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EXISTENCE OF SOLUTIONS TO PERTURBED FRACTIONAL NIRENBERG PROBLEMS

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Abstract. In this article we study a fractional Nirenberg problem with a small perturbation of a constant. Under a flatness assumption around the critical points, we prove an existence result in terms of Euler-Hopf index. Our method hinges on a revisited version of the celebrated critical points at infinity approach which goes back to Bahri.

1. Introduction and statement of main result

Fractional calculus is a growing area because of it is a widespread applications in many domains ranging from Medicine to Engineering. Fractional partial differential equations have become important in modeling of modern technology, e.g. for semiconductor and grapheme crystals, which are fundamentally important for photovoltaic and nano-technological applications. They model the flow of glaciers and snow/mud avalanches, the formation of sand dunes as well as the quantum mechanical dynamics of ultra-cold gases, and they are used to describe processes in material science and in the collective motion of biological cells and biological species. Strikingly analogous fractional fluid-type PDE can be used for the modelling of the motion of human crowds, in image processing, in visualization and oil-reservoir modeling, and in water supplies. Important oil exploration and extraction models are based on fractional reaction diffusion equations with porous-medium type nonlinear diffusion processes describing typical geological features [10, 14]. Reaction diffusion equations model chemical reactions and pattern formation in biological systems. Fractional desalination processes can be modelled with systems of reaction-diffusion equations, where diffusion describes random effects (Levy processes) and reaction (local nonlinearities) to model instantaneous chemical processes, [7, 8, 9, 15].

One of the most important fractional PDEs is certainly the Fractional Nirenberg problem. In the classical setting, this problem has attracted many mathematicians in the previous decades. Quite recently some aspects regarding the fractional versions have been addressed, [1, 2, 11, 12]. The aim of this article is to study a

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perturbed version of the fractional Nirenberg equation. More precisely, let $\mathbb{S}^n$ be the unit sphere with its standard metric $g_{\mathbb{S}^n}$. We sutdy the existence of solutions $u : \mathbb{S}^n \to \mathbb{R}$ of the f nonlinear equation

$$P_\sigma u = c(n,\sigma) K u^{\frac{n+2\sigma}{n-2\sigma}}, \quad u > 0, \quad \text{on } \mathbb{S}^n, \tag{1.1}$$

where $\sigma \in (0,1)$, K is a C^1-positive function defined on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ and

$$P_\sigma = \frac{\Gamma(B + \frac{1}{2} + \sigma)}{\Gamma(B + \frac{1}{2} - \sigma)}, \quad B = \sqrt{-\Delta_{g_{\mathbb{S}^n}} + \Big(\frac{n-1}{2}\Big)^2},$$

Γ is the Gamma function, $c(n,\sigma) = \Gamma(\frac{n}{2} + \sigma)/\Gamma(\frac{n}{2} - \sigma)$, and $\Delta_{g_{\mathbb{S}^n}}$ is the Laplace-Beltrami operator on $(\mathbb{S}^n, g_{\mathbb{S}^n})$.

We assume that K satisfy the so-called β-flatness condition $(f)_\beta$; that is for each critical point y of K, there exists a real number $\beta = \beta(y) \in]1, n]$ such that in some geodesic normal coordinate system centered at y, we have

$$K(x) = K(y) + \sum_{k=1}^{n} b_k |(x-y)_k|^\beta + R(x-y), \tag{1.2}$$

where $b_k = b_k(y) \in \mathbb{R}^*, \sum_{k=1}^n b_i \neq 0$ and $\sum_{s=0}^{[\beta]} |\nabla^s R(z)||z|^{s-\beta} = o(1)$ as z tends to zero. Here ∇^s denotes a derivative of order s and $[\beta]$ is the integer part of β. Let

$$\mathcal{K} = \{y \in S^n, \nabla K(y) = 0\}, \quad \mathcal{K}^+ = \{y \in \mathcal{K}, -\sum_{k=1}^n b_k(y) > 0\},$$
$$\widetilde{i}(y) = \sharp\big\{b_k = b_k(y),\ 1 \le k \le n \text{ such that } b_k < 0\big\}.$$

Our main result is the following Theorem which states the existence of a solution of (1.1) under the β-flatness condition for any β and with a very simple index formula. We do believe that this is the very first result in this direction. It is quite surprising to get such assumptions for a very complicated critical problem. Our method hinges on the celebrated critical points at infinity method which goes back to Bahri [3].

Theorem 1.1. *Let K be a C^1-function satisfying the β-flatness condition,* (1.2), $1 < \beta \le n$. *If*

$$\sum_{y \in \mathcal{K}^+} (-1)^{n - \widetilde{i}(y)} \neq 1,$$

then problem (1.1) *has a solution provided K is close to* 1.

This article is organized as follows. In section 2, we show some preliminary results. This prepares the field for applying Bahri's approach. In section 3, we prove Theorem 3.1 which gives sufficient conditions for the existence of critical points at infinity. Then we construct the pseudo gradients fields and get the compactness of the sequences. This is a technical part. In section 4, we shall prove Theorem 1.1.

2. Preliminary results

Problem (1.1) has a variational structure, see [12, sec. 3] and [1]. The Euler-Lagrange functional associated with (1.1) is

$$J(u) = \frac{\|u\|^2}{\big(\int_{\mathbb{S}^n} K u^{\frac{2n}{n-2\sigma}}\big)^{(n-2\sigma)/n}}, \quad u \in H^\sigma(\mathbb{S}^n), \tag{2.1}$$

where $H^\sigma(\mathbb{S}^n)$ is the completion of $C^\infty(\mathbb{S}^n)$ by means of the norm

$$\|u\| = \Big(\int_{\mathbb{S}^n} P_\sigma uu\Big)^{1/2}. \tag{2.2}$$

Problem (1.1) is equivalent to finding critical points of J subjected to the constraint $u \in \Sigma^+$, where

$$\Sigma^+ = \{u \in \Sigma : u \geq 0\}, \quad \Sigma = \{u \in H^\sigma(\mathbb{S}^n) : \|u\| = 1\}.$$

The exponent $2n(n-2\sigma)$ is critical for the Sobolev embedding $H^\sigma(\mathbb{S}^n) \to L^q(\mathbb{S}^n)$. This embedding is continuous and not compact. The functional J does not satisfy the Palais-Smale condition on Σ^+, but the sequences which violate the Palais-Smale condition are known. To describe them, let us introduce some notation. For $a \in \mathbb{S}^n$ and $\lambda > 0$, let

$$\delta_{a,\lambda}(x) = \bar{c}\frac{\lambda^{\frac{n-2\sigma}{2}}}{\big(1 + \frac{\lambda^2-1}{2}(1-\cos(d(x,a)))\big)^{\frac{n-2\sigma}{2}}}, \tag{2.3}$$

where $d(\cdot,\cdot)$ is the distance induced by the standard metric of $\mathbb{S}^n$ and $\bar{c}$ is chosen so that $\delta_{a,\lambda}$ is the family of the solutions for

$$P_\sigma u = u^{\frac{n+2\sigma}{n-2\sigma}}, \quad u > 0, \quad \text{on } \mathbb{S}^n, \tag{2.4}$$

see [11, page 3]. For $\varepsilon > 0$, $p \in \mathbb{N}^*$, we define the following set of potential critical points at infinity.

