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UNBOUNDED SOLUTIONS FOR SCHRÖDINGER QUASILINEAR ELLIPTIC PROBLEMS WITH PERTURBATION BY A POSITIVE NON-SQUARE DIFFUSION TERM

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Abstract. In this article, we present a version of Keller-Osserman condition for the Schrödinger quasilinear elliptic problem

$$\begin{gathered}-\Delta u+\frac{k}{2}u\Delta u^2=a(x)g(u) \quad \text{in } \mathbb{R}^N,\\ u>0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty}u(x)=\infty\,,\end{gathered}$$

where $a:\mathbb{R}^N\to[0,\infty)$ and $g:[0,\infty)\to[0,\infty)$ are suitable continuous functions, $N\geq 1$, and $k>0$ is a parameter. By combining a dual approach and this version of Keller-Osserman condition, we show the existence and multiplicity of solutions.

1. Introduction

In this article, we consider the problem

$$\begin{gathered}-\Delta u+\frac{k}{2}u\Delta u^2=a(x)g(u) \quad \text{in } \mathbb{R}^N,\\ u>0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty}u(x)\to\infty,\end{gathered} \tag{1.1}$$

where Δ is the *Laplacian* operator, $a(x)$ is a nonnegative continuous function, $g:[0,\infty)\to[0,\infty)$ is a nondecreasing continuous function that satisfies $g(s)>0$, $s>0$, $N\geq 1$ and $k>0$ is a parameter.

Equation (1.1) is a modified nonlinear Schrödinger equation by the quasilinear and nonconvex term $u\Delta u^2$, which is called of square diffusion. A solution of (1.1) is related to standing wave solutions for the quasilinear Schrödinger equation

$$iz_t+\Delta z-\omega(x)z+\kappa\Delta(h(|z|^2))h'(|z|^2)z+\eta(x,z)=0, \quad x\in\mathbb{R}^N, \tag{1.2}$$

where ω is a potential function, h and η are real functions and κ is a real constant. This connection is established by the fact that $z(t,x)=e^{-i\beta t}u(x)$ is a solution to (1.2) if and only if u satisfies the equation in (1.1) for suitable constants ω, h, η and κ. This kind of equations appears in several applications: superfluid film in plasma physics [10]; in models of the self-channeling of a high-power ultrashort laser in

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matter [9] and [16]; in the theory of Heidelberg ferromagnetism and magnus [7]; in dissipative quantum mechanics [1]; and in condensed matter theory [6].

Even for bounded solutions, there are only a few results in the literature studying existence and multiplicity of such solutions to the equation in (1.1) with positive perturbation; that is, $k > 0$. One important result, that shows the existence of solutions for a related operator of the equation in (1.1), is due to Alves, Wang and Shen [3] who showed the existence of bounded solutions satisfying

$$\sup_{x\in\mathbb{R}^N} |u(x)| \le \sqrt{1/k}$$

for each $0 < k < k_0$, for some $k_0 > 0$. In fact, they considered the equation

$$-\Delta u + V(x)u + \frac{k}{2}u\Delta u^2 = a(x)g(u) \quad \text{in } \mathbb{R}^N$$

for some appropriate potential V. For more references on this direction, we refer the reader to [4, 2, 20, 19] and references therein.

On the other hand, after these papers, we wondered whether it is possible to exist unbounded solutions for (1.1); that is, solutions $u(x) \to \infty$ as $|x| \to \infty$. Surprisingly, under an appropriate version of Keller-Osserman condition to this operator, we were able to show the existence of an infinite number of solutions to (1.1). Our solutions satisfy

$$\inf_{x\in\mathbb{R}^N} |u(x)| \ge \sqrt{1/k}$$

for a given $k > 0$.

Research about existence of explosive solutions (or unbounded solutions) is motivated principally by its applications in models of population dynamical, subsonic motion of a gas, non-Newtonian fluids, non-Newtonian filtration as well as in the theory of the electric potential in a glowing hollow metal body. Remarkable work about unbounded solutions was done by Keller [8] and Osserman [15], both in 1957. They established necessary and sufficient conditions for the existence of solutions and sub solutions to the semilinear and autonomous problem (that is, $a \equiv 1$)

$$\begin{gathered}\Delta u = a(x)g(u) \quad \text{in } \mathbb{R}^N, \\ u \ge 0 \quad \text{in } \mathbb{R}^N, \ \lim_{|x|\to\infty} u(x) \to \infty,\end{gathered} \tag{1.3}$$

where g is a non-decreasing continuous function. This is done under the condition

$$\int_1^{+\infty} \frac{dt}{G(t)^{1/2}} = \infty, \quad \text{where } G(t) = \int_0^t g(s)ds, \quad t > 0. \tag{1.4}$$

After these works, a function g satisfies the well-known Keller-Osserman condition, for the Laplacian operator, if

$$\int_1^{+\infty} \frac{dt}{G(t)^{1/2}} < \infty\,.$$

Recently, there have been a number of papers trying to obtain "Keller-Osserman conditions" for various operators. The authors have also considered this question for ϕ-Laplacian operator in [18].

