m) + \int_\Omega a_0 \, dx \,.\end{aligned}$$

Because of **(A1)**, for each $m \in \mathbb{N}$,

$$\begin{aligned}\nu(q-2-\gamma)\|Du^m\|_2^2 &\leq (q-2-\gamma)\int_\Omega \sum_{i,j=1}^{n}\sum_{h=1}^{N} a_{ij}^h(x,u^m)D_iu_h^m D_ju_h^m\,dx \\ &\leq 2\|w^m\|_{H^{-1}(\Omega,\mathbb{R}^N)}\|u^m\|_{H_0^1(\Omega,\mathbb{R}^N)} + 2qf(u^m) + 2\int_\Omega a_0\,dx\,.\end{aligned}$$

Since (w^m) converges to 0 in $H^{-1}(\Omega,\mathbb{R}^N)$, we conclude that (u^m) is a bounded sequence in $H_0^1(\Omega,\mathbb{R}^N)$. □

Lemma 5. *If condition* (6) *holds, then the map*

$$\begin{aligned} H_0^1(\Omega,\mathbb{R}^N) &\longrightarrow L^{2n/(n+2)}(\Omega,\mathbb{R}^N) \\ u &\longmapsto g(x,u) \end{aligned}$$

is completely continuous.

Proof. This is a direct consequence of [9, Theorem 2.2.7]. □

3. Compactness of concrete Palais-Smale sequences

The next result is crucial for the $(CPS)_c$ condition to hold for our elliptic system.

Lemma 6. *Assume* **(A1)** *and* **(A2)**, *let* (u^m) *be a bounded sequence in* $H_0^1(\Omega,\mathbb{R}^N)$, *and set*

$$\begin{aligned}\langle w^m, v\rangle &= \int_\Omega \sum_{i,j=1}^{n}\sum_{h=1}^{N} a_{ij}^h(x,u^m)D_iu_h^m D_jv_h\,dx + \\ &+\frac{1}{2}\int_\Omega \sum_{i,j=1}^{n}\sum_{h=1}^{N} D_sa_{ij}^h(x,u^m)\cdot vD_iu_h^m D_ju_h^m\,dx\end{aligned}$$

for all $v \in C_c^\infty(\Omega,\mathbb{R}^N)$. *If* (w^m) *is strongly convergent to some* w *in* $H^{-1}(\Omega,\mathbb{R}^N)$, *then* (u^m) *admits a strongly convergent subsequence in* $H_0^1(\Omega,\mathbb{R}^N)$.

Proof. Since (u^m) is bounded, we have $u^m \rightharpoonup u$ for some u up to a subsequence. Each component u_k^m satisfies (2.5) in [5], so we may suppose that $D_iu_k^m \to D_iu_k$ a.e. in Ω for all $k = 1,\dots,N$ (see also [12]). We first prove that

$$\begin{aligned}&\int_\Omega \sum_{i,j=1}^{n}\sum_{h=1}^{N} a_{ij}^h(x,u)D_iu_h D_ju_h\,dx + \\ &\qquad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n}\sum_{h=1}^{N} D_sa_{ij}^h(x,u)\cdot uD_iu_h D_ju_h\,dx = \langle w,u\rangle. \quad (10)\end{aligned}$$

Let ψ be as in assumption **(A2)** and consider the following test functions

$$v^m = \varphi(\sigma_1\exp[\sigma_1(\psi(u_1)-\psi(u_1^m))],\dots,\sigma_N\exp[\sigma_N(\psi(u_N)-\psi(u_N^m))]),$$

where $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$ and $\sigma_l = \pm 1$ for all l. Therefore, since we have

$$D_jv_k^m = (\sigma_kD_j\varphi + (\psi'(u_k)D_ju_k - \psi'(u_k^m)D_ju_k^m)\varphi)\exp[\sigma_k(\psi(u_k)-\psi(u_k^m))],$$

