Electronic Journal of Differential Equations, Vol. 2005(2005), No. 35, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

BOUNDENESS AND STEPANOV'S ALMOST PERIODICITY OF SOLUTIONS

ZUOSHENG HU

Abstract. In this paper, we establish a necessary condition of Stepanov's almost periodicity of solutions for general linear almost periodic systems, and then we construct an example of linear system in which all solutions are bounded, but any non-trivial solution is not Stepanov's almost periodic.

1. Introduction

It is well-known that for (Bohr) almost periodic differential equations

$$x' = A(t)x + f(t), \tag{1.1}$$

boundedness of solutions does not imply their almost periodicity. Conley and Miller [2] gave an example of an equation (1.1) with $n = 1$ where a bounded solution is not almost periodic. In [5], Mingarelli, Pu and Zheng constructed an example, for each $n > 1$, of an equation (1.1) with almost periodic coefficients in which there exists a bounded solution which is not almost periodic. In [4], Hu and Mingarelli constructed a class of linear almost periodic systems in which all solutions are bounded but there still exists no any non-trivial solution which is almost periodic. The question arising naturally is whether boundedness of solutions can imply their Stepanov's almost periodicity which is a weaker almost periodicity defined by Stepanov (see [1] for the details). As far as we know, this is an open problem. In this paper, in order to solve this open problem, we establish a necessary condition of Stepanov's almost periodicity of solutions for general linear almost periodic systems, and then we construct an example of linear system in which all solutions are bounded, but any non-trivial solution is not Stepanov's almost periodic. So, the answer of this problem is negative.

For completeness, we recall the definition of the Stepanov norm $S_l(f)$ of a function $f \in L_1^{\mathrm{loc}}(\mathbb{R}, X)$. The quantity

$$S_l(f) = \sup_{t \in \mathbb{R}} \frac{1}{l} \int_t^{t+l} \|f(s)\| ds$$

where $l > 0$ is some constant, is the Stepanov norm (or S_l-norm) of f.

Replacing the supremum norm by the S_l-norm in the definition of continuity (respectively, uniform continuity, boundedness) of f, we can introduce the concept

2000 *Mathematics Subject Classification.* 34C27, 34A30.
Key words and phrases. Boundedness; Stepanov's almost periodicity.

Submitted January 17, 2005. Published March 24, 2005.

of S$_l$-continuity (respectively, S$_l$-uniform continuity, S$_l$-boundedness) of f. For example, we call $f \in L_1^{\rm loc}(\mathbb{R}, X)$ S$_l$-bounded if there exists a constant $M > 0$ such that $S_l(f) \leq M$. It is easy to show that S_l-boundedness (S_l-continuity, S$_l$-uniform continuity) is not dependent on the constant l (see [1]). So, we simply call such functions S-bounded, S-continuous, and S-uniformly continuous whenever these notions apply.

We define $S_l(t, f)$ as follows:

$$S_l(t, f) = \frac{1}{l} \int_t^{t+l} \|f(s)\| ds \quad \text{for all } t \in \mathbb{R}. \tag{1.2}$$

From (1.2) we have that for any $t, s \in \mathbb{R}$,

$$S_l(t, f_s) = S_l(t + s, f).$$

where f_s is the translate of f. We use $S_lC(\mathbb{R}, X)$ to denote the set of all S$_l$-continuous functions. Obviously, $C(\mathbb{R}, X) \subset S_lC(\mathbb{R}, X)$. Because l can be taken any positive real number, we simply denote $S_lC(\mathbb{R}, X)$ by $SC(\mathbb{R}, X)$. As in the case of Bohr almost periodic functions, we introduce the definition of a Stepanov almost periodic function.

Definition 1.1. Let $f \in S_lC(\mathbb{R}, X)$. If for any sequence $\{\alpha_n\} \subset \mathbb{R}$, there exist a subsequence $\{\alpha'_n\}$ of $\{\alpha_n\}$ and a function $g \in S_lC(\mathbb{R}, X)$ such that

$$\lim_{n\to\infty} S_l(t, f_{\alpha'_n} - g) = 0, \quad \text{uniformly on } \mathbb{R},$$

then f is called S$_l$-almost periodic on $\mathbb{R}$.