For (1.3) non-autonomous, it has arisen an important issues on existence of solutions, namely, "how radial" is $a(x)$ at infinity; that is, how big is the function

$$a_{\rm osc}(r) := \overline{a}(r) - \underline{a}(r), \quad r \ge 0,$$

where

$$\underline{a}(r) = \min\{a(x) : |x| = r\}, \quad \overline{a}(r) = \max\{a(x) : |x| = r\}, \quad r \geq 0. \tag{1.5}$$

As a consequence of this, we have that $a_{\rm osc}(r) = 0$, $r \geq r_0$,if and only if, a is symmetric radially for $|x| \geq r_0$, for some $r_0 \geq 0$. In particular, if $r_0 = 0$ we say that $a(x)$ is radially symmetric.

Considering $a_{\rm osc} \equiv 0$, Lair and Wood in [12] proved that

$$\int_1^\infty ra(r)dr = \infty \tag{1.6}$$

is a sufficient condition for (1.3) to have radial solution. They considered $g(u) = u^\gamma$, $u \geq 0$ with $0 < \gamma \leq 1$, that is, g satisfies (1.4).

In 2003, Lair [11] allowed $a(x)$ to be not necessarily radial in the whole space, but he did not allow $a(x)$ to have $a_{\rm osc}$ too big. More exactly, he assumed

$$\int_0^\infty ra_{\rm osc}(r)\exp(\underline{A}(r))dr < \infty, \quad \text{where } \underline{A}(r) = \int_0^r s\underline{a}(s)ds, \ r \geq 0$$

and proved that (1.3), with suitable g that includes u^γ for $0 < \gamma \leq 1$, admits a solution, if and only if, (1.6) holds with $\underline{a}$ in place of a.

In this way, Mabroux and Hansen [13] in 2007 improved the above results, considering the hypothesis

$$\int_0^\infty ra_{\rm osc}(r)(1 + \underline{A}(r))^{\gamma/(1-\gamma)}dr < \infty\,.$$

For a more general operator, Rhouma and Drissi [5] in 2014 proved similar results.

Before stating our main results, we dfine a solution of (1.1) as a positive function $u \in C^1(\mathbb{R}^N)$ such that $u \to \infty$ as $|x| \to \infty$, and

$$\begin{aligned} &\int_{\mathbb{R}^N} (1 - ku^2)\nabla u \nabla\varphi dx - k\int_{\mathbb{R}^N} |\nabla u|^2 u\varphi dx \\ &= \int_{\mathbb{R}^N} a(x)g(u)\varphi dx \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N). \end{aligned}$$

Throughout this article we assume that $g : [0, \infty) \to [0, \infty)$ is a nondecreasing continuous function with $g(s) > 0$ for $s > 0$. Also we use the condition

$$\liminf_{t\to\infty} \frac{g(t)}{t} > 0. \tag{1.7}$$

Our first result reads as follows.

Theorem 1.1. *Assume that* (1.7) *is satisfied and*

$$\int_1^{+\infty} \frac{dt}{G_0(t)^{1/2}} = \infty, \quad \text{where } G_0(t) = \int_0^t g(\sqrt{s})ds, \ t > 0\,. \tag{1.8}$$

If $a_{\rm osc} \equiv 0$ *and*

$$\int_0^\infty \Big(s^{1-N}\int_0^s t^{N-1}a(t)dt\Big)ds = \infty, \tag{1.9}$$

then for each $\sigma > 1$ *and* $k > 0$*, there exists a solution* $u = u_{\sigma,k} \in C^1(\mathbb{R}^N)$ *to problem* (1.1)*. Furthermore,*

$$\inf_{x\in\mathbb{R}^N} u(x) \geq \sqrt{\sigma/k}\,.$$

For non-radial potentials $a(x)$, we need to control the size of this non-radiality. So, for each $\sigma > 1$, let us assume that $\mathcal{G} = \mathcal{G}_\sigma : (0,\infty) \to (0,\infty)$, defined by