$$\begin{aligned} V(p,\varepsilon) = \Big\{ & u \in \Sigma : \exists a_1,\dots,a_p \in \mathbb{S}^n, \exists \alpha_1,\dots,\alpha_p > 0 \text{ and} \\ & \exists \lambda_1,\dots,\lambda_p > \varepsilon^{-1} \text{ with } \|u - \sum_{i=1}^p \alpha_i \delta_{a_i,\lambda_i}\| < \varepsilon,\ \varepsilon_{ij} < \varepsilon\ \forall i \neq j, \\ & \text{and } |J(u)^{\frac{n}{n-2\sigma}} \alpha_i^{\frac{2}{n-2\sigma}} K(a_i) - 1| < \varepsilon\ \forall i,j = 1,\dots,p, \end{aligned}$$

where

$$\varepsilon_{ij} = \Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2\Big)^{\frac{2\sigma-n}{2}}.$$

Following [13] and [6], the failure of the Palais-Smale condition can be described as follows.

Proposition 2.1. *Assume that J has no critical points Σ^+. Let (u_k) be a sequence in Σ^+ such that $J(u_k)$ is bounded and $\partial J(u_k)$ goes to zero. Then there exist an integer $p \in \mathbb{N}^*$, a sequence $(\varepsilon_k) > 0$, ε_k tends to zero, and an extracted subsequence of u_k's, again denoted (u_k), such that $u_k \in V(p,\varepsilon_k)$.*

If u is a function in $V(p,\varepsilon)$, one can find an optimal representation, following the ideas introduced in [4]. Namely, we have the following result.

Proposition 2.2. *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon \leq \varepsilon_p$ and $u \in V(p,\varepsilon)$, then the minimization problem*

$$\min_{\alpha_i>0,\lambda_i>0,a_i\in\mathbb{S}^n} \big\|u - \sum_{i=1}^p \alpha_i \delta_{(a_i,\lambda_i)}\big\|,$$

has a unique solution (α,λ,a) up to a permutation.

If we denote

$$v := u - \sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)},$$

then v belongs to $H^\sigma(\mathbb{S}^n)$ and satisfies, arguing as in [3, page 175], the condition:

(V0) $\langle v, \varphi_i \rangle = 0$ for $i = 1, \dots, p$, and $\varphi_i = \delta_i$, $\partial\delta_i/\partial\lambda_i$, $\partial\delta_i/\partial a_i$, where, $\delta_i = \delta_{a_i,\lambda_i}$ and the inner product in $H^\sigma(\mathbb{S}^n)$ defined by

$$\langle u, v \rangle = \int_{\mathbb{S}^n} v P_\sigma u.$$

We will say that $v \in (V_0)$ if v satisfies (V0).

The following Morse lemma eliminates the v-contributions.

Proposition 2.3. *There is a $\mathcal{C}^1$-mapping that to each triplete $(\alpha_i, a_i, \lambda_i)$ with $\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)}$ belonging to $V(p, \varepsilon)$ associates $\overline{v} = \overline{v}(\alpha, a, \lambda)$, such that $\overline{v}$ is unique and satisfies*

$$J\Big(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \overline{v}\Big) = \min_{v \in (V_0)} \Big\{ J\Big(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + v\Big)\Big\}.$$

Moreover, there exists a change of variables $v - \overline{v} \to V$ such that

$$J\Big(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + v\Big) = J\Big(\sum_{i=1}^{p} \alpha_i \delta_{(a_i,\lambda_i)} + \overline{v}\Big) + \| V \|^2 .$$

Furthermore, under the β-flatness condition, with $1 < \beta \le n$, there exists $c > 0$ such that

$$\begin{aligned}\overline{v}\| \le c \sum_{i=1}^{p} \Big[\frac{1}{\lambda_i^{\frac{n}{2}}} + \frac{1}{\lambda_i^{\beta}} + \frac{|\nabla K(a_i)|}{\lambda_i} + \frac{(\log \lambda_i)^{\frac{n+2\sigma}{2n}}}{\lambda_i^{\frac{n+2\sigma}{2}}} \Big] \\ + c \begin{cases} \sum_{k \neq r} \varepsilon_{k\,r}^{\frac{n+2\sigma}{2(n-2\sigma)}} \Big(\log \varepsilon_{kr}^{-1}\Big)^{\frac{n+2\sigma}{2n}}, & \text{if } n \ge 3 \\ \sum_{k \neq r} \varepsilon_{k\,r} \Big(\log \varepsilon_{kr}^{-1}\Big)^{\frac{n-2\sigma}{n}}, & \text{if } n < 3. \end{cases}\end{aligned}$$

At the end of this section, we state the definition of critical point at infinity.

Definition 2.4. A critical point at infinity of J on Σ^+ is a limit of a flow line $u(s)$ of the equation

$$\begin{gathered}\frac{\partial u}{\partial s} = -\partial J(u(s)) \\ u(0) = u_0,\end{gathered}$$

such that $u(s)$ remains in $V(p, \varepsilon(s))$ for $s \ge s_0$. Here $\varepsilon(s) > 0$ and $\to 0$ when $s \to +\infty$. Using proposition 2.2, $u(s)$ can be written as

$$u(s) = \sum_{i=1}^{p} \alpha_i(s) \delta_{(a_i(s),\lambda_i(s))} + v(s).$$

Denoting $\widetilde{\alpha}_i := \lim_{s\to+\infty} \alpha_i(s)$ and $\widetilde{y}_i := \lim_{s\to+\infty} a_i(s)$, we denote by

$$\sum_{i=1}^{p} \widetilde{\alpha}_i \delta_{(\widetilde{y}_i,\infty)} \quad \text{or} \quad (\widetilde{y}_1, \dots, \widetilde{y}_p)_\infty,$$

a critical point at infinity.

We point out that the topological argument that we will use in the proof avoid all critical points at infinity which are in $V(p,\varepsilon)$ with $p \geq 2$. For this, we will interest in the next section to characterize the critical points at infinity in $V(1,\varepsilon)$. Next we identify the function K and its composition with the stereographic projection Π. We will also identify a point x of S^n and its image by Π.