We use the notation in [3]. $\alpha = \{\alpha_n\}$ is a sequence of real numbers. $\alpha' \subset \alpha$ means that $\alpha' = \{\alpha'_n\} \subset \{\alpha_n\}$ is a subsequence of α. For $f, g \in \mathbb{R}, ST_\alpha f = g$ means that there exist a sequence α and a real number $l > 0$ such that

$$\lim_{n\to\infty} \frac{1}{l} \int_t^{t+l} \|f(s + \alpha_n) - g(s)\| ds = 0, \tag{1.3}$$

pointwise for $t \in \mathbb{R}$ and $UST_\alpha f = g$ means that (1.3) holds uniformly on $t \in \mathbb{R}$.

Now we give the definition of the *uniform Stepanov hull* of a Stepanov almost periodic function.

Definition 1.2. Let $f \in SC(\mathbb{R}, X)$. The set

$$\{g \in SC(\mathbb{R}, X) : \text{ there exists a sequence } \{\alpha_n\} \subset \mathbb{R} \text{ such that } UST_\alpha f = g\}$$

is called the *uniform Stepanov hull*, or simply *uniform S-hull*, and is denoted by $\mathcal{SH}(f)$.

Obviously, for any $f \in SC(\mathbb{R}, X)$, $\mathcal{SH}(f)$ is not empty since $f \in \mathcal{SH}(f)$.

2. Necessary Condition

Consider the general system of linear differential equations

$$x' = A(t)x + f(t), \tag{2.1}$$

where $x \in \mathbb{R}^n$, $A(t)$ is an $n \times n$ matrix function, and $f(t)$ is an n-dimensional vector function, defined on $\mathbb{R}$. Throughout this paper, we assume that $A(t)$ and $f(t)$ are all Bohr's almost periodic on $\mathbb{R}$.

Lemma 2.1. *Suppose that $\phi(t)$ is a solution of* (2.1) *and that there exist a sequence $\alpha = \{\alpha_n\}$, $g \in SH(f)$, $B \in SH(A)$ and $\varphi \in SH(\phi)$ such that $UST_\alpha A = B$, $UST_\alpha f = g$ and $UST_\alpha \phi = \varphi$. Then there exist a subsequence α' of α and a solution $\tilde{\varphi}(t)$ of*

$$x' = B(t)x + g(t), \tag{2.2}$$

such that $UST_{\alpha'}\phi = \tilde{\varphi}$ on $\mathbb{R}$. If ϕ is bounded, so is $\tilde{\varphi}$ with the same bound.

Proof. By the assumption, there is a constant $l > 0$ such that

$$\begin{gathered}
\lim_{n\to\infty} \frac{1}{l}\int_t^{t+l} |A(s+\alpha_n) - B(s)|ds = 0, \\
\lim_{n\to\infty} \frac{1}{l}\int_t^{t+l} |f(s+\alpha_n) - g(s)|ds = 0, \\
\lim_{n\to\infty} \frac{1}{l}\int_t^{t+l} |\phi(s+\alpha_n) - \varphi(s)|ds = 0,
\end{gathered}$$

uniformly on $\mathbb{R}$. In particular, for each $t \in \mathbb{R}$ it follows that $\lim_{n\to\infty}\int_0^l |\phi(s+t+\alpha_n) - \varphi(t+s)|ds = 0$; in other words, the sequence $\{\phi(t+s+\alpha_n)\}$ in $L^1[0,l]$ converges to $\varphi(t+s)$, and hence the sequence converges to $\varphi(t+s)$ in the sense of measure on $[0,l]$. Since $t \in \mathbb{R}$ is arbitrary, the sequence of measurable functions $\{\phi(\tau+\alpha_n)\}$ converges to $\varphi(\tau)$ in the sense of measure on $\mathbb{R}$; hence from a well known result in the theory of Lebesgue measure we know that there is a subsequence α' of α such that $\lim_{n\to\infty}\phi(\tau+\alpha'_n) = \varphi(\tau)$ a.e. on $\mathbb{R}$. Take a point t_0 in $\mathbb{R}$ such that $\lim_{n\to\infty}\phi(t_0+\alpha'_n) = \varphi(t_0)$. Notice that