we deduce that for all $m \in \mathbb{N}$,

$$\begin{aligned}
&\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m (\sigma_h D_j\varphi + \psi'(u_h) D_j u_h \varphi) \exp[\sigma_h(\psi(u_h) - \psi(u_h^m))]\,dx + \\
&\quad + \int_\Omega \sum_{i,j=1}^n \sum_{h,l=1}^N \frac{\sigma_l}{2} D_{s_l} a_{ij}^h(x,u^m) \exp[\sigma_l(\psi(u_l) - \psi(u_l^m))] D_i u_h^m D_j u_h^m \varphi\,dx + \\
&\quad - \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \psi'(u_h^m) \exp[\sigma_h(\psi(u_h) - \psi(u_h^m))] \varphi\,dx = \\
&\qquad\qquad = \langle w^m, v^m\rangle\,.
\end{aligned}$$

Let us study the behavior of each term of the previous equality as $m \to \infty$. First of all, if $v = (\sigma_1\varphi, \dots, \sigma_N\varphi)$, we have that $v^m \rightharpoonup v$ implies

$$\lim_m \langle w^m, v^m\rangle = \langle w, v\rangle. \tag{11}$$

Since $u^m \rightharpoonup u$, by Lebesgue's Theorem we obtain

$$\begin{aligned}
&\lim_m \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m (D_j(\sigma_h\varphi) + \\
&\qquad\qquad + \varphi\psi'(u_h) D_j u_h) \exp[\sigma_h(\psi(u_h) - \psi(u_h^m))]\,dx = \\
&= \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h (D_j v_h + \varphi\psi'(u_h) D_j u_h)\,dx\,.
\end{aligned} \tag{12}$$

Finally, note that by assumption **(A2)** we have

$$\begin{aligned}
&\sum_{i,j=1}^n \sum_{h=1}^N \Big(\sum_{l=1}^N \frac{\sigma_l}{2} D_{s_l} a_{ij}^h(x,u^m) \exp[\sigma_l(\psi(u_l) - \psi(u_l^m))] + \\
&- a_{ij}^h(x,u^m) \psi'(u_h^m) \exp[\sigma_h(\psi(u_h) - \psi(u_h^m))] \Big) D_i u_h^m D_j u_h^m \quad \le \quad 0\,.
\end{aligned}$$

Hence, we can apply Fatou's Lemma to obtain

$$\begin{aligned}
&\limsup_m \Big\{ \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h,l=1}^N D_{s_l} a_{ij}^h(x,u^m) \exp[\sigma_l(\psi(u_l) - \psi(u_l^m))] D_i u_h^m D_j u_h^m (\sigma_l\varphi)\,dx + \\
&- \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \psi'(u_h^m) \exp[\sigma_h(\psi(u_h) - \psi(u_h^m))] \varphi\,dx \Big\} \le \\
&\le \quad \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h,l=1}^N D_{s_l} a_{ij}^h(x,u) D_i u_h D_j u_h (\sigma_l\varphi)\,dx + \\
&\quad - \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \psi'(u_h) \varphi\,dx\,,
\end{aligned}$$

which, together with (11) and (12), yields

$$\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j v_h \, dx + \\ + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u) \cdot v D_i u_h D_j u_h \, dx \geq \langle w, v \rangle$$

for all test functions $v = (\sigma_1 \varphi, \dots, \sigma_N \varphi)$ with $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$, $\varphi \geq 0$. Since we may exchange v with $-v$ we get

$$\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j v_h \, dx + \\ + \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u) \cdot v D_i u_h D_j u_h \, dx = \langle w, v \rangle$$

for all test functions $v = (\sigma_1 \varphi, \dots, \sigma_N \varphi)$, and since every function $v \in C_c^\infty(\Omega, \mathbb{R}^N)$ can be written as a linear combination of such functions, taking into account Lemma 3, we infer (10). Now, let us prove that

$$\limsup_m \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \, dx \leq \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \, dx. \tag{13}$$