3. Critical points at infinity in $V(1,\varepsilon)$

Theorem 3.1. *Assume that K is close to a positive constant and satisfies the β-flatness condition with $1 < \beta \leq n$. Then the only critical points at infinity of J in $V(1,\varepsilon)$ are*

$$(y)_\infty := \frac{1}{K(y)^{\frac{n-2\sigma}{2}}}\delta_{(y,\infty)}, \quad y \in \mathcal{K}^+.$$

Such critical point at infinity has an index equal to $n - \widetilde{i}(y)$.

Proof of Theorem 3.1. Our argument uses a careful analysis and precise expansion of the gradient of J. Namely, we perform a construction of suitable pseudo-gradient in $V(1,\varepsilon)$, for which the Palais-Smale condition is satisfied along the decreasing flow-lines as long as these flow-lines do not enter in the neighborhood of critical points y of K such that $y \in \mathcal{K}^+$. The following proposition describes the concentration phenomenon of J in $V(1,\varepsilon)$. Its proof will be given later.

Proposition 3.2. *Let $\widetilde{\beta} := \max\{\beta(y), y \in \mathcal{K}\}$. Under the assumptions of Theorem 3.1, there exists a pseudo-gradient W in $V(1,\varepsilon)$ such that for any $u = \alpha\delta_{(a,\lambda)} \in V(1,\varepsilon)$ we have*

(i)
$$\langle \partial J(u), W(u)\rangle \leq -c\Big(\frac{1}{\lambda^{\widetilde{\beta}}} + \frac{\nabla K(a)}{\lambda}\Big),$$

(ii)
$$\langle \partial J(u+\bar{v}), W(u) + \frac{\partial \bar{v}}{\partial(\alpha,a,\lambda)}(W(u))\rangle \leq -c\Big(\frac{1}{\lambda^{\widetilde{\beta}}} + \frac{\nabla K(a)}{\lambda}\Big).$$

Furthermore, $|W|$ is bounded and the only case where $\lambda(s)$ increases is when $a(s)$ goes to $y \in \mathcal{K}^+$.

In the above proposition, we observe that when the concentration point $a(s)$ of a flow line of W do not enter to some neighborhood of $y \in \mathcal{K}^+$, the associated $\lambda(s)$ decreases and therefore no concentration phenomenon in this region. However, if $a(s)$ goes to $y \in \mathcal{K}^+$, $\lambda(s)$ increase and goes to $+\infty$. Thus, we obtain a critical point at infinity. In this statement, the functional J can be expressed (after a suitable change of variables) as

$$J(\alpha\delta_{(a,\lambda)} + \bar{v}) = J(\widetilde{\alpha}\delta_{(\widetilde{a},\widetilde{\lambda})}) = \frac{S_n}{\widetilde{\alpha}^{\frac{4\sigma}{n-2\sigma}}K(\widetilde{a})^{\frac{n-2\sigma}{2}}}\Big\{1 + \frac{1}{\widetilde{\lambda}^{\beta}}\Big\},$$

where

$$S_n = \bar{c}^{\frac{2n}{n-1}}\int_{\mathbb{R}^n}\frac{dx}{(1+|x|^2)^n}.$$

Thus, the index of such critical points at infinity is $n - \widetilde{i}(y)$. Since J behaves in this region as $1/K^{\frac{n-2\sigma}{2}}$. This concludes the proof of Theorem 3.1. □

Proof of Proposition 3.2. To construct the required pseudo-gradient, we dived the set $V(1,\varepsilon)$ into three regions, then construct an appropriate pseudo-gradient in each region, and then glue them through convex combinations. Let $\rho > 0$ small enough such that for any $y \in \mathcal{K}$, the expansion $(f)_\beta$ holds in $B(y,\rho)$ and let

$$\begin{gathered}
V_1(1,\varepsilon) = \{u = \alpha\delta_{(a,\lambda)} \in V(1,\varepsilon) : a \in B(y,\rho),\ y \in \mathcal{K} \text{ with } \beta = \beta(y) < n\},\\
V_2(1,\varepsilon) = \{u = \alpha\delta_{(a,\lambda)} \in V(1,\varepsilon) : a \in B(y,\rho),\ y \in \mathcal{K} \text{ with } \beta = \beta(y) = n\},\\
V_3(1,\varepsilon) = \{u = \alpha\delta_{(a,\lambda)} \in V(1,\varepsilon) : a \notin \cup_{y\in\mathcal{K}} B(y,\rho)\}.
\end{gathered}$$

Pseudo-gradient in $V_1(1,\varepsilon)$.

Lemma 3.3. *For $u = \alpha\delta_{(a,\lambda)} \in V_1(1,\varepsilon)$, we have the expansion*

$$\Big\langle \partial J(u), \alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\Big\rangle = c\alpha^{\frac{2n}{n-2\sigma}} J(u)^{\frac{2(n-\sigma)}{n-2\sigma}} \frac{(\sum_{k=1}^n b_k)}{\lambda^\beta} + O(|a-y|^\beta) + o\Big(\frac{1}{\lambda^\beta}\Big).$$

Proof. For $u = \alpha\delta_{(a,\lambda)} \in V_1(1,\varepsilon)$, we have

$$\begin{aligned}
&\Big\langle \partial J(u), \alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\Big\rangle \\
&= 2J(u)\Big[\Big\langle u, \alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\Big\rangle - J(u)^{\frac{n}{n-2\sigma}} \int_{S^n} K u^{\frac{n+2\sigma}{n-2\sigma}} \alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\Big].
\end{aligned} \tag{3.1}$$

Observe that

$$\Big\langle \delta_{(a,\lambda)}, \frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\Big\rangle = \int_{S^n} \delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}} \frac{\partial\delta_{(a,\lambda)}}{\partial\lambda} = 0. \tag{3.2}$$

Therefore,

$$\begin{aligned}
\Big\langle \partial J(u), \alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\Big\rangle &= -2\alpha^{\frac{2n}{n-2\sigma}} J(u)^{\frac{2(n-\sigma)}{n-2\sigma}} \int_{S^n} K(x)\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda} \\
&= -2\alpha^{\frac{2n}{n-2\sigma}} J(u)^{\frac{2(n-\sigma)}{n-2\sigma}} \int_{S^n} (K(x)-K(y))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}.
\end{aligned}$$

Elementary computations show that

$$\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda} = \frac{n-2\sigma}{2} c_0^{\frac{2n}{n-2\sigma}} \lambda^n \frac{1-\lambda^2|x-a|^2}{(1+\lambda^2|x-a|^2)^{n+1}}. \tag{3.3}$$