$$\begin{gathered}
\lim_{n\to\infty}\int_{t_0}^{t} |A(s+\alpha'_n) - B(s)|ds = 0, \\
\lim_{n\to\infty}\int_{t_0}^{t} |f(s+\alpha'_n) - g(s)|ds = 0, \\
\lim_{n\to\infty}\int_{t_0}^{t} |\phi(s+\alpha'_n) - \varphi(s)|ds = 0,
\end{gathered}$$

locally uniformly for $t \in \mathbb{R}$ (which means the uniformity of convergence on any finite interval in $\mathbb{R}$). Since ϕ is a solution of (2.1), we get the following relations:

$$\phi(t+\alpha'_n) = \phi(t_0+\alpha'_n) + \int_{t_0}^{t}\{A(s+\alpha'_n)\phi(s+\alpha'_n) + f(s+\alpha'_n)\}ds, \tag{2.3}$$

for $t \in \mathbb{R}$, $n = 1, 2, \dots$. Note that $A(t)$ and $f(t)$ are bound on $\mathbb{R}$ because of Bohr's almost periodicity of $A(t)$ and $f(t)$. Then, by Gronwall's inquality one can see that $\{\phi(t+\alpha'_n)\}$ is uniformly bounded on each finite interval in $\mathbb{R}$. From these facts we see that

$$\lim_{n\to\infty}\int_{t_0}^{t}\{A(s+\alpha'_n)\phi(s+\alpha'_n) + f(s+\alpha'_n)\}ds = \int_{t_0}^{t}\{B(s)\varphi(s) + g(s)\}ds,$$

locally uniformly for $t \in \mathbb{R}$. Define a (continuous) function $\tilde{\varphi}(t)$ by

$$\tilde{\varphi}(t) = \varphi(t_0) + \int_{t_0}^{t}\{B(s)\varphi(s) + g(s)\}ds, \quad t \in \mathbb{R}.$$

From (2.3) we see that $\lim_{n\to\infty}\phi(t+\alpha'_n)=\tilde\varphi(t)$, locally uniformly for $t\in\mathbb{R}$. Consequently, it follows that $\varphi(t)\equiv\tilde\varphi(t)$ a.e. on $\mathbb{R}$. Hence

$$\lim_{n\to\infty}\frac{1}{l}\int_t^{t+l}|\phi(s+\alpha'_n)-\tilde\varphi(s)|ds=\lim_{n\to\infty}\frac{1}{l}\int_t^{t+l}|\phi(s+\alpha'_n)-\varphi(s)|ds=0,$$

uniformly on $\mathbb{R}$, which shows $UST_{\alpha'}\phi=\tilde\varphi$. Furthermore, we get

$$\begin{aligned}\tilde\varphi(t)&=\varphi(t_0)+\int_{t_0}^t\{B(s)\varphi(s)+g(s)\}ds\\&=\tilde\varphi(t_0)+\int_{t_0}^t\{B(s)\tilde\varphi(s)+g(s)\}ds,\quad t\in\mathbb{R},\end{aligned}$$

which shows that $\tilde\varphi$ is a solution of (2.2). The last conclusion is obvious. This completes the proof of Lemma. □

Remark. According to Lemma 2.1, if ϕ is a solution of (2.1) and assumptions are satisfied, we can simply say that $UST_\alpha\phi$ is a solution of (2.2).