$$\mathcal{G}(t) = \frac{(\sigma-1)\sqrt{k}}{8\sqrt{\sigma}} t^2/g(t), \quad t > 0,$$

is non-decreasing and is invertible; such that

$$\begin{aligned} 0 \le \overline{H} := \frac{1}{\sqrt{\sigma - 1}} \int_0^\infty &\Big(s^{1-N} \int_0^s t^{N-1} a_{\rm osc}(t) dt\Big) \\ &\times \Big[g\Big(\mathcal{G}^{-1}\Big(s\Big(\int_0^s \overline{a}(t)dt\Big)\Big)\Big)\Big] ds < \infty\,. \end{aligned} \tag{1.10}$$

Our second result reads as follows.

Theorem 1.2. *Assume g satisfies (1.8) and that for $t > 0$ the function $g(t)/t^\delta$ is non-decreasing for some $\delta \ge \sigma/(\sigma - 1)$. Also suppose that $a(x)$ is such that $\underline{a}$ satisfies (1.9) and $\overline{a}$ satisfies (1.10). Then there exists a solution $u = u_{\alpha,\sigma,k,\varepsilon} \in C^1(\mathbb{R}^N)$ of problem (1.1) satisfying*

$$\inf_{x\in\mathbb{R}^N} u(x) \ge \sqrt{\sigma/k} \quad \text{and} \quad \alpha \le u(0) \le (\alpha + \varepsilon) + \overline{H}$$

for each $\sigma > 1$ and $\alpha, k, \varepsilon > 0$ given so that $\alpha > \sqrt{\sigma/k}$.

We note that this work contributes to the literature of quasilinear Schrödinger equation in at least two aspects: Firstly, as far as we know, there are no results considering this kind of operators (a positive perturbation) in the context of unbounded solutions. We mention the authors have already considered in [17] a negative perturbation, that is, $k < 0$ in the problem (1.1). Secondly, we present a version of "Keller-Orsemann condition" for this kind of operator that "captures" the influence of the perturbation term.

We organized this article the following way: in section 2, we establish an equivalent problem to the (1.1), via a very specific change of variable. In the last section we complete the proof of Theorems 1.1 and 1.2.

2. Auxiliary results

In this section, a change of variables allows us to transform problem (1.1) into a new problem. In the new problem, we establish a version of Keller-Orsemann condition and show the existence of an entire solution that is unbounded.

First, we note that the problem (1.1) is equivalent to the modified quasilinear Schrödinger problem

$$\begin{gathered} \operatorname{div}(l^2(u)\nabla u) - l(u)l'(u)|\nabla u|^2 = a(x)g(u), \quad x \in \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) \to \infty, \end{gathered} \tag{2.1}$$

whenever $u(x) > \sqrt{\sigma/k}$ for $x \in \mathbb{R}^N$, where $l(t) = \sqrt{kt^2 - 1}$ for $t > \sqrt{\sigma/k}$ for each $k > 0$ and $\sigma > 1$ given. In these situations, we conclude that the solutions obtained to (2.1) are solutions of the original problem (1.1).

So, we look for by a positive function $u \in C^1(\mathbb{R}^N)$ that satisfies $u \to \infty$ as $|x| \to \infty$ and

$$-\int_{\mathbb{R}^N} l^2(u)\nabla u \nabla\varphi dx - \int_{\mathbb{R}^N} l(u)l'(u)|\nabla u|^2 \varphi dx = \int_{\mathbb{R}^N} a(x)g(u)\varphi dx$$

for all $\varphi \in C_0^\infty(\mathbb{R}^N)$; that is, this $u \in C^1(\mathbb{R}^N)$ will be an unbounded solution to (2.1).

To do this, first let us define $l : [1/\sqrt{k\sigma}, \infty) \to [0,\infty)$ by

$$l(t) = l_{\sigma,k}(t) = \begin{cases} \frac{\sqrt{k\sigma}}{\sqrt{\sigma-1}}t - \frac{1}{\sqrt{\sigma-1}} & \text{if } \frac{1}{\sqrt{k\sigma}} \le t \le \sqrt{\frac{\sigma}{k}}, \\ \sqrt{kt^2-1} & \text{if } t > \sqrt{\frac{\sigma}{k}}, \end{cases}$$

for each $\sigma > 1$, and set

$$L(t) = L_{\sigma,k}(t) = \int_{1/\sqrt{k\sigma}}^t l(s)ds \quad \text{for } t \ge 1/\sqrt{k\sigma}. \tag{2.2}$$

It is a consequence of the above definitions that the function $L : [1/\sqrt{k\sigma}, \infty) \to [0,\infty)$ is a C^2-injective function; that is, the inverse function $L^{-1} : [0,\infty) \to [1/\sqrt{k\sigma}, \infty)$ is well-defined and L^{-1} is a C^2-function as well. After this, we are able to prove more Lemmas. The first lemma follows from the definitions and properties of l and L.

