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A second order ODE with a nonlinear final condition *

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Abstract

We study a semilinear second-order ordinary differential equation with initial condition $u(0) = u_0$. We prove the existence of solutions satisfying a nonlinear final condition $u(T) = h'(u(T))$, under a certain growth condition. Also we state conditions ensuring that any solution with Cauchy data $u(0) = u_0$, $u'(0) = v_0$ is defined on the whole interval $[0, T]$.

1 Introduction

We study the differential equation

$$u''(t) + r(t)u'(t) + g(t, u(t)) = f(t) \tag{1.1}$$

with initial condition $u(0) = u_0$.

In the first section, we state the basic assumptions and results concerning the Dirichlet problem associated with (1.1). In the second section, we define a fixed point setting for solving a problem with final value $u(T)$ depending on the velocity at time T. We prove that if g satisfies a growth condition that holds for example when g is *sublinear*, then there exist a class of functions h such that (1.1) admits at least one solution u with $u(0) = u_0$, $u(T) = h(u'(T))$. A physical example of this equation is the forced pendulum equation, for which existence results under Dirichlet and periodic conditions are known, see [3, 5, 6] and their references. For nonexistence results, see e.g. [1, 8]. Finally, in the third section we prove the existence of a continuous real function $\psi = \psi_{u_0}$ such that a solution of (1.1) with initial value u_0 is defined over $[0, T]$ if and only if the equation $\psi(s) = u'(0)$ is solvable. Furthermore, if g is locally Lipschitz on u the union over u_0 of the sets $\{u_0\} \times \mathrm{Range}(\psi_{u_0})$ is a simply connected open subset of $\mathbb{R}^2$.

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2 Basic assumptions and unique solvability of the Dirichlet problem

Let $S : H^2(0,T) \to L^2(0,T)$ be the semi-linear operator $Su = u'' + ru' + g(t,u)$. We assume throughout this paper that g is continuous and satisfies the condition

$$\frac{g(t,u)-g(t,v)}{u-v} \le c < \big(\frac{\pi}{T}\big)^2 \quad \text{for all } t \in [0,T], u, v \in \mathbb{R}, u \neq v \tag{2.1}$$

Moreover, we shall assume that the friction coefficient $r \in H^1(0,T)$ is non-decreasing.

Concerning the Dirichlet problem for (1.1), we recall the following results whose proofs can be found in [2]. For related results and a general overview of this problem, we refer the reader to [4, 7].

Lemma 2.1 *Let $u, v \in H^2(0,T)$ with $u - v \in H_0^1(0,T)$. Then*

$$\|Su - Sv\|_2 \ge \big((\frac{\pi}{T})^2 - c\big)\|u - v\|_2$$

and

$$\|Su - Sv\|_2 \ge \frac{(\pi/T)^2 - c}{\pi/T}\|u' - v'\|_2$$

Theorem 2.2 *The Dirichlet problem*

$$\begin{gathered} Su = f(t) \quad \textit{in } (0,T) \\ u(0) = u_0, \quad u(T) = u_T \end{gathered}$$

is uniquely solvable in $H^2(0,T)$ for any $f \in L^2(0,T)$, $u_0, u_T \in \mathbb{R}$.

Theorem 2.3 *Let $f \in L^2(0,T)$ and $\mathcal{S} = S^{-1}(f)$ with the topology induced by the H^2-norm. Then the trace function,* $\mathrm{Tr} : \mathcal{S} \to \mathbb{R}^2$*, given by* $\mathrm{Tr}(u) = (u(0), u(T))$ *is an homeomorphism.*

3 Nonlinearities at the endpoint

In this section we study the problem

$$\begin{gathered} u'' + ru' + g(t,u) = f \quad \text{in } (0,T) \\ u(0) = u_0, \quad u(T) = h(u'(T)) \end{gathered} \tag{3.1}$$

for $f \in L^2(0,T)$ and h continuous. First we transform the problem in a one-dimensional fixed point problem: Indeed, for $s \in \mathbb{R}$, we define u_s as the unique solution of the problem

$$\begin{gathered} u'' + ru' + g(t,u) = f \quad \text{in } (0,T) \\ u(0) = u_0, \quad u(T) = h(s) \end{gathered}$$