Because of (4), Fatou's Lemma implies that

$$\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u \cdot D_s a_{ij}^h(x,u) D_i u_h D_j u_h \, dx \leq \\ \leq \liminf_m \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u^m \cdot D_s a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \, dx \,.$$

Combining this fact with (10), we deduce that

$$\begin{aligned} &\limsup_m \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \, dx = \\ &= \limsup_m \Big[-\frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u^m \cdot D_s a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m \, dx + \langle w^m, u^m \rangle \Big] \leq \\ &\leq -\frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N u \cdot D_s a_{ij}^h(x,u) D_i u_h D_j u_h \, dx + \langle w, u \rangle = \\ &= \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \, dx \,, \end{aligned}$$

so that (13) is proved. Finally, by (3) we have

$$\begin{aligned} &\nu \| Du^m - Du \|_2^2 \leq \\ &\leq \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) \left(D_i u_h^m D_j u_h^m - 2 D_i u_h^m D_j u_h + D_i u_h D_j u_h \right) dx. \end{aligned}$$

Hence, by (13) we obtain

$$\limsup_m \|Du^m - Du\|_2 \le 0$$

which proves that $u^m \to u$ in $H_0^1(\Omega, \mathbb{R}^N)$. □

We now come to one of the main tools of this paper, the $(CPS)_c$ condition for system (1).

Theorem 7. *Assume* **(A1)**, **(A2)**, **(G1)**, **(G2)**, **(AG)**. *Then f satisfies $(CPS)_c$ condition for each $c \in \mathbb{R}$.*

Proof. Let (u^m) be a $(CPS)_c$ sequence for f. Since (u^m) is bounded in $H_0^1(\Omega, \mathbb{R}^N)$, from Lemma 5 we deduce that, up to a subsequence, $(g(x, u^m))$ is strongly convergent in $H^{-1}(\Omega, \mathbb{R}^N)$. Applying Lemma 6, we conclude the present proof. □

4. Existence of multiple solutions for elliptic systems

We now prove the main result, which is an extension of theorems of [7, 9] and a generalization of [4, Theorem 3.2] to systems in diagonal form.

Proof of Theorem 1. We want to apply [9, Theorem 2.1.6]. First of all, because of Theorem 7, f satisfies $(CPS)_c$ for all $c \in \mathbb{R}$. Whence, (c) of [9, Theorem 2.1.6] is satisfied. Moreover we have

$$\begin{aligned} \frac{\nu}{2}\int_\Omega |Du|^2\,dx - \int_\Omega G(x,u)\,dx &\le f(u) \le \\ &\le \frac{1}{2}nNC\int_\Omega |Du|^2\,dx - \int_\Omega G(x,u)\,dx. \end{aligned}$$

We want to prove that assumptions (a) and (b) of [9, Theorem 2.1.6] are also satisfied. Let us observe that, instead of (b) of [9, Theorem 2.1.6], it is enough to find a sequence (W_n) of finite dimensional subspaces with $\dim(W_n) \to +\infty$ satisfying the inequality of (b) (see also [17, Theorem 1.2]). Let W be a finite dimensional subspace of $H_0^1(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$. From (7) we deduce that for all $s \in \mathbb{R}^N$ with $|s| \ge R$

$$G(x,s) \ge \frac{G\left(x, R\frac{s}{|s|}\right)}{R^q}|s|^q \ge b_0(x)|s|^q,$$

where

$$b_0(x) = R^{-q}\inf\{G(x,s) : |s| = R\} > 0$$

a.e. $x \in \Omega$. Therefore there exists $a_0 \in L^1(\Omega)$ such that

$$G(x,s) \ge b_0(x)|s|^q - a_0(x) \tag{14}$$

a.e. $x \in \Omega$ and for all $s \in \mathbb{R}^N$. Since $b_0 \in L^1(\Omega)$, we may define a norm $\|\cdot\|_G$ on W by

