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POSITIVE SOLUTIONS OF A NONLINEAR THREE-POINT BOUNDARY-VALUE PROBLEM

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Abstract. We study the existence of positive solutions to the boundary-value problem

$$u'' + a(t)f(u) = 0, \quad t \in (0,1)$$
$$u(0) = 0, \quad \alpha u(\eta) = u(1)\,,$$

where $0 < \eta < 1$ and $0 < \alpha < 1/\eta$. We show the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem in cones.

1. Introduction

The study of multi-point boundary-value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [7, 8]. Then Gupta [5] studied three-point boundary-value problems for nonlinear ordinary differential equations. Since then, the more general nonlinear multi-point boundary value problems have been studied by several authors by using the Leray-Schauder Continuation Theorem, Nonlinear Alternatives of Leray-Schauder, and coincidence degree theory. We refer the reader to [1-3, 6, 10-12] for some recent results of nonlinear multi-point boundary value problems.

In this paper, we consider the existence of positive solutions to the equation

$$u'' + a(t)f(u) = 0, \quad t \in (0,1) \tag{1.1}$$

with the boundary condition

$$u(0) = 0, \quad \alpha u(\eta) = u(1)\,, \tag{1.2}$$

where $0 < \eta < 1$. Our purpose here is to give some existence results for positive solutions to (1.1)-(1.2), assuming that $\alpha\eta < 1$ and f is either superlinear or sublinear. Our proof is based upon the fixed point theorem in a cone.

From now on, we assume the following:
(A1) $f \in C([0,\infty),[0,\infty))$;
(A2) $a \in C([0,1],[0,\infty))$ and there exists $x_0 \in [\eta,1]$ such that $a(x_0) > 0$

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Set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. By the positive solution of (1.1)-(1.2) we understand a function $u(t)$ which is positive on $0 < t < 1$ and satisfies the differential equation (1.1) and the boundary conditions (1.2).

The main result of this paper is the following

Theorem 1. *Assume (A1) and (A2) hold. Then the problem (1.1)-(1.2) has at least one positive solution in the case*
(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear) or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

The proof of above theorem is based upon an application of the following well-known Guo's fixed point theorem [4].

Theorem 2. *Let* E *be a Banach space, and let* $K \subset E$ *be a cone. Assume* Ω_1, Ω_2 *are open subsets of* E *with* $0 \in \Omega_1, \quad \overline{\Omega}_1 \subset \Omega_2$, *and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that
(i) $\| Au \| \leq \| u \|, \quad u \in K \cap \partial\Omega_1$, *and* $\| Au \| \geq \| u \|, \quad u \in K \cap \partial\Omega_2$; *or*
(ii) $\| Au \| \geq \| u \|, \quad u \in K \cap \partial\Omega_1$, *and* $\| Au \| \leq \| u \|, \quad u \in K \cap \partial\Omega_2$.
Then A *has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. The Preliminary Lemmas

Lemma 1. *Let* $\alpha\eta \neq 1$ *then for* $y \in C[0,1]$, *the problem*

$$u'' + y(t) = 0, \quad t \in (0,1) \tag{2.1}$$

$$u(0) = 0, \quad \alpha u(\eta) = u(1) \tag{2.2}$$

has a unique solution

$$u(t) = -\int_0^t (t-s)y(s)ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)y(s)ds + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)y(s)\,ds$$

The proof of this lemma can be found in [6].

Lemma 2. *Let* $0 < \alpha < \frac{1}{\eta}$. *If* $y \in C[0,1]$ *and* $y \geq 0$, *then the unique solution* u *of the problem (2.1)-(2.2) satisfies*

$$u \geq 0, \quad t \in [0,1]$$

Proof From the fact that $u''(x) = -y(x) \leq 0$, we know that the graph of $u(t)$ is concave down on (0,1). So, if $u(1) \geq 0$, then the concavity of u and the boundary condition $u(0) = 0$ imply that $u \geq 0$ for $t \in [0,1]$.

If $u(1) < 0$, then we have that

$$u(\eta) < 0 \tag{2.3}$$

and

$$u(1) = \alpha u(\eta) > \frac{1}{\eta} u(\eta). \tag{2.4}$$

This contradicts the concavity of u.