Hence, when $\varphi_s(t) = \frac{h(s)-u_0}{T}t + u_0$, we have

$$u_s(t) - \varphi_s(t) = \int_0^T (f - ru_s' - g(\theta, u_s'))G(t,\theta)d\theta$$

where G is the Green function associated with the second order differential operator. Namely,

$$G(t,\theta) = \begin{cases} \frac{t(\theta-T)}{T} & \text{if } \theta \geq t \\ \frac{\theta(t-T)}{T} & \text{if } \theta \leq t \end{cases}$$

By simple computation we obtain

$$u_s'(T) = \frac{h(s)-u_0}{T} + \int_0^T (f - ru_s' - g(\theta, u_s))\frac{\theta}{T}d\theta$$

and from Theorem 2.2 we have

Theorem 3.1 *Let $\xi : \mathbb{R} \to \mathbb{R}$ with*

$$\xi(s) = \frac{h(s)-u_0}{T} + \int_0^T (f - ru_s' - g(\theta, u_s))\frac{\theta}{T}d\theta\,.$$

Then ξ is a continuous fixed point operator for (3.1), i.e. u is a solution of (3.1) if and only if $u = u_s$ for some $s \in \mathbb{R}$ such that $\xi(s) = s$.

Proof Continuity of ξ follows immediately from the continuity of $\mathrm{Tr}^{-1} : \mathbb{R}^2 \to S^{-1}(f)$. Moreover, if $\xi(s) = s$, then $u_s(T) = h(u_s'(T))$, proving that u_s is a solution of (3.1). Conversely, if u is a solution of (3.1), then $u = u_s$ for $s = u'(T)$. $\square$

We establish an existence result for (3.1) assuming that the graph of h crosses the constant u_0.

Theorem 3.2 *Assume that (2.1) holds and that $h - u_0$ has nonconstant sign on $\mathbb{R}$. Then (3.1) admits a solution for T small enough.*

Proof First we give a slightly different formulation of the equality $\xi(s) = s$. Integrating by parts, we see that

$$\int_0^T r(\theta)u_s'(\theta)\theta d\theta = r(T)Th(s) - \int_0^T [r(\theta) + \theta r'(\theta)]u_s(\theta)d\theta$$

and then

$$\xi(s) = (\frac{1}{T} - r(T))h(s) + \frac{1}{T}\left[\int_0^T \theta f(\theta)d\theta - u_0\right] + \frac{1}{T}\int_0^T (r + \theta r')u_s - \theta g(\theta, u_s)d\theta$$

Hence, s is a fixed point of ξ if and only if

$$sT = (1 - r(T)T)h(s) - u_0 + \int_0^T (r + \theta r')u_s - \theta g(\theta, u_s)d\theta + \int_0^T \theta f(\theta)d\theta \quad (3.2)$$

¿From Lemma 2.1,

$$\|u_s-\varphi_s\|_2 \le \frac{T^2}{\pi^2-cT^2}\|Su_s-S\varphi_s\|_2 = \frac{T^2}{\pi^2-cT^2}\|f-r\varphi_s'-g(\cdot,\varphi_s)\|_2$$

and

$$\|u_s-\varphi_s\|_\infty \le \frac{\pi T^{3/2}}{\pi^2-cT^2}\|f-r\varphi_s'-g(\cdot,\varphi_s)\|_2$$

Moreover,

$$\|\varphi_s\|_2 = \sqrt{\frac{T}{3}(h(s)^2+h(s)u_0+u_0^2)} := c(s)\sqrt{T}$$

and as

$$\|\varphi_s\|_\infty = \max\{|u_0|,|h(s)|\}, \qquad \varphi_s' = \frac{h(s)-u_0}{T}$$

then letting $T\to 0$ for fixed s we have that $\|u_s\|_2\to 0$ and $\|u_s\|_\infty$ is bounded. Hence, we conclude that the right-hand side of (3.2) converges to $h(s)-u_0$.