For $\mu > 0$ such that $B(a,\mu) \subset B(y,\rho)$, we have

$$\begin{aligned}
&\int_{S^n} (K(x)-K(y))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda} \\
&= \frac{n-2\sigma}{2} c_0^{\frac{2n}{n-2\sigma}} \int_{B(a,\mu)} (K(x)-K(y))\lambda^n \frac{1-\lambda^2|x-a|^2}{(1+\lambda^2|x-a|^2)^{n+1}} dx \\
&\quad + \int_{C_{B(a,\mu)}} (K(x)-K(y))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\Big],
\end{aligned}$$

$$\begin{aligned}
&\int_{C_{B(a,\mu)}} (K(x)-K(y))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda} \\
&\le c\sup_{S^n} |K(x)-K(y)| \int_{C_{B(0,\lambda\mu)}} \frac{|1-|z|^2|}{(1+|z|^2)^{n+1}} dz,
\end{aligned}$$

taking $z = \lambda(x-a)$. Hence,

$$\int_{C_{B(a,\mu)}} (K(x)-K(y))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda \frac{\partial \delta_{(a,\lambda)}}{\partial \lambda} = O\Big(\frac{\sup_{S^n}|K(x)-K(y)|}{\lambda^n}\Big), \tag{3.4}$$

$$\begin{aligned}
&\int_{S^n} (K(x)-K(y))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}} \lambda \frac{\partial \delta_{(a,\lambda)}}{\partial \lambda} \\
&= \frac{n-2\sigma}{2} c_0^{\frac{2n}{n-2\sigma}} \int_{B(a,\mu)} (K(x)-K(y))\lambda^n \frac{1-\lambda^2|x-a|^2}{(1+\lambda^2|x-a|^2)^{n+1}} dx \qquad (3.5)\\
&\quad + O\Big(\frac{\sup_{S^n}|K(x)-K(y)|}{\lambda^n}\Big).
\end{aligned}$$

Using the β-flatness condition, and a change of variable $z = \lambda(x-a)$, we obtain

$$\begin{aligned}
&\int_{B(a,\mu)} (K(x)-K(y))\lambda^n \frac{1-\lambda^2|x-a|^2}{(1+\lambda^2|x-a|^2)^{n+1}} dx \\
&= \frac{1}{\lambda^\beta}\sum_{k=1}^n b_k \int_{B(0,\lambda\mu)} |z_k + \lambda(a-y)_k|^\beta \frac{1-|z|^2}{(1+|z|^2)^{n+1}} dz \\
&\quad + o\Big(\frac{1}{\lambda^\beta}\int_{B(0,\lambda\mu)} |z_k|^\beta \frac{|1-|z|^2|}{(1+|z|^2)^{n+1}} dz\Big) \\
&\quad + o\Big(\frac{1}{\lambda^\beta}|\lambda(a-y)|^\beta \int_{\mathbb{R}^n} \frac{|1-|z|^2|}{(1+|z|^2)^{n+1}} dz\Big) \qquad (3.6)\\
&= \frac{1}{\lambda^\beta}\sum_{k=1}^n b_k \Big[\int_{B(0,\lambda\mu)} |z_k|^\beta \frac{1-|z|^2}{(1+|z|^2)^{n+1}} dz \\
&\quad + O\Big(|\lambda(a-y)_k| \int_{B(0,\lambda\mu)} |z_k|^{\beta-1} \frac{1-|z|^2}{(1+|z|^2)^{n+1}} dz\Big) + O(|\lambda(a-y)_k|^\beta)\Big] \\
&\quad + o\Big(\frac{1}{\lambda^\beta}\int_{B(0,\lambda\mu)} |z_k|^\beta \frac{|1-|z|^2|}{(1+|z|^2)^{n+1}} dz\Big) + o(|(a-y)|^\beta).
\end{aligned}$$

Observe that for $\beta < n$, we have

$$\begin{aligned}
\int_{B(0,\lambda\mu)} |z_k|^\beta \frac{1-|z|^2}{(1+|z|^2)^{n+1}} dz &= \int_{\mathbb{R}^n} |z_k|^\beta \frac{1-|z|^2}{(1+|z|^2)^{n+1}} dz + O(\frac{1}{\lambda^{n-\beta}}) \\
&= -c_1 + O(\frac{1}{\lambda^{n-\beta}}),
\end{aligned}$$

where $c_1 > 0$ and

$$\int_{B(0,\lambda\mu)} |z_k|^{\beta-1} \frac{1-|z|^2}{(1+|z|^2)^{n+1}} dz = O(\frac{1}{\lambda^{n+1-\beta}}).$$

This completes the proof. □

Lemma 3.4. *Let $u = \alpha\delta_{(a,\lambda)} \in V_1(1,\varepsilon)$. For $k = 1,\dots,n$, we have*

(i)

$$\begin{aligned}
\Big\langle \partial J(u), \alpha\frac{1}{\lambda}\frac{\partial \delta_{(a,\lambda)}}{\partial a_k}\Big\rangle &= -c\alpha^{\frac{2n}{n-2\sigma}} J(u)^{\frac{2(n-\sigma)}{n-2\sigma}} \frac{b_k}{\lambda} \operatorname{sign}(a-y)_k |(a-y)_k|^{\beta-1} \\
&\quad + O\Big(\sum_{j=2}^{[\beta]} \frac{|a-y|^{\beta-j}}{\lambda^j},\ \textit{if } \beta \ge 2\Big) + O(\frac{1}{\lambda^\beta}).
\end{aligned}$$

Moreover, if $\lambda|a-y|$ is bounded, we have

(ii)

$$\begin{aligned}&\big\langle\partial J(u),\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\big\rangle\\&=-c\alpha^{\frac{2n}{n-2\sigma}}J(u)^{\frac{2(n-\sigma)}{n-2\sigma}}\frac{b_k}{\lambda^\beta}\int_{\mathbb{R}^n}|z_k+\lambda(a-y)_k|^\beta\frac{z_k}{(1+|z|^2)^{n+1}}dz+o(\frac{1}{\lambda^\beta}).\end{aligned}$$

Proof. As in the proof of Lemma 3.3, we have

$$\big\langle\partial J(u),\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\big\rangle=-2\alpha^{\frac{2n}{n-2\sigma}}J(u)^{\frac{2(n-\sigma)}{n-2\sigma}}\int_{S^n}(K(x)-K(a))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}}\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k},$$

$$\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}}\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}=(n-2\sigma)c_0^{\frac{2n}{n-2\sigma}}\frac{\lambda^{n+1}(x-a)_k}{(1+\lambda^2|x-a|^2)^{n+1}}.$$

Let $\mu>0$ such that $B(a,\mu)\subset B(y,\rho)$. Then

$$\begin{aligned}\int_{C_{B(a,\mu)}}(K(x)-K(a))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}}\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}&=O\Big(\int_{C_{B(a,\mu)}}\frac{\lambda|x-a|}{(1+|z|^2)^{n+1}}\lambda^n dx\Big)\\&=O\Big(\frac{1}{\lambda^{n+1}}\Big).\end{aligned}\tag{3.7}$$