$$\|u\|_G = \left(\int_\Omega b_0|u|^q\,dx\right)^{1/q}.$$

Since W is finite dimensional and $q > 2$, from (14) it follows

$$\lim_{\substack{\|u\|_G \to +\infty \\ u \in W}} f(u) = -\infty$$

and condition (b) of [9, Theorem 2.1.6] is clearly fulfilled too for a sufficiently large $R > 0$. Let now (λ_h, u_h) be the sequence of eigenvalues and eigenvectors for the problem

$$\begin{gathered}\Delta u = -\lambda u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,.\end{gathered}$$

Let us prove that there exist $h_0, \alpha > 0$ such that

$$\forall u \in V^+ : \|Du\|_2 = 1 \Longrightarrow f(u) \geq \alpha,$$

where $V^+ = \overline{\operatorname{span}}\left\{u_h \in H^1_0(\Omega, \mathbb{R}^N) : h \geq h_0\right\}$. In fact, given $u \in V^+$ and $\varepsilon > 0$, we find

$$a^{(1)}_\varepsilon \in C^\infty_c(\Omega), \;\; a^{(2)}_\varepsilon \in L^{\frac{2n}{n+2}}(\Omega),$$

such that $\|a^{(2)}_\varepsilon\|_{\frac{2n}{n+2}} \leq \varepsilon$ and

$$|g(x,s)| \leq a^{(1)}_\varepsilon(x) + a^{(2)}_\varepsilon(x) + \varepsilon |s|^{\frac{n+2}{n-2}}.$$

If $u \in V^+$, it follows that

$$\begin{aligned} f(u) &\geq \frac{\nu}{2}\|Du\|_2^2 - \int_\Omega G(x,u)\,dx \\ &\geq \frac{\nu}{2}\|Du\|_2^2 - \int_\Omega \left(\left(a^{(1)}_\varepsilon + a^{(2)}_\varepsilon\right)|u| + \frac{n-2}{2n}\varepsilon |u|^{\frac{2n}{n-2}}\right)dx \\ &\geq \frac{\nu}{2}\|Du\|_2^2 - \|a^{(1)}_\varepsilon\|_2\|u\|_2 - c_1\|a^{(2)}_\varepsilon\|_{\frac{2n}{n+2}}\|Du\|_2 - \varepsilon c_2\|Du\|_2^{\frac{2n}{n-2}} \\ &\geq \frac{\nu}{2}\|Du\|_2^2 - \|a^{(1)}_\varepsilon\|_2\|u\|_2 - c_1\varepsilon\|Du\|_2 - \varepsilon c_2\|Du\|_2^{\frac{2n}{n-2}}. \end{aligned}$$

Then if h_0 is sufficiently large, from the fact that (λ_h) diverges, for all $u \in V^+$, $\|Du\|_2 = 1$ implies

$$\|a^{(1)}_\varepsilon\|_2\|u\|_2 \leq \frac{\nu}{6}\,.$$

Hence, for $\varepsilon > 0$ small enough, $\|Du\|_2 = 1$ implies that $f(u) \geq \nu/6$.

Finally, set $V^- = \overline{\operatorname{span}}\left\{u_h \in H^1_0(\Omega, \mathbb{R}^N) : h < h_0\right\}$, we have the decomposition

$$H^1_0(\Omega; \mathbb{R}^N) = V^+ \oplus V^-.$$

Therefore, since the hypotheses for [9, Theorem 2.1.6] are fulfilled, we can find a sequence (u^m) of weak solution of system (1) such that

$$\lim_m f(u^m) = +\infty,$$

and the theorem is now proven.