Setting $s_\pm\in\mathbb{R}$ such that $h(s_+)<u_0<h(s_-)$, it follows, for small T, that

$$T\xi(s_+)\le h(s_+)-u_0+B(s_+)$$

and

$$T\xi(s_-)\ge h(s_-)-u_0+B(s_-)$$

for some B such that $B(s_\pm)\to 0$. Hence it suffices to take T such that

$$h(s_+)-u_0+B(s_+)\le Ts_+, \quad h(s_-)-u_0+B(s_-)\ge Ts_-$$

□

For the next existence result, we assume that g grows at most linearly, i.e.

$$|g(t,x)|\le \alpha|x|+\beta \tag{3.3}$$

for some positive constants α, β. We remark that (2.1) and (3.3) are independent: for example, $g(x)=-x^3$ satisfies (2.1) but not (3.3). Conversely, $g(x)=\sin(Kx)$ does not satisfy (2.1) for $K\ge\left(\frac{\pi}{T}\right)^2$. For simplicity we define the constants

$$c_T=\sqrt{\frac{T}{3}}+\frac{T^2}{\pi^2-cT^2}\Big(\alpha\sqrt{\frac{T}{3}}+\frac{\|r\|_2}{T}\Big), \quad M=\Big(\|r+\theta r'\|_2+\sqrt{\frac{T^3}{3}}\alpha\Big)c_T$$

and the functions

$$C_\pm(s)=\Big((1-r(T)T)\operatorname{sgn}\big(\frac{h(s)}{s}\big)\pm M\Big)\Big|\frac{h(s)}{s}\Big|.$$

Theorem 3.3 *Assume that (2.1) and (3.3) hold. Then (3.1) admits at least one solution $u\in H^2(0,T)$ in each of the following cases*
Case A: $M<|1-r(T)T|$, with

$$T<\limsup_{s\to+\infty} C_-(s) \quad \text{or} \quad T>\liminf_{s\to-\infty} C_+(s) \tag{3.4}$$

and

$$T < \limsup_{s\to-\infty} C_-(s) \quad or \quad T > \liminf_{s\to+\infty} C_+(s) \tag{3.5}$$

Case B: $M > |1-r(T)T|$, *with* $T > \liminf_{s\to\pm\infty} C_+(s)$
Case C: $M = |1-r(T)T|$, *and there exist sequences* $s_j^- \to -\infty$, $s_j^+ \to +\infty$ *such that* $T > C_+(s_j^\pm)$ *for every* j, *each one of them satisfying one of the following conditions:*

$$\operatorname{sgn}\big(\frac{h(s_j)}{s_j}\big) = \operatorname{sgn}(1-r(T)T) \quad \textit{for every } j \tag{3.6}$$

or

$$\lim_{j\to\infty} \frac{h(s_j)}{s_j^2} = 0 \tag{3.7}$$

Remarks: i) The left-hand-side in condition 3.4 (resp. 3.5) implies

$$\limsup_{s\to+\infty} \frac{h(s)}{s}\operatorname{sgn}(1-r(T)T) > \frac{T}{|1-r(T)T|-M} \qquad (\text{resp. } s\to-\infty)$$

ii) The following assumptions are sufficient for the right-hand-side in condition 3.4 (resp. 3.5) to be satisfied.

$$\liminf_{s\to-\infty} \Big|\frac{h(s)}{s}\Big| < \frac{T}{M+|1-r(T)T|} \qquad (\text{resp. } s\to+\infty)$$

or

$$\operatorname{sgn}\big(\frac{h(s_j)}{s_j}\big) = -\operatorname{sgn}(1-r(T)T)$$

for a sequence $s_j \to -\infty$ (resp. $s_j \to +\infty$).
iii) Conditions in case B are not fulfilled when

$$|h(s)| \ge a|s| + b, \quad \text{with} \quad a \ge \frac{T}{M-|1-r(T)T|}$$

In the same way, conditions in case C imply

$$\liminf_{|s|\to\infty} \Big|\frac{h(s)}{s}\Big| < \frac{T}{2M}$$

Proof of Theorem 3.3 As in the previous theorem,

$$\|u_s\|_2 \le \sqrt{T}c(s) + \frac{T^2}{\pi^2 - cT^2}\big(\alpha\sqrt{T}c(s) + |h(s)-u_0|\frac{\|r\|_2}{T} + \|f\|_2 + \beta\big) := A(s)$$

and then

$$\|u_s\|_2 \le c_T|h(s)| + \gamma|h(s)|^{1/2} + \delta$$

for some constants $\gamma, \delta \in \mathbb{R}$. Moreover,

$$\Big|\int_0^T (r+\theta r')u_s - \theta g(\theta,u_s)d\theta\Big| \le \Big(\|r+\theta r'\|_2 + \sqrt{\frac{T^3}{3}}\alpha\Big)c_T|h(s)| + R(s)$$