A change of variable $z=\lambda(x-a)$ yields

$$\begin{aligned}&\int_{B(a,\mu)}(K(x)-K(a))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}}\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\\&=(n-2\sigma)c_0^{\frac{2n}{n-2\sigma}}\int_{B(0,\lambda\mu)}(K(a+z/\lambda)-K(a))\frac{z_k}{(1+|z|^2)^{n+1}}dz\\&=(n-2\sigma)c_0^{\frac{2n}{n-2\sigma}}\frac{1}{\lambda}\int_{B(0,\lambda\mu)}\frac{DK(a)(z)z_k}{(1+|z|^2)^{n+1}}dz\\&\quad+O\Big(\sum_{j=2}^n\frac{1}{\lambda^j}\int_{\mathbb{R}^n}\frac{|D^jK(a)||z|^{j+1}}{(1+|z|^2)^{n+1}}dz\Big)+O\Big(\frac{1}{\lambda^\beta}\Big).\end{aligned}\tag{3.8}$$

Observe that

$$\begin{aligned}\int_{B(0,\lambda\mu)}\frac{DK(a)(z)z_k}{(1+|z|^2)^{n+1}}dz&=\frac{\partial K}{\partial x_k}(a)\int_{B(0,\lambda\mu)}\frac{z_k^2}{(1+|z|^2)^{n+1}}dz\\&=\frac{1}{n}\frac{\partial K}{\partial x_k}(a)\int_{B(0,\lambda\mu)}\frac{|z|^2}{(1+|z|^2)^{n+1}}dz\\&=\frac{1}{n}\frac{\partial K}{\partial x_k}(a)\Big(\int_{\mathbb{R}^n}\frac{|z|^2}{(1+|z|^2)^{n+1}}dz+O\Big(\frac{1}{\lambda^n}\Big)\Big).\end{aligned}$$

Since $\int_{B(0,\lambda\mu)}\frac{z_jz_k}{(1+|z|^2)^{n+1}}dz=0$ for all $j\neq k$. Using that $a\in B(a,\rho)$, we derive from the β-flatness condition, that

$$\begin{aligned}\frac{\partial K}{\partial x_k}(a)&=b_k\beta\,\mathrm{sign}(a-y)_k|(a-y)_k|^{\beta-1}+\frac{\partial R}{\partial x_k}(a-y)\\&=b_k\beta\,\mathrm{sign}(a-y)_k|(a-y)_k|^{\beta-1}+o(|a-y|^{\beta-1}).\end{aligned}$$

Moreover, for $j=2,\dots[\beta]$, where $[\beta]$ denotes the integer part of β, we have

$$|D^jK(a)|=O\big(|a-y|^{\beta-j}\big).$$

Thus,

$$\begin{aligned}&\int_{B(a,\mu)}(K(x)-K(a))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}}\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\\&=cb_k\operatorname{sign}(a-y)_k\frac{|(a-y)_k|^{\beta-1}}{\lambda}+o\Big(\frac{|a-y|^{\beta-1}}{\lambda}\Big)+O\Big(\sum_{j=2}^{n}\frac{|a-y|^{\beta-j}}{\lambda^j}\Big)+o\Big(\frac{1}{\lambda^\beta}\Big).\end{aligned}$$

This concludes the proof of (i) of Lemma 3.4. Now, we prove (ii). For $u=\alpha\delta_{(a,\lambda)}\in V_1(1,\varepsilon)$, we have

$$\big\langle\partial J(u),\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\big\rangle=-2J(u)^{\frac{2(n-\sigma)}{n-2\sigma}}\alpha^{\frac{2n}{n-2\sigma}}\int_{S^n}(K(x)-K(y))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}}\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}.$$

Using (3.7) and the β-flatness condition, we obtain

$$\begin{aligned}&\big\langle\partial J(u),\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\big\rangle\\&=-2J(u)^{\frac{2(n-\sigma)}{n-2\sigma}}\alpha^{\frac{2n}{n-2\sigma}}\int_{B(0,\lambda\mu)}\sum_{j=1}^{n}b_j|(x-y)_j|^\beta\frac{(n-2\sigma)\lambda(x-a)_k}{(1+\lambda^2|x-a|^2)^{n+1}}\lambda^n dx\\&\quad+o\Big(\int_{B(0,\lambda\mu)}\frac{|x-y|^\beta|\lambda(x-a)_k|}{(1+\lambda^2|x-a|^2)^{n+1}}\lambda^n dx\Big)+O\Big(\frac{1}{\lambda^{n+1}}\Big).\end{aligned}$$

Observe that for $j\neq k$, we have

$$\int_{B(0,\lambda\mu)}\frac{|(x-y)_j|^\beta\lambda(x-a)_k}{(1+\lambda^2|x-a|^2)^{n+1}}\lambda^n dx=0.$$

Thus, after the change of variables $z=\lambda(x-a)$, we obtain

$$\begin{aligned}&\big\langle\partial J(u),\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\big\rangle\\&=-2(n-2\sigma)J(u)^{\frac{2(n-\sigma)}{n-2\sigma}}\alpha^{\frac{2n}{n-2\sigma}}\frac{b_k}{\lambda^\beta}\int_{B(0,\lambda\mu)}|z_k+\lambda(a-y)_k|^\beta\frac{z_k}{(1+|z|^2)^{n+1}}dz\\&\quad+o\Big(\frac{1}{\lambda^\beta}\int_{\mathbb{R}^n}|z_k+\lambda(a-y)_k|^\beta\frac{|z_k|}{(1+|z|^2)^{n+1}}dz\Big)+o(\frac{1}{\lambda^\beta}).\end{aligned}$$

Using that $\lambda|a-y|$ is bounded, we obtain

$$\begin{gathered}\int_{\mathbb{R}^n}|z_k+\lambda(a-y)_k|^\beta\frac{z_k}{(1+|z|^2)^{n+1}}dz=O(1),\\\int_{C_{B(0,\lambda\mu)}}|z_k+\lambda(a-y)_k|^\beta\frac{z_k}{(1+|z|^2)^{n+1}}dz=o\Big(\frac{1}{\lambda^{n+1-\beta}}\Big).\end{gathered}$$