Lemma 2.1. *Under the above conditions, the functions l and L^{-1} satisfy:*

(1) $0 \le l(t) \le \frac{\sqrt{k\sigma}}{\sqrt{\sigma-1}}t$ *for all* $t \in [1/\sqrt{k\sigma}, \sqrt{\sigma/k}]$ *and* $\frac{\sqrt{(\sigma-1)k}}{\sqrt{\sigma}}t \le l(t) \le \sqrt{k}t$ *for all* $t > \sqrt{\sigma/k}$,

(2) $0 \le L(t) \le \frac{\sqrt{k\sigma}}{\sqrt{\sigma-1}}t^2$ *for all* $t \in [1/\sqrt{k\sigma}, \sqrt{\sigma/k}]$ *and*
$$\frac{1}{2}\frac{\sqrt{(\sigma-1)k}}{\sqrt{\sigma}}t^2 - \frac{1}{2}\frac{\sqrt{(\sigma-1)\sigma}}{\sqrt{k}} \le L(t) \le \sqrt{k}t^2,$$
for all $t > \sqrt{\sigma/k}$,

(3) $L^{-1}(t) \le \sqrt{\frac{2t\sqrt{\sigma}}{\sqrt{(\sigma-1)k}} + \frac{\sigma}{k}}$ *for* $t > 0$, *and*
$$L^{-1}(t) \ge \begin{cases} \sqrt[4]{\frac{\sigma-1}{k\sigma}}\sqrt{t} & \text{for } \frac{1}{\sqrt{k\sigma(\sigma-1)}} \le t \le \frac{\sqrt{\sigma^3}}{\sqrt{k(\sigma-1)}}, \\ \sqrt{\frac{t}{\sqrt{k}}} & \text{for } t \ge \frac{\sigma}{\sqrt{k}}, \end{cases}$$

(4) *for* $t > \sqrt{\sigma/k}$, *the function* $\frac{t^\delta}{l(t)}$ *is nondecreasing for each* $\delta \ge \frac{\sigma}{\sigma-1}$.

In the sequel, we use the definitions and properties of l, L and L^{-1} to provide solutions (2.1) by establishing solutions to (2.3) below.

Lemma 2.2. *Assume $u = L^{-1}(w)$, where $w \in C^1(\mathbb{R}^N)$ is a solution of the problem*

$$\begin{gathered} \Delta w = a(x)\frac{g(L^{-1}(w))}{l(L^{-1}(w))} \quad \text{in } \mathbb{R}^N, \\ w > \sigma/\sqrt{k} \quad \text{in } \mathbb{R}^N, \quad \lim_{|x|\to\infty} w(x) = \infty. \end{gathered} \tag{2.3}$$

Then $u \in C^1(\mathbb{R}^N)$ is a solution of problem (1.1) that satisfies $u(x) \ge \sqrt{\sigma/k}$ for all $x \in \mathbb{R}^N$.

Proof. First, note that $u \ge \sqrt{\sigma/k}$ is a consequence of $w \ge \sigma/\sqrt{k}$. By the regularity of L, we obtain $u \in C^1(\mathbb{R}^N)$, because $w \in C^1(\mathbb{R}^N)$. Besides this, it follows from the behavior of L and L^{-1} that $w(x) \to +\infty$ as $|x| \to +\infty$ if and only if $u(x) \to +\infty$ as $|x| \to +\infty$.

Since
$$w = L(u) = \int_{1/\sqrt{k\sigma}}^{u} l(s)ds,$$
it follows that
$$\nabla w = l(u)\nabla u = (ku^2 - 1)^{1/2}\nabla u;$$
that is,
$$\nabla u = (ku^2 - 1)^{-1/2}\nabla w.$$
Thus, for each $\varphi \in C_0^1(\mathbb{R}^N)$, we have
$$(1 - ku^2)\nabla u\nabla\varphi = -(ku^2 - 1)^{1/2}\nabla w\nabla\varphi. \tag{2.4}$$