with $R(s) \leq C_1|h(s)|^{1/2} + C_2$ for some constants C_1, C_2. We remark that $\frac{R(s)}{s} \to 0$ for $|s| \to \infty$ if h is *subquadratic* (i.e. $\frac{h(s)}{s^2} \to 0$ for $|s| \to \infty$). Hence,

$$\begin{aligned}
&[(1 - r(T)T) - M \operatorname{sgn}(h(s))]h(s) - R(s) \\
&\quad \leq T\xi(s) \\
&\quad \leq [(1 - r(T)T) + M \operatorname{sgn}(h(s))]h(s) + R(s)
\end{aligned}$$

and it suffices to find $s_\pm$ satisfying:

$$s_- T \leq [(1 - r(T)T) - M \operatorname{sgn}(h(s_-))]h(s_-) - R(s_-) \tag{3.8}$$

$$s_+ T \geq [(1 - r(T)T) + M \operatorname{sgn}(h(s_+))]h(s_+) + R(s_+) \tag{3.9}$$

Assuming that $s_- > 0$ then (3.8) is equivalent to

$$T \leq \Big[\operatorname{sgn}\big(\frac{h(s_-)}{s_-}\big)(1 - r(T)T) - M\Big]\Big|\frac{h(s_-)}{s_-}\Big| - \frac{R(s_-)}{s_-}$$

Hence, if $M < |1 - r(T)T|$ then left-hand-side of (3.4) is a sufficient condition for (3.8): indeed, if $T < k\Big|\frac{h(s_j)}{s_j}\Big|$ for $s_j \to +\infty$ and some $k > 0$, then

$$k\Big|\frac{h(s_j)}{s_j}\Big| - \frac{R(s_j)}{s_j} = \Big|\frac{h(s_j)}{s_j}\Big|\Big(k - \frac{R(s_j)}{|h(s_j)|}\Big)$$

As $|h(s_j)| \to \infty$, we have that $R(s_j)/|h(s_j)| \to 0$ and the result follows.

In the same way, if we assume that $s_- < 0$, then (3.8) is equivalent to

$$T \geq \Big[\operatorname{sgn}\Big(\frac{h(s_-)}{s_-}\Big)(1 - r(T)T) + M\Big]\Big|\frac{h(s_-)}{s_-}\Big| - \frac{R(s_-)}{s_-}$$

and right-hand-side of (3.4) is sufficient, as well as conditions in cases B and C. The same conclusions can be obtained for (3.9), which completes the proof. $\square$

Example We consider the forced pendulum equation

$$u''(t) + \sin u = f(t) \tag{3.10}$$

for which it is clear that (3.3) holds, and (2.1) holds when $T < \pi$. In this case $c_T = \sqrt{\frac{T}{3}}$, $M = 0$, and $C_-(s) = C_+(s) = \frac{h(s)}{s}$. If we assume, further, that

$$\lim_{s \to \pm\infty} \frac{h(s)}{s} = L_\pm$$

then (3.1) is solvable, unless

$$L_- \leq T \leq L_+ \quad \text{or} \quad L_+ \leq T \leq L_-$$

In particular, (3.1) is solvable when h is sublinear or superlinear (and obviously when h is linear, $h(s) = as + b$, for $T \neq a$).

It is well known that (3.10) admits T-periodic solutions when f is T-periodic and $\int_0^T f = 0$. Furthermore, in [3] it has been proved that for any 2π-periodic $f_0 \in L^2(0, 2\pi)$ such that $\int_0^{2\pi} f_0 = 0$ there exist two numbers $d(f_0) \leq 0 \leq D(f_0)$ such that (P) admits 2π-periodic solutions for $f(t) = f_0(t) + f_1$ if and only if

$$d(f_0) \leq f_1 \leq D(f_0)$$

Remark Assuming (2.1) and (3.3) we may define the functions $\xi^{\pm} : \mathbb{R} \to \mathbb{R}$ as

$$\begin{aligned}\xi^{\pm}(s) =& \frac{1}{T}\Big((1 - r(T)T)h(s) \pm \Big[\|r + \theta r'\|_2 A(s) + \sqrt{\frac{T^3}{3}}(\alpha A(s) + \beta)\Big] \\ &+ \int_0^T \theta f(\theta) d\theta - u_0\Big)\end{aligned}$$

with $A(s)$ as in the previous proof. Then a sufficient condition for the solvability of (3.1) is the existence of $s_{\pm} \in \mathbb{R}$ such that $s_- \leq \xi^-(s_-)$ and $\xi^+(s_+) \leq s_+$. Indeed, from the previous computations we have

$$|\int_0^T (r + \theta r')u_s - \theta g(\theta, u_s) d\theta| \leq \|r + \theta r'\|_2 A(s) + \sqrt{\frac{T^3}{3}}(\alpha A(s) + \beta)$$