Lemma 3. *Let* $\alpha\eta > 1$*. If* $y \in C[0,1]$ *and* $y(t) \geq 0$ *for* $t \in (0,1)$*, then (2.1)-(2.2) has no positive solution.*

Proof Assume that (2.1)-(2.2) has a positive solution u.
If $u(1) > 0$, then $u(\eta) > 0$ and

$$\frac{u(1)}{1} = \frac{\alpha u(\eta)}{1} > \frac{u(\eta)}{\eta} \tag{2.5}$$

this contradicts the concavity of u.
If $u(1) = 0$ and $u(\tau) > 0$ for some $\tau \in (0,1)$, then

$$u(\eta) = u(1) = 0, \quad \tau \neq \eta \tag{2.6}$$

If $\tau \in (0,\eta)$, then $u(\tau) > u(\eta) = u(1)$, which contradicts the concavity of u. If $\tau \in (\eta, 1)$, then $u(0) = u(\eta) < u(\tau)$ which contradicts the concavity of u again.

In the rest of the paper, we assume that $\alpha\eta < 1$. Moreover, we will work in the Banach space $C[0,1]$, and only the sup norm is used.

Lemma 4. *Let* $0 < \alpha < \frac{1}{\eta}$*. If* $y \in C[0,1]$ *and* $y \geq 0$*, then the unique solution* u *of the problem (2.1)-(2.2) satisfies*

$$\inf_{t\in[\eta,1]} u(t) \geq \gamma\|u\|,$$

where $\gamma = \min\{\alpha\eta, \frac{\alpha(1-\eta)}{1-\alpha\eta}, \eta\}$.

Proof. We divide the proof into two steps.
Step 1. We deal with the case $0 < \alpha < 1$. In this case, by Lemma 2, we know that

$$u(\eta) \geq u(1). \tag{2.7}$$

Set

$$u(\bar{t}) = \|u\|. \tag{2.8}$$

If $\bar{t} \leq \eta < 1$, then

$$\min_{t\in[\eta,1]} u(t) = u(1) \tag{2.9}$$

and

$$\begin{aligned} u(\bar{t}) &\leq u(1) + \frac{u(1) - u(\eta)}{1-\eta}(0-1) \\ &= u(1)[1 - \frac{1-\frac{1}{\alpha}}{1-\eta}] \\ &= u(1)\frac{1-\alpha\eta}{\alpha(1-\eta)}. \end{aligned}$$

This together with (2.9) implies that

$$\min_{t\in[\eta,1]} u(t) \geq \frac{\alpha(1-\eta)}{1-\alpha\eta}\|u\|. \tag{2.10}$$

If $\eta < \bar{t} < 1$, then

$$\min_{t\in[\eta,1]} u(t) = u(1) \tag{2.11}$$

From the concavity of u, we know that

$$\frac{u(\eta)}{\eta} \geq \frac{u(\bar{t})}{\bar{t}} \tag{2.12}$$

Combining (2.12) and boundary condition $\alpha u(\eta) = u(1)$, we conclude that

$$\frac{u(1)}{\alpha\eta} \geq \frac{u(\bar{t})}{\bar{t}} \geq u(\bar{t}) = \|u\|.$$

This is

$$\min_{t\in[\eta,1]} u(t) \geq \alpha\eta\|u\|. \tag{2.13}$$

Step 2. We deal with the case $1 \leq \alpha < \frac{1}{\eta}$. In this case, we have

$$u(\eta) \leq u(1). \tag{2.14}$$

Set

$$u(\bar{t}) = \|u\| \tag{2.15}$$

then we can choose $\bar{t}$ such that

$$\eta \leq \bar{t} \leq 1 \tag{2.16}$$

(we note that if $\bar{t} \in [0,1] \setminus [\eta,1]$, then the point $(\eta, u(\eta))$ is below the straight line determined by $(1, u(1))$ and $(\bar{t}, u(\bar{t}))$. This contradicts the concavity of u. From (2.14) and the concavity of u, we know that

$$\min_{t\in[\eta,1]} u(t) = u(\eta). \tag{2.17}$$

Using the concavity of u and Lemma 2, we have that

$$\frac{u(\eta)}{\eta} \geq \frac{u(\bar{t})}{\bar{t}}. \tag{2.18}$$

This implies

$$\min_{t\in[\eta,1]} u(t) \geq \eta\|u\|. \tag{2.19}$$

This completes the proof.