This completes the proof. □

Let δ be a small positive constant and let $\theta_1,\theta_2,\theta_3:\mathbb{R}\to\mathbb{R}$ be the cut-off functions:

$$\theta_1(t)=\begin{cases}1 & \text{if } |t|\le\frac{\delta}{2}\\0 & \text{if } |t|\ge\delta;\end{cases}$$

$$\theta_2(t)=\begin{cases}1 & \text{if } \frac{\delta}{2}\le|t|\le\frac{1}{\delta}\\0 & \text{if } |t|\in[0,\frac{\delta}{4}]\cup[\frac{2}{\delta},+\infty[\,;\end{cases}$$

$$\theta_3(t) = \begin{cases} 1 & \text{if } |t| \geq \frac{1}{\delta} \\ 0 & \text{if } |t| \leq \frac{1}{2\delta}. \end{cases}$$

Let W_1 be the following vector field. For $u = \alpha\delta_{(a,\lambda)} \in V_1(1,\varepsilon)$,

$$\begin{aligned} &W_1(u) \\ &= -\theta_1(\lambda|a-y|)(\sum_{k=1}^n b_k)\alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda} + \theta_3(\lambda|a-y|)\sum_{k=1}^n b_k \operatorname{sign}(a-y)_k\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k} \\ &\quad + \theta_2(\lambda|a-y|)\sum_{k=1}^n b_k\int_{\mathbb{R}^n}|z_k+\lambda(a-y)_k|^\beta\frac{z_k}{(1+|z|^2)^{n+1}}dz\ \alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}. \end{aligned}$$

Observe that, using Lemma 3.3, if $\lambda|a-y| \leq \delta$, we have

$$\langle\partial J(u), -(\sum_{k=1}^n b_k)\alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\rangle \leq -c_1\ \frac{(\sum_{k=1}^n b_k)^2}{\lambda^\beta},\ \leq -c\Big(\frac{1}{\lambda^\beta}+\frac{|\nabla K(a)|}{\lambda}\Big).$$

If $\lambda|a-y| \in [\frac{\delta}{4},\frac{2}{\delta}]$, by the identity (ii) of Lemma 3.4, we obtain

$$\begin{aligned} &\Big\langle\partial J(u), \sum_{k=1}^n b_k\int_{\mathbb{R}^n}|z_k+\lambda(a-y)_k|^\beta\frac{z_k}{(1+|z|^2)^{n+1}}dz\ \alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\Big\rangle \\ &= -\frac{c}{\lambda^\beta}\sum_{k=1}^n\Big(\int_{\mathbb{R}^n}|z_k+\lambda(a-y)_k|^\beta\frac{z_k}{(1+|z|^2)^{n+1}}dz\Big)^2 + o(\frac{1}{\lambda^\beta}) \\ &\leq -\frac{c}{\lambda^\beta}\Big(\int_{\mathbb{R}^n}|z_{k_a}+\lambda(a-y)_{k_a}|^\beta\frac{z_{k_a}}{(1+|z|^2)^{n+1}}dz\Big)^2 + o(\frac{1}{\lambda^\beta}), \end{aligned}$$

where $|(a-y)_{k_a}| = \max_{1\leq k\leq n}|(a-y)_k|$. Since $\lambda|(a-y)_{k_a}| \geq \frac{1}{\sqrt{n}}\frac{\delta}{4}$, we obtain

$$\Big(\int_{\mathbb{R}^n}|z_{k_a}+\lambda(a-y)_{k_a}|^\beta\frac{z_{k_a}}{(1+|z|^2)^{n+1}}dz\Big)^2 \geq c_\delta > 0.$$

Thus,

$$\begin{aligned} &\Big\langle\partial J(u), \sum_{k=1}^n b_k\int_{\mathbb{R}^n}|z_k+\lambda(a-y)_k|^\beta\frac{z_k}{(1+|z|^2)^{n+1}}dz\ \alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\Big\rangle \\ &\leq -\frac{c_1}{\lambda^\beta} \\ &\leq -c\Big(\frac{1}{\lambda^\beta}+\frac{|\nabla K(a)|}{\lambda}\Big). \end{aligned}$$

Lastly, if $\lambda|a-y| \geq \frac{1}{2\delta}$, by the identity (i) of Lemma 3.4, we have

$$\begin{aligned} \langle\partial J(u), \sum_{k=1}^n b_k \operatorname{sign}(a-y)_k\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\rangle &\leq -c_1\frac{|a-y|^{\beta-1}}{\lambda}, \\ &\leq -c\Big(\frac{1}{\lambda^\beta}+\frac{|\nabla K(a)|}{\lambda}\Big). \end{aligned}$$

Therefore W_1 satisfies the required estimation in $V_1(1,\varepsilon)$.

Pseudo-gradient in $V_2(1,\varepsilon)$.

Lemma 3.5. *For $u=\alpha\delta_{(a,\lambda)}\in V_2(1,\varepsilon)$, we have the expansion*
$$\begin{aligned}\big\langle \partial J(u),\alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\big\rangle &= c\alpha^{\frac{2n}{n-2\sigma}}J(u)^{\frac{2(n-\sigma)}{n-2\sigma}}\frac{(\sum_{k=1}^n b_k)\ \log\lambda}{\lambda^\beta}+O(|a-y|^\beta)\\ &\quad+O\Big(\frac{|a-y|}{\lambda^\beta}\Big)+o\Big(\frac{\log\lambda}{\lambda^\beta}\Big).\end{aligned}$$

Proof. Using (3.1), (3.5) and (3.6), the proof follows from the estimates
$$\begin{aligned}&\int_{B(0,\lambda\mu)}|z_k+\lambda(a-y)_k|^\beta\frac{1-|z|^2}{(1+|z|^2)^{n+1}}dz\\ &=-c\log\lambda+O(|a-y|)+O(\lambda|a-y|^\beta)+o(\log\lambda).\end{aligned}$$
□

Lemma 3.6. *Let $u=\alpha\delta_{(a,\lambda)}\in V_2(1,\varepsilon)$. For any $k=1,\dots,n$, we have*
$$\begin{aligned}\big\langle \partial J(u),\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\big\rangle &= -c\alpha^{\frac{2n}{n-2\sigma}}J(u)^{\frac{2(n-\sigma)}{n-2\sigma}}\frac{b_k}{\lambda}\operatorname{sign}(a-y)_k|(a-y)_k|^{\beta-1}\\ &\quad+o\Big(\sum_{j=2}^{n-1}\frac{|a-y|^{\beta-j}}{\lambda^j}\Big)+O(\frac{1}{\lambda^n}).\end{aligned}$$

The proof of the above lemma proceed as in the proof of Lemma 3.4; we omit it.