On the other hand, since $w \in C^1(\mathbb{R}^N)$ is a solution of problem (2.3), we have
$$\begin{aligned}
\int_{\mathbb{R}^N}(ku^2-1)^{1/2}\nabla w\nabla\varphi &= \int_{\mathbb{R}^N}\nabla w\nabla\{(ku^2-1)^{1/2}\varphi\} - \int_{\mathbb{R}^N}\frac{ku}{ku^2-1}|\nabla w|^2\varphi \\
&= -\int_{\mathbb{R}^N} a(x)\frac{g(u)}{l(u)}(ku^2-1)^{1/2}\varphi - \int_{\mathbb{R}^N} ku|\nabla u|^2\varphi \\
&= -\int_{\mathbb{R}^N} a(x)g(u)\varphi - \int_{\mathbb{R}^N} ku|\nabla u|^2\varphi.
\end{aligned}$$
Then using (2.4), we have $u \in C^1(\mathbb{R}^N)$ is a solution to (1.1). This completes the proof. $\square$

3. Proof of main results

Below, we show the existence of a solution to (2.3) and after this, by using the second Lemma above, we are able to show the existence of a solution to Problem (1.3). This arguments relies on ideas found in [14].

Proof of Theorem 1.1. Since $a_{\rm osc} \equiv 0$, that is, $a(x) = a(|x|)$ for all $x \in \mathbb{R}^N$, we have that a radial solution for (2.3) can be obtained by solving the problem
$$\begin{gathered}
(r^{N-1}w')' = r^{N-1}a(r)\frac{g(L^{-1}(w))}{l(L^{-1}(w))} \quad \text{in } (0,\infty), \\
w'(0) = 0, \quad w(0) = \alpha \ge 0,
\end{gathered} \tag{3.1}$$
where $r = |x| \ge 0$ and $\alpha > \sigma/\sqrt{k}$ is a real number, for each $\sigma > 1$ and $k > 0$.

Since a, g, l and L^{-1} are continuous functions, we can follow the approach in [21] to show that there exist a right maximal extreme $\Gamma(\alpha) > 0$, and a function $w_\alpha \in C^2(0,\Gamma(\alpha)) \cap C^1([0,\Gamma(\alpha)))$ solution of the problem (3.1) on $(0,\Gamma(\alpha))$, for each $\alpha > \sigma/\sqrt{k}$ given.

If we assumed that $\Gamma(\alpha) < \infty$ for some $\alpha > \sigma/\sqrt{k}$, then we would obtain, by standard arguments of ordinary differential equations, that either $w_\alpha(r) \to \infty$ as $r \to \Gamma(\alpha)^-$ or $w'_\alpha(r) \to \infty$ as $r \to \Gamma(\alpha)^-$; that is, $w_\alpha(|x|)$ would satisfy the problem
$$\begin{gathered}
(r^{N-1}w')' = r^{N-1}a(r)\frac{g(L^{-1}(w))}{l(L^{-1}(w))} \quad \text{in } (0,\Gamma(\alpha)), \\
w'(0) = 0,\ w(0) = \alpha > 0, \\
\lim_{r\to\Gamma(\alpha)^-} w_\alpha(r) = \infty \quad \text{or} \quad \lim_{r\to\Gamma(\alpha)^-} w'_\alpha(r) = \infty.
\end{gathered} \tag{3.2}$$

So, using that a solution w of (3.2) is non-decreasing and $l(t) \geq \sqrt{\sigma - 1}$ for all $t \geq \sqrt{\sigma/k}$, we obtain that w satisfies

$$\begin{gathered} (r^{N-1}w')' \leq \frac{a_\infty}{\sqrt{\sigma-1}} r^{N-1} g(L^{-1}(w)) \quad \text{in } (0, \Gamma(\alpha)), \\ w(0) = \alpha > 0, \quad w'(0) = 0, \\ \lim_{r \to \Gamma(\alpha)^-} w_\alpha(r) = \infty \quad \text{or} \quad \lim_{r \to \Gamma(\alpha)^-} w'_\alpha(r) = \infty, \end{gathered} \tag{3.3}$$

where $a_\infty = \max_{\bar{B}_{\Gamma(\alpha)}} a(x)$.

By integrating the inequality above over $(0, r)$ with $0 < r < \Gamma(\alpha)$ and assuming $\|w_\alpha\|_\infty \leq C < \infty$ for some $C > 0$, we obtain

$$\limsup_{r \to \Gamma(\alpha)^-} w'(r) \leq \Gamma(\alpha)^{1-N} \int_0^{\Gamma(\alpha)} t^{N-1} g(L^{-1}(w(t))) dt < \infty$$

by the continuity of all involved functions. So, from now on, we assume that $w_\alpha(x) \to \infty$ as $r \to \Gamma(\alpha)^-$.