3 Proof of main theorem

Proof of Theorem 1. Superlinear case. Suppose then that $f_0 = 0$ and $f_\infty = \infty$. We wish to show the existence of a positive solution of (1.1)-(1.2). Now (1.1)-(1.2) has a solution $y = y(t)$ if and only if y solves the operator equation

$$\begin{aligned} y(t) = &-\int_0^t (t-s)a(s)f(y(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(y(s))ds \\ &+ \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\ &\stackrel{\text{def}}{=} Ay(t). \end{aligned} \tag{3.1}$$

Denote

$$K = \{y \mid y \in C[0,1], y \geq 0, \min_{\eta \leq t \leq 1} y(t) \geq \gamma \|y\|\}. \tag{3.2}$$

It is obvious that K is a cone in $C[0,1]$. Moreover, by Lemma 4, $AK \subset K$. It is also easy to check that $A : K \to K$ is completely continuous.

Now since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(y) \leq \epsilon y$, for $0 < y < H_1$, where $\epsilon > 0$ satisfies

$$\frac{\epsilon}{1-\alpha\eta} \int_0^1 (1-s)a(s)ds \leq 1. \tag{3.3}$$

Thus, if $y \in K$ and $\|y\| = H_1$, then from (3.1) and (3.3), we get

$$\begin{aligned} Ay(t) &\leq \frac{t}{1-\alpha\eta} \int_0^1 (1-s)a(s)f(y(s))ds \\ &\leq \frac{t}{1-\alpha\eta} \int_0^1 (1-s)a(s)\epsilon y(s)ds \\ &\leq \frac{\epsilon}{1-\alpha\eta} \int_0^1 (1-s)a(s)ds\|y\| \\ &\leq \frac{\epsilon}{1-\alpha\eta} \int_0^1 (1-s)a(s)dsH_1. \end{aligned} \tag{3.4}$$

Now if we let

$$\Omega_1 = \{y \in C[0,1] \mid \|y\| < H_1\}, \tag{3.5}$$

then (3.4) shows that $\|Ay\| \leq \|y\|$, for $y \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\hat{H}_2 > 0$ such that $f(u) \geq \rho u, \quad$ for $u \geq \hat{H}_2$, where $\rho > 0$ is chosen so that

$$\rho \frac{\eta\gamma}{1-\eta\alpha} \int_\eta^1 (1-s)a(s)ds \geq 1. \tag{3.6}$$

Let $H_2 = \max\{2H_1, \frac{\hat{H}_2}{\gamma}\}$ and $\Omega_2 = \{y \in C[0,1] \mid \|y\| < H_2\}$, then $y \in K$ and $\|y\| = H_2$ implies

$$\min_{\eta \leq t \leq 1} y(t) \geq \gamma\|y\| \geq \hat{H}_2,$$

and so

$$\begin{aligned}
Ay(\eta) =& -\int_0^\eta (\eta - s)a(s)f(y(s))dt - \frac{\alpha\eta}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(y(s))ds \\
&+ \frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\
=& -\frac{1}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(y(s))ds + \frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\
=& -\frac{1}{1-\alpha\eta}\int_0^\eta \eta a(s)f(y(s))ds + \frac{1}{1-\alpha\eta}\int_0^\eta sa(s)f(y(s))ds \\
&+ \frac{\eta}{1-\alpha\eta}\int_0^1 a(s)f(y(s))ds - \frac{\eta}{1-\alpha\eta}\int_0^1 sa(s)f(y(s))ds \qquad (3.7)\\
=& \frac{\eta}{1-\alpha\eta}\int_\eta^1 a(s)f(y(s))ds + \frac{1}{1-\alpha\eta}\int_0^\eta sa(s)f(y(s))ds \\
&- \frac{\eta}{1-\alpha\eta}\int_0^1 sa(s)f(y(s))ds \\
\geq& \frac{\eta}{1-\alpha\eta}\int_\eta^1 a(s)f(y(s))ds - \frac{\eta}{1-\alpha\eta}\int_\eta^1 sa(s)f(y(s))ds \quad (\text{by } \eta < 1) \\
=& \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)f(y(s))ds.
\end{aligned}$$