Then $\xi^- \leq \xi \leq \xi^+$ and the result the result follows from Theorem 3.1. $\square$

4 Blow-up results

In this section we study the behavior of the solutions of the Cauchy problem

$$\begin{gathered} u'' + ru' + g(t, u) = f \quad \text{in } (0, T) \\ u(0) = u_0, \quad u'(0) = v_0 \end{gathered} \tag{4.1}$$

As a simple remark, under condition (2.1) we see that if g is locally Lipschitz on u, then there exists an interval $I(u_0)$ such that $v_0 \in I(u_0)$ if and only if u is defined over $[0, T]$. Indeed, it suffices to show that the set

$$I := \{v_0 : \text{ the local solution of (4.1) does not blow up on } [0, T]\}$$

is connected. Let $v_0, v_2 \in I$ and $v_1 \notin I$ such that $v_0 < v_1 < v_2$. Then the corresponding solution u_1 intersects u_0 or u_2 in $(0, T]$, and from the uniqueness in Theorem 2.2, a contradiction yields.

Remark It is well known that if the growth condition (3.3) holds, then any solution of (4.1) is defined over $\mathbb{R}$ for every u_0. In other words, the solutions may blow up only when $|g|$ grows faster than linearly.

Example Let $g(t,u) = -2u^3$ and $f = 0$. Then (2.1) holds, and for $u_0 = 0 \neq v_0$ we have that

$$u' = \operatorname{sgn}(v_0)\sqrt{v_0^2 + u^4}$$

Assume for example that u is defined over $[0,1]$. Then, as $|u'| > |v_0|$ for $t > 0$, we have that $|u(\frac{1}{2})| > \frac{v_0}{2}$. Moreover, $|u'| > u^2$, and hence

$$\frac{1}{|u(\frac{1}{2})|} - \frac{1}{|u(1)|} > \frac{1}{2}$$

Thus,

$$\frac{2}{|v_0|} - \frac{1}{2} > \frac{1}{|u(1)|}$$

proving that $|v_0| < 4$. This shows that $I(0) \subset (-4,4)$.

The following theorem shows that the Lipschitz condition is not necessary in order to prove the existence of $I(u_0)$. Further, we give an explicit expression for $I(u_0)$ as the range of a continuous function.

Theorem 4.1 *Assume that (2.1) holds. Then there exists an interval $I(u_0)$ such that the following two conditions are equivalent:*
i) $v_0 \in I(u_0)$
ii) At least one local solution of (4.1) is defined over $[0,T]$.
Moreover, if $h(s) = u_0 + sT$ and $\psi : \mathbb{R} \to \mathbb{R}$ given by

$$\psi(s) = s + \int_0^T (f - ru_s' - g(\theta, u_s))\frac{\theta - T}{T} d\theta,$$

then $I(u_0) = \operatorname{Range}(\psi)$.

Proof As in Section 3, we have

$$u_s(t) - \varphi_s(t) = \int_0^T (f - ru_s' - g(\theta, u_s))G(t,\theta)d\theta$$

with $\varphi_s(t) = st + u_0$. By simple computation, $u_s'(0) = \psi(s)$, and the proof is complete. $\square$

Remark In particular, if g is locally Lipschitz on u then ψ is injective and hence $I(u_0)$ is open.

Theorem 4.2 *Assume (2.1) and that g is locally Lipschitz on u. Then the set*

$$\bigcup_{u_0 \in \mathbb{R}} \{u_0\} \times I(u_0)$$

is open and simply connected in $\mathbb{R}^2$.

Proof Let $\mathcal{S} = S^{-1}(f)$ and consider the continuous mapping $\rho : \mathcal{S} \to \mathbb{R}^2$, $\rho(u) = (u(0), u'(0))$. Then $v_0 \in I(u_0)$ if and only if $(u_0, v_0) \in \mathrm{Range}(\rho)$. As g is locally Lipschitz, ρ is injective, and hence $\mathrm{Range}(\rho) = \rho \circ \mathrm{Tr}^{-1}(\mathbb{R}^2)$ is open and simply connected. $\square$

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