By using $w' \geq 0$ again, we can rewrite the inequality in (3.3) as

$$w'' \leq \frac{a_\infty}{\sqrt{\sigma-1}} (g \circ L^{-1})(w) \quad \text{for all } 0 < r < \Gamma(\alpha)$$

this lead us, after multiplying this inequality by w' and integrating it on $(0, r)$, to

$$\frac{1}{2}\Big(w'(r)\Big)^2 \leq \int_0^r \frac{a_\infty}{\sqrt{\sigma-1}} (g \circ L^{-1})(w(s)) w'(s) ds = \frac{a_\infty}{\sqrt{\sigma-1}} \int_\alpha^{w(r)} (g \circ L^{-1})(s) ds;$$

that is,

$$\Big(\int_\alpha^{w(r)} (g \circ L^{-1})(s) ds\Big)^{-1/2} w'(r) \leq \sqrt{2}\sqrt{a_\infty/\sqrt{\sigma-1}} \quad \text{for all } 0 < r < \Gamma(\alpha).$$

Now, by integrating in the above inequality over $(0, \Gamma(\alpha))$ and reminding that $w_\alpha(x) \to \infty$ as $r \to \Gamma(\alpha)^-$, we obtain

$$\int_\alpha^\infty \Big(\int_\alpha^t (g \circ L^{-1})(s) ds\Big)^{-1/2} dt \leq \sqrt{2}\sqrt{a_\infty/\sqrt{\sigma-1}}\,\Gamma(\alpha) < \infty. \tag{3.4}$$

On the other hand, from Lemma 2.1-(3) and the monotonicity of g, it follows that

$$(g \circ L^{-1})(t) \leq g\Big(\sqrt{\frac{2t\sqrt{\sigma}}{\sqrt{(\sigma-1)k}} + \frac{\sigma}{k}}\Big) \quad \text{for all } t > \sigma/\sqrt{k};$$

that is,

$$\int_\alpha^t (g \circ L^{-1})(s) ds \leq \int_\alpha^t g\Big(\sqrt{\frac{2t\sqrt{\sigma}}{\sqrt{(\sigma-1)k}} + \frac{\sigma}{k}}\Big) ds \text{ for all } t > \alpha.$$

As a consequence of this and (3.4), we have

$$\int_\alpha^\infty \Big\{\int_\alpha^t g \; Big(\sqrt{\frac{2s\sqrt{\sigma}}{\sqrt{(\sigma-1)k}} + \frac{\sigma}{k}}\Big) ds\Big\}^{-1/2} dt \leq \int_\alpha^\infty \Big\{\int_\alpha^t (g \circ L^{-1})(s) ds\Big\}^{-1/2} dt.$$

So, by estimating in the last inequality and using (3.4) again, we obtain

$$\int_1^\infty G_0(t)^{-1/2} dt \leq C\Big(\sqrt{a_\infty/\sqrt{\sigma-1}}\Big)\Gamma(\alpha) < \infty,$$

for some real constant $C > 0$. This is impossible, because we are assuming that g satisfies (1.8).

It follows from Lemma 2.1-(1), hypothesis (1.7) and $L^{-1}(w_\alpha) \geq \sqrt{\sigma/k}$, that

$$\frac{g(L^{-1}(w_\alpha(r)))}{l(L^{-1}(w_\alpha(r)))} \geq \frac{g(L^{-1}(w_\alpha(r)))}{\sqrt{k}L^{-1}(w_\alpha(r))} \geq M > 0 \quad \text{for all } r > 0, \tag{3.5}$$

and for each $\alpha > \sigma/\sqrt{k}$ given and for some $M > 0$, because $w_\alpha(r) \geq \alpha$ for all $r \geq 0$.

Since, w_α satisfies

$$w_\alpha(r) = \alpha + \int_0^r \Big(t^{1-N}\int_0^t s^{N-1}a(s)\frac{g(L^{-1}(w_\alpha))}{l(L^{-1}(w_\alpha))}ds\Big)dt, \quad r \geq 0, \tag{3.6}$$

it follows from (3.5), that

$$w_\alpha(r) \geq \alpha + M\int_0^r \Big(t^{1-N}\int_0^t s^{N-1}a(s)ds\Big)dt \to \infty, \quad \text{as } r \to \infty. \tag{3.7}$$