Hence, for $y \in K \cap \partial\Omega_2$,

$$\|Ay\| \geq \rho\frac{\eta\gamma}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)ds\|y\| \geq \|y\|.$$

Therefore, by the first part of the Fixed Point Theorem, it follows that A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, such that $H_1 \leq \|u\| \leq H_2$. This completes the superlinear part of the theorem.

Sublinear case. Suppose next that $f_0 = \infty$ and $f_\infty = 0$. We first choose $H_3 > 0$ such that $f(y) \geq My$ for $0 < y < H_3$, where

$$M\gamma(\frac{\eta}{1-\alpha\eta})\int_\eta^1 (1-s)a(s)ds \geq 1. \tag{3.8}$$

By using the method to get (3.7), we can get that

$$\begin{aligned}
Ay(\eta) =& -\int_0^\eta (\eta - s)a(s)f(y(s))dt - \frac{\alpha\eta}{1-\alpha\eta}\int_0^\eta (\eta - s)a(s)f(y(s))ds \\
&+ \frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\
\geq& \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)f(y(s))ds \qquad (3.9)\\
\geq& \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)My(s)ds \\
\geq& \frac{\eta}{1-\alpha\eta}\int_\eta^1 (1-s)a(s)M\gamma ds\|y\| \\
\geq& H_3
\end{aligned}$$

Thus,we may let $\Omega_3 = \{y \in C[0,1] \mid \|y\| < H_3\}$ so that

$$\|Ay\| \geq \|y\|, \qquad y \in K \cap \partial\Omega_3.$$

Now,since $f_\infty = 0$, there exists $\hat{H}_4 > 0$ so that $f(y) \leq \lambda y$ for $y \geq \hat{H}_4$,where $\lambda > 0$ satisfies

$$\frac{\lambda}{1-\alpha\eta}[\int_0^1 (1-s)a(s)ds] \leq 1. \tag{3.10}$$

We consider two cases:
Case (i). Suppose f is bounded, say $f(y) \leq N$ for all $y \in [0,\infty)$. In this case choose

$$H_4 = \max\{2H_3, \frac{N}{1-\alpha\eta}\int_0^1 (1-s)a(s)ds\}$$

so that for $y \in K$ with $\|y\| = H_4$ we have

$$\begin{aligned} Ay(t) =& -\int_0^t (t-s)a(s)f(y(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)a(s)f(y(s))ds \\ &+ \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\ \leq& \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\ \leq& \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)Nds \\ \leq& H_4 \end{aligned}$$

and therefore $\|Ay\| \leq \|y\|$.

Case (ii). If f is unbounded, then we know from (A1) that there is $H_4 : H_4 > \max\{2H_3, \frac{1}{\gamma}\hat{H}_4\}$ such that

$$f(y) \leq f(H_4) \quad \text{for} \quad 0 < y \leq H_4.$$

(We are able to do this since f is unbounded). Then for $y \in K$ and $\|y\| = H_4$ we have

$$\begin{aligned} Ay(t) =& -\int_0^t (t-s)a(s)f(y(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)a(s)f(y(s))ds \\ &+ \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(y(s))ds \\ \leq& \frac{t}{1-\alpha\eta}\int_0^1 (1-s)a(s)f(H_4)ds \\ \leq& \frac{1}{1-\alpha\eta}\int_0^1 (1-s)a(s)\lambda H_4 ds \\ \leq& H_4. \end{aligned}$$

Therefore, in either case we may put

$$\Omega_4 = \{y \in C[0,1] \mid \|y\| < H_4\},$$

and for $y \in K \cap \partial\Omega_4$ we may have $\|Ay\| \leq \|y\|$. By the second part of the Fixed Point Theorem, it follows that BVP (1.1)-(1.2) has a positive solution. Therefore, we have completed the proof of Theorem 1.

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