5. Regularity of weak solutions for elliptic systems

Assume conditions **(A1)** and **(G1)**, and consider the nonlinear elliptic system

$$\int_\Omega \sum_{i,j=1}^n \sum_{h,k=1}^N a^{hk}_{ij}(x,u) D_i u_h D_j v_k\,dx = \int_\Omega b(x,u,Du)\cdot v\,dx \tag{15}$$

for all $v \in H^1_0(\Omega; \mathbb{R}^N)$. For $l = 1, .., N$, we choose

$$b_l(x,u,Du) = \left\{ -\sum_{i,j=1}^n \sum_{h,k=1}^N D_{s_l} a^{hk}_{ij}(x,u) D_i u_h D_j u_k + g_l(x,u) \right\}.$$

Assume that there exist $c > 0$ and $q < \frac{n+2}{n-2}$ such that for all $s \in \mathbb{R}^N$ and a.e. in Ω

$$|g(x,s)| \le c\,(1+|s|^q)\,. \tag{16}$$

Then it follows that for every $M > 0$, there exists $C(M) > 0$ such that for a.e. $x \in \Omega$, for all $\xi \in \mathbb{R}^{nN}$ and $s \in \mathbb{R}^N$ with $|s| \le M$

$$|b(x,s,\xi)| \le c(M)\left(1+|\xi|^2\right)\,. \tag{17}$$

A nontrivial regularity theory for quasilinear systems (see, [14, Chapter VI]) yields the following :

Theorem 8. *For every weak solution $u \in H^1(\Omega,\mathbb{R}^N) \cap L^\infty(\Omega,\mathbb{R}^N)$ of the system (1) there exist an open subset $\Omega_0 \subseteq \Omega$ and $s > 0$ such that*

$$\forall p \in (n,+\infty) : u \in C^{0,1-\frac{n}{p}}(\Omega_0;\mathbb{R}^N),$$
$$\mathcal{H}^{n-s}(\Omega \backslash \Omega_0) = 0\,.$$

Proof. For the proof, see [14, Chapter VI]. □

We now consider the particular case when $a_{ij}^{hk}(x,s) = \alpha_{ij}(x,s)\delta^{hk}$, and provide an almost everywhere regularity result.

Lemma 9. *Assume condition (17). Then the weak solutions $u \in H_0^1(\Omega,\mathbb{R}^N)$ of the system*

$$\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}(x,u) D_i u_h D_j v_h \,dx + \\ + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}(x,u)\cdot v D_i u_h D_j u_h \,dx = \int_\Omega g(x,u)\cdot v\,dx \tag{18}$$

for all $v \in C_c^\infty(\Omega,\mathbb{R}^N)$, belong to $L^\infty(\Omega,\mathbb{R}^N)$.

Proof. By [20, Lemma 3.3], for each $(CPS)_c$ sequence (u^m) there exist $u \in H_0^1 \cap L^\infty$ and a subsequence (u^{m_k}) with $u^{m_k} \rightharpoonup u$. Then, given a weak solution u, consider the sequence (u^m) such that each element is equal to u and the assertion follows. □

We can finally state a partial regularity result for our system.

Theorem 10. *Assume condition (17) and let $u \in H_0^1(\Omega,\mathbb{R}^N)$ be a weak solution of the system*

$$\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}(x,u) D_i u_h D_j v_h \,dx + \\ + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}(x,u)\cdot v D_i u_h D_j u_h \,dx = \int_\Omega g(x,u)\cdot v\,dx \tag{19}$$

for all $v \in C_c^\infty(\Omega,\mathbb{R}^N)$. Then there exist an open subset $\Omega_0 \subseteq \Omega$ and $s > 0$ such that

$$\forall p \in (n,+\infty) : u \in C^{0,1-\frac{n}{p}}(\Omega_0;\mathbb{R}^N),$$
$$\mathcal{H}^{n-s}(\Omega \backslash \Omega_0) = 0.$$

Proof. It suffices to combine the previous Lemma with Theorem 8. □

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Marco Squassina
Dipartimento di Matematica, Milan University, Via Saldini 50, 20133 Milano, Italy
E-mail address: squassin@ares.mat.unimi.it