This completes the proof. □

Proof of Theorem 1.2. Set $\beta > \alpha > \sigma/\sqrt{k}$. Since the hypothesis $g(t)/t^\delta$ being non-increasing implies (1.7), from Theorem 1.1 there exist positive and radially symmetric solutions $w_\alpha, w_\beta \in C^1(\mathbb{R}^N)$ to the problems

$$\begin{gathered} \Delta w_\alpha = \overline{a}(|x|)\frac{g(L^{-1}(w_\alpha))}{l(L^{-1}(w_\alpha))} \quad \text{in } \mathbb{R}^N, \\ w_\alpha(0) = \alpha, \quad \lim_{|x|\to\infty} w_\alpha(x) = \infty, \end{gathered}$$

and

$$\begin{gathered} \Delta w_\beta = \underline{a}(|x|)\frac{g(L^{-1}(w_\beta))}{l(L^{-1}(w_\beta))} \quad \text{in } \mathbb{R}^N, \\ w_\beta(0) = \beta, \quad \lim_{|x|\to\infty} w_\beta(x) = \infty, \end{gathered}$$

respectively, where $\underline{a}$ and $\overline{a}$ were defined in (1.5).

Besides this, from (3.6), (3.7), w_α, g, L^{-1} be non-decreasing and Lemma 2.1-(3), it follows that

$$\begin{aligned} w_\alpha(r) &\leq 2\int_0^r \Big(\int_0^t \overline{a}(s)\frac{g(L^{-1}(w_\alpha))}{l(L^{-1}(w_\alpha))}ds\Big)dt \\ &\leq 2g(L^{-1}(w_\alpha(r)))\int_0^r \Big(\int_0^t \frac{\overline{a}(s)}{l(L^{-1}(w_\alpha))}ds\Big)dt \\ &\leq 2g\Big(\sqrt{2\sqrt{\frac{\sigma}{(\sigma-1)k}}w_\alpha(r) + \frac{\sigma}{k}}\Big)\int_0^r \Big(\int_0^t \frac{\overline{a}(s)}{l(L^{-1}(w_\alpha))}ds\Big)dt \\ &\leq 2g\Big(2\sqrt[4]{\frac{\sigma}{(\sigma-1)k}}\sqrt{w_\alpha}\Big)\int_0^r \Big(\int_0^t \frac{\overline{a}(s)}{l(L^{-1}(w_\alpha))}ds\Big)dt \\ &\leq \frac{2}{\sqrt{\sigma-1}}g\Big(2\sqrt[4]{\frac{\sigma}{(\sigma-1)k}}\sqrt{w_\alpha}\Big)\Big[r\Big(\int_0^r \overline{a}(t)dt\Big) - \int_0^r t\overline{a}(t)dt\Big] \\ &\leq \frac{2}{\sqrt{\sigma-1}}g\Big(2\sqrt[4]{\frac{\sigma}{(\sigma-1)k}}\sqrt{w_\alpha}\Big)r\int_0^r \overline{a}(t)dt. \end{aligned}$$

for all $r > 0$ sufficiently large. That is, it follows from the definition of $\mathcal{G}$, that

$$2\sqrt[4]{\frac{\sigma}{(\sigma-1)k}}\sqrt{w_\alpha} \le \mathcal{G}^{-1}\Big(r\int_0^r \overline{a}(t)dt\Big) \quad \text{for all } r >> 0.$$

Now, setting

$$0 < S(\beta) = \sup\{r > 0 : w_\alpha(r) < w_\beta(r)\} \le \infty,$$

we claim that $S(\beta) = \infty$ for all $\beta > \alpha + \overline{H}$ and for each $\alpha > \sigma/\sqrt{k}$ given. In fact, by assuming this is not true, then there exists a $\beta_0 > \alpha+\overline{H}$ such that $w_\alpha(S(\beta_0)) = w_\beta(S(\beta_0))$. So, by using that $g(t)/t^\delta$ is non-decreasing for $\delta > \sigma/(\sigma-1)$, Lemma 2.1 and $w_\alpha \le w_\beta$ on $[0, S(\beta_0)]$, we obtain