Let δ be a positive constant small enough and let $\phi:\mathbb{R}\to\mathbb{R}$ be the cut-off function
$$\phi(t)=\begin{cases}1 & \text{if } |t|<\frac{1}{2\delta}\\ 0 & \text{if } |t|\ge\frac{1}{\delta}.\end{cases}$$
Let W_2 be the following vector field. For $u=\alpha\delta_{(a,\lambda)}\in V_2(1,\varepsilon)$,
$$\begin{aligned}W_2(u)&=-\Big(\sum_{k=1}^n b_k\Big)\phi(\lambda|a-y|)\alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\\ &\quad+\Big(1-\phi(\lambda|a-y|)\Big)\sum_{k=1}^n b_k\operatorname{sign}(a-y)_k\ \alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}.\end{aligned}$$
Observe that, if $\lambda|a-y|<\frac{1}{\delta}$ in the expansion of Lemma 3.5, we have
$$O\Big(|a-y|^\beta\Big)=O\Big(\frac{1}{\lambda^\beta}\Big)=o\Big(\frac{\log\lambda}{\lambda^\beta}\Big),\quad \text{as }\lambda\to+\infty.$$
Therefore,
$$\langle\partial J(u),-(\sum_{k=1}^n b_k)\alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\rangle\le -c_1\ \frac{(\sum_{k=1}^n b_k)^2\log\lambda}{\lambda^\beta}.$$
Also, under the β-flatness condition, we have
$$\frac{|\nabla K(a)|}{\lambda}=o\Big(\frac{\log\lambda}{\lambda^\beta}\Big),\quad\text{as }\lambda\to+\infty.$$
Thus,
$$\langle\partial J(u),-\Big(\sum_{k=1}^n b_k\Big)\alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\rangle\le -c\ \Big(\frac{\log\lambda}{\lambda^\beta}+\frac{|\nabla K(a)|}{\lambda}\Big)$$

$$\leq -c\Big(\frac{1}{\lambda^\beta}+\frac{|\nabla K(a)|}{\lambda}\Big).$$

Now, if $\lambda|a-y|>1/(2\delta)$, by Lemma 3.6 we obtain

$$\begin{aligned}&\Big\langle \partial J(u),\sum_{k=1}^n b_k\operatorname{sign}(a-y)_k\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\Big\rangle\\&\leq -c_1\frac{|a-y|^{\beta-1}}{\lambda}+O\Big(\sum_{j=2}^{n-1}\frac{|a-y|^{\beta-j}}{\lambda^j}\Big)+O(\frac{1}{\lambda^n}).\end{aligned}$$

Observe that for $j=2,\dots,n-1$, we have

$$\frac{|a-y|^{\beta-j}}{\lambda^j}=o\Big(\frac{|a-y|^{\beta-1}}{\lambda}\Big),$$

for δ small enough. Also,

$$\frac{1}{\lambda^n}=o\Big(\frac{|a-y|^{\beta-1}}{\lambda}\Big),$$

for δ small enough. Then, we obtain

$$\begin{aligned}\langle \partial J(u),\sum_{k=1}^n b_k\operatorname{sign}(a-y)_k\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\rangle &\leq -c_2\frac{|a-y|^{\beta-1}}{\lambda}\\&\leq -c\Big(\frac{1}{\lambda^\beta}+\frac{|\nabla K(a)|}{\lambda}\Big),\end{aligned}$$

since

$$\frac{|\nabla K(a)|}{\lambda}\leq c_3\frac{|a-y|^{\beta-1}}{\lambda}$$

and in our statement

$$\frac{1}{\lambda^\beta}\leq c_4\frac{|a-y|^{\beta-1}}{\lambda}.$$

Hence,

$$\langle\partial J(u),W_2(u)\rangle\leq -c\Big(\frac{1}{\lambda^\beta}+\frac{\nabla K(a)}{\lambda}\Big).$$

3.1. **Pseudo-gradient in** $V_3(1,\varepsilon)$.

Lemma 3.7. *Let $u=\alpha\delta_{(a,\lambda)}\in V_3(1,\varepsilon)$. Then*

$$\Big\langle\partial J(u),\alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\Big\rangle=o\Big(\frac{1}{\lambda}\Big),\quad \text{as } \lambda\to+\infty.$$

Proof.

$$\begin{aligned}&\Big\langle\partial J(u),\alpha\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\Big\rangle\\&=-2\alpha^{\frac{2n}{n-2\sigma}}J(u)^{\frac{2(n-\sigma)}{n-2\sigma}}\int_{S^n}(K(x)-K(a))\delta_{(a,\lambda)}^{\frac{n+2\sigma}{n-2\sigma}}\lambda\frac{\partial\delta_{(a,\lambda)}}{\partial\lambda}\\&=-(n-2\sigma)\alpha^{\frac{2n}{n-2\sigma}}J(u)^{\frac{2(n-\sigma)}{n-2\sigma}}\int_{\mathbb{R}^n}\Big(K(a+z/\lambda)-K(a)\Big)\frac{1-|z|^2}{(1+|z|^2)^{n+1}}dz.\end{aligned}$$

Let $\mu_1>0$ very small. Since K is of class C^1, for $z\in B(0,\lambda\mu_1)$ we have

$$K(a+z/\lambda)-K(a)=DK(a)(z/\lambda)+o\Big(\frac{|z|}{\lambda}\Big).$$

Using that

$$\int_{B(0,\lambda\mu_1)} DK(a)(z/\lambda)\frac{1-|z|^2}{(1+|z|^2)^{n+1}}dz = 0,$$

$$\int_{C_{B(0,\lambda\mu_1)}} \Big(K(a+z/\lambda)-K(a)\Big)\frac{1-|z|^2}{(1+|z|^2)^{n+1}}dz = O\Big(\frac{1}{\lambda^n}\Big),$$

the statement of the lemma follows. □

Lemma 3.8. *Let $u=\alpha\delta_{(a,\lambda)}\in V_3(1,\varepsilon)$. Then for $k=1,\dots,n$, we have*

$$\big\langle\partial J(u),\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\big\rangle = -c\alpha^{\frac{2n}{n-2\sigma}}J(u)^{\frac{2(n-\sigma)}{n-2\sigma}}\frac{\frac{\partial K}{\partial x_k}(a)}{\lambda}+o(\frac{1}{\lambda}).$$

Proof. Note that

$$\begin{aligned}&\big\langle\partial J(u),\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}\big\rangle\\ &= -2(n-2\sigma)\alpha^{\frac{2n}{n-2\sigma}}J(u)^{\frac{2(n-\sigma)}{n-2\sigma}}\int_{\mathbb{R}^n}\Big(K(a+z/\lambda)-K(a)\Big)\frac{z_k}{(1+|z|^2)^{n+1}}dz.\end{aligned}$$