$$\begin{aligned}
&\beta_0 \\
&= \alpha + \int_0^{S(\beta_0)} t^{1-N}\Big[\int_0^t s^{N-1}\Big(\overline{a}(s)\frac{g(L^{-1}(w_\alpha))}{l(L^{-1}(w_\alpha))} - \underline{a}(s)\frac{g(L^{-1}(w_\beta))}{l(L^{-1}(w_\beta))}\Big)ds\Big]dt \\
&= \alpha + \int_0^{S(\beta_0)} t^{1-N}\Big[\int_0^t s^{N-1}\Big(\overline{a}(s)\frac{g(L^{-1}(w_\alpha))}{l(L^{-1}(w_\alpha))} \\
&\quad - \underline{a}(s)\frac{g(L^{-1}(w_\beta))}{L^{-1}(w_\beta)^\delta}\frac{L^{-1}(w_\beta)^\delta}{l(L^{-1}(w_\beta))}\Big)ds\Big]dt \\
&\le \alpha + \int_0^{S(\beta_0)} t^{1-N}\Big[\int_0^t s^{N-1}\Big(\overline{a}(s)\frac{g(L^{-1}(w_\alpha))}{l(L^{-1}(w_\alpha))} - \underline{a}(s)\frac{g(L^{-1}(w_\alpha))}{l(L^{-1}(w_\alpha))}\Big)ds\Big]dt\,.
\end{aligned} \tag{3.8}$$

On the other hand, from g, l and w_α being non-decreasing, it follows that

$$\begin{aligned}
&0 \le t^{1-N}\Big[\int_0^t s^{N-1}\Big(\overline{a}(s)\frac{g(L^{-1}(w_\alpha))}{l(L^{-1}(w_\alpha))} - \underline{a}(s)\frac{g(L^{-1}(w_\alpha))}{l(L^{-1}(w_\alpha))}\Big)ds\Big]\chi_{[0,S(\beta)]}(t) \\
&= t^{1-N}\Big[\int_0^t s^{N-1}a_{\rm osc}(s)\frac{g(L^{-1}(w_\alpha))}{l(L^{-1}(w_\alpha))}ds\Big] \\
&\le \frac{1}{\sqrt{\sigma-1}}\Big(t^{1-N}\int_0^t s^{N-1}a_{\rm osc}(s)ds\Big)g\Big(\mathcal{G}^{-1}\Big(t\int_0^t \overline{a}(s)ds\Big)\Big) := \mathcal{H}(t),
\end{aligned}$$

for $t \gg 0$, where $\chi_{[0,S(\beta)]}$ stands for the characteristic function of $[0, S(\beta)]$.

So, from the hypothesis (1.10) and (3.8), it follows that

$$\beta_0 \le \alpha + \int_0^\infty \mathcal{H}(s)ds \le \alpha + \overline{H},$$

but this is impossible.

Now, by setting $\beta = (\alpha+\epsilon) + \overline{H}$, for each $\alpha > \sigma/\sqrt{k}$ and $\epsilon > 0$ given, and by considering the problem

$$\begin{gathered}
\Delta w = a(x)\frac{g(L^{-1}(w))}{l(L^{-1}(w))} \quad \text{in } B_n(0), \\
w \ge 0 \text{ in } B_n(0), \quad w = w_\alpha \quad \text{on } \partial B_n(0),
\end{gathered} \tag{3.9}$$

we can infer by standard methods of sub and super solutions that there exists a $w_n = w_{n,\alpha} \in C^1(\overline{B}_n)$ solution of (3.9) satisfying $\sigma/\sqrt{k} < \alpha \le w_\alpha \le w_n \le w_\beta$ in B_n for all $n \in \mathbb{N}$.

So, by defining $w_m^n = w_m|_{B_n}$ for $m > n$ and for each $n \in \mathbb{N}$ given, where w_m is a solution of Problem (3.9) in the ball $B_m(0)$, we obtain that $\{w_m^n\}$ is a bounded m-sequence in $C^{1,\nu_n}(\bar{B}_n)$ for some $0 < \nu_n \le 1$ by Regularity theory.

Hence, we can extract subsequences of $\{w_m^n\}$ such that

$$\begin{gathered} w_2^1, w_3^1, w_4^1, \dots \xrightarrow{C^1(\bar{B}_1)} w^1, \\ w_3^2, w_4^2, w_5^2, \dots \xrightarrow{C^1(\bar{B}_2)} w^2, \\ w_4^3, w_5^3, w_6^3, \dots \xrightarrow{C^1(\bar{B}_3)} w^3, \\ \dots \end{gathered}$$

So, the function $w : \mathbb{R}^N \to (0, \infty)$ given by $w(x) = w^n(x)$ for $x \in B_n$ is well-defined and the sequence $\{w_{2n}^n\}$ satisfies $w_{2n}^n \to w$ in $C^1(K)$ for any compact set $K \subset \mathbb{R}^N$ with $\sigma/\sqrt{k} < \alpha \le w_\alpha \le w \le w_\beta$; that is, $w \in C^1(\mathbb{R}^N)$ and is a solution of (1.1). $\square$

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