Let $\mu>0$ be very small. Then

$$\begin{aligned}&\int_{\mathbb{R}^n}\Big(K(a+z/\lambda)-K(a)\Big)\frac{z_k}{(1+|z|^2)^{n+1}}dz\\ &= \int_{B(0,\lambda\mu)} DK(a)(\frac{z}{\lambda})\frac{z_k}{(1+|z|^2)^{n+1}}dz+o(\frac{1}{\lambda})\\ &= \sum_{j=1}^n\frac{\partial K}{\partial x_j}(a)\frac{1}{\lambda}\int_{B(0,\lambda\mu)}\frac{z_jz_k}{(1+|z|^2)^{n+1}}dz+o(\frac{1}{\lambda})\\ &= \frac{\partial K}{\partial x_k}(a)\frac{1}{n\lambda}\int_{\mathbb{R}^n}\frac{|z|^2}{(1+|z|^2)^{n+1}}dz+o(\frac{1}{\lambda}),\end{aligned}$$

since

$$\int_{B(0,\lambda\mu)}\frac{z_jz_k}{(1+|z|^2)^{n+1}}dz = 0$$

for all $j\neq k$. This completes the proof. □

Let W_3 the following vector field. For $u=\alpha\delta_{(a,\lambda)}\in V_3(1,\varepsilon)$,

$$W_3(u) = \frac{1}{|\nabla K(a)|}\sum_{k=1}^n\frac{\partial K}{\partial x_k}(a)\alpha\frac{1}{\lambda}\frac{\partial\delta_{(a,\lambda)}}{\partial a_k}.$$

By Lemma 3.8, we have

$$\big\langle\partial J(u),W_3(u)\big\rangle\le -c_1\frac{|\nabla K(a)|}{\lambda}\le -c\Big(\frac{1}{\lambda^\beta}+\frac{|\nabla K(a)|}{\lambda}\Big),$$

since $\frac{1}{\lambda^\beta}=o\Big(\frac{|\nabla K(a)|}{\lambda}\Big)$ in $V_3(1,\varepsilon)$.

Finally, the required pseudo-gradient W in $V(1,\varepsilon)$ is defined by a convex combination of W_i, $i=1,\dots,3$. W satisfies claim (i) of Theorem 3.1. Concerning claim (ii), it follows (as in [4, Appendix 2]) from (i) and the fact that $\|\bar v\|^2$ is small with respect to the absolute value of the upper bound of claim (i), see Proposition 2.3. Observe that $|W|$ is bounded, since $|Z|$ and $|X_k|$, $k=1,\dots,n$ are bounded. This completes the proof of Proposition 3.2. □

4. Proof of Theorem 1.1

Let

$$J_1(u)=\frac{1}{\left(\int_{S^n} u^{\frac{2n}{n-2\sigma}}\,dx\right)^{\frac{n-2\sigma}{n}}},\quad u\in\Sigma$$

be the Euler Lagrange functional associated to Yamabe problem on S^n. It is known that J_1 possesses a $(n+1)$-dimensional manifold Z of critical points, giving by

$$Z=\left\{\delta_{(a,\lambda)}: a\in S^n,\lambda>0\right\}.$$

Let $\tilde{S}_n$ be the best Sobolev constant. We have

$$\tilde{S}_n=\inf\{J_1(u)=J_1(\delta_{(a,\lambda)}),\ \text{for any } \tilde{\delta}_{(a,\lambda)}\in Z\}.$$

Given $a,b\in\mathbb{R}$, we set

$$J^a=\left\{u\in\Sigma^+,\ J(u)\le a\right\},\quad J^a_b=\left\{u\in\Sigma^+,\ b\le J(u)\le a\right\}.$$

Lemma 4.1. *Let $\eta>0$, if K is close to 1, we have*

$$J^{\tilde{S}_n+\eta}\subset J_1^{\tilde{S}_n+2\eta}\subset J^{\tilde{S}_n+3\eta}.$$

The above lemma follows from $J(u)=J_1(u)\big(1+O\big(\|K-1\|_{L^\infty(S^n)}\big)\big)$, with $O\big(\|K-1\|_{L^\infty(S^n)}\big)$ being independent of u.

Let $(y_1,\dots,y_q)_\infty$ be a critical point at infinity of q masses and let $C_\infty(y_1,\dots,y_q)$ be the level of J at $(y_1,\dots,y_q)_\infty$. By [2], we have

$$C_\infty(y_1,\dots,y_q)=\tilde{S}_n\Big(\sum_{i=1}^{q}\frac{1}{K(y_i)^{(n-2\sigma)/2}}\Big)^{2/n}.$$

Hence it goes to $q\tilde{S}_n$ when $\|K-1\|_{L^\infty(S^n)}$ is small.

Let $\eta=\widetilde{S}_n/4$, we can therefore assume that K is close to 1 so that all the critical points at infinity of J of two masses or more are above the level $\widetilde{S}_n+3\eta$, and the critical points at infinity of J of one mass are below $\widetilde{S}_n+\eta$. Therefore,

$$J \text{ has no critical points at infinity in } J^{(\widetilde{S}_n+3\eta)}_{(\widetilde{S}_n+\eta)}. \tag{4.1}$$

To prove the existence result, we argue by contradiction and we assume that J has no critical points. It follows from (4.1) that

$$J^{\widetilde{S}_n+3\eta}\simeq J^{\widetilde{S}_n+\eta},$$

where $\simeq$ denotes retracts by deformation. Thus by lemma 4.1, we obtain that

$$J_1^{\widetilde{S}_n+2\eta}\simeq J^{\widetilde{S}_n+\eta}. \tag{4.2}$$

Let $u_0\in J_1^{\widetilde{S}_n+2\eta}$, we solve

$$\begin{gathered}\frac{\partial u}{\partial s}=-\partial J_1(u),\\ u(0)=u_o.\end{gathered}$$

Let $u(s,u_0)$ be the solution for $s>0$. Using [3], we know that the Palais-Smale condition is satisfied for the above differential equation, up to $s=+\infty$. When s tends to $+\infty$, $u(s,u_0)$ converges to a single mass in Z. Hence

$$J_1^{\widetilde{S}_n+2\eta}\simeq Z. \tag{4.3}$$

Observe that Z is a contractible set, thus by (4.2) and (4.3) we derive that $J^{\widetilde{S}_n+\eta}$ is a contractible set. Now we use the gradient flow of $(-\partial J)$ to deform $J^{\widetilde{S}_n+\eta}$. As mentioned above the only critical points at infinity of J under the level $\widetilde{S}_n+\eta$ are $(y)_\infty$, $y \in \mathcal{K}^+$. Thus

$$J^{\widetilde{S}_n+\eta} \simeq \cup_{y\in\mathcal{K}^+} W_u^\infty(y), \tag{4.4}$$

see [5, sections 7 and 8]. We apply now the Euler-Poincaré characteristic of both sides of (4.4), we obtain

$$1 = \sum_{y\in\mathcal{K}^+} (-1)^{n-\widetilde{i}(y)}. \tag{4.5}$$

Hence if (4.5) is not satisfied, then (1.1) has a solution.

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