Theorem 2.2. *Let $A(t)$ be Bohr's almost periodic on $\mathbb{R}$. If $x(t)$ is a non-trivial Stepanov's almost periodic solution of the equation*

$$x'=A(t)x, \tag{2.4}$$

then for any $l>0$,

$$\inf_{t\in\mathbb{R}}\frac{1}{l}\int_t^{t+l}\|x(s)\|ds>0 \tag{2.5}$$

Proof. On the contrary, suppose that there exists a real number $l>0$ such that (2.5) does not hold, i. e.

$$\lim_{t\in\mathbb{R}}\frac{1}{l}\int_t^{t+l}\|x(s)\|ds=0. \tag{2.6}$$

Then we will find a contradiction. From (2.6), we can pick up a sequence $\alpha=\{\alpha_n\}$ such that

$$\lim_{n\to\infty}\frac{1}{l}\int_{\alpha_n}^{\alpha_n+l}\|x(s)\|ds=0$$

or

$$\lim_{n\to\infty}\frac{1}{l}\int_0^l\|x(s+\alpha_n)\|ds=0. \tag{2.7}$$

Since $A(t)$ is almost periodic on $\mathbb{R}$ and $x(t)$ is Stepanov's almost periodic on $\mathbb{R}$, we can extract a subsequence $\alpha'\subset\alpha$, $B(t)\in\mathcal{H}(A)$ and $y(t)\in\mathcal{SH}(x)$ such that $y=UST_{\alpha'}x$ and $B=UT_{\alpha'}A$. By Lemma 2.1, there exist a subsequence $\alpha''\subset\alpha'$ and a solution $\tilde y$ of the equation

$$y'=B(t)y \tag{2.8}$$

such that

$$\tilde y=UST_{\alpha''}x \tag{2.9}$$

on $\mathbb{R}$. On the other hand, we have

$$\frac{1}{l}\int_o^l|\tilde y(s)|ds\le\frac{1}{l}\int_0^l|\tilde y(s)-x(s+\alpha''_n)|ds+\frac{1}{l}\int_0^l|x(s+\alpha''_n)|ds \tag{2.10}$$

Let $n \to \infty$, we obtain that

$$\frac{1}{l} \int_0^l |\tilde{y}(s)| ds = 0$$

from (2.7) and (2.9). And thus, there exists at least one $t_0 \in [0, l]$ such that $\tilde{y}(t_0) = 0$. Since $\tilde{y}(t)$ is a solution of (2.8), we have $\tilde{y}(t) = 0$ for all $t \in \mathbb{R}$. Then (2.9) implies

$$\lim_{n\to\infty} \frac{1}{l} \int_\tau^{\tau+l} \|x(s + \alpha_n'')\| ds = 0$$

uniformly for $\tau \in \mathbb{R}$; hence

$$\begin{aligned} \frac{1}{l} \int_0^l \|x(t+s)\| ds &= \lim_{n\to\infty} \frac{1}{l} \int_0^l \|x(t - \alpha'' + s + \alpha_n'')\| ds \\ &= \lim_{n\to\infty} \frac{1}{l} \int_{t-\alpha''}^{t-\alpha''+l} \|x(s + \alpha_n'')\| ds = 0 \end{aligned}$$

and consequently, $x(t + s) = 0$ on $[0,\ l]$ for any $t \in \mathbb{R}$. Since $t \in \mathbb{R}$ is arbitrary, we must have $x(t) \equiv 0$ on $\mathbb{R}$, which is a contradiction to the fact that $x(t)$ is a non-trivial solution of (2.8). This completes the proof of this theorem. □

Corollary 2.3. *Let $a(t)$ be a scalar almost periodic function defined on $\mathbb{R}$. If each bounded solution of the equation*

$$x' = a(t)x \tag{2.11}$$

is Stepanov's almost periodic on $\mathbb{R}$, then

$$\sup_{t\in\mathbb{R}} \int_0^t a(s) ds < \infty \tag{2.12}$$

implies that for any real number $l > 0$,

$$\inf_{t\in\mathbb{R}} \Big(\sup_{s\in[t,t+l]} \int_0^s a(\tau) d\tau \Big) > -\infty. \tag{2.13}$$

Proof. Suppose that (2.12) holds. Then $x(t) = \exp\big(\int_0^t a(s)ds\big)$ is a non-trivial bounded solution of (2.11), so it is Stepanov's almost periodic. By Theorem 2.2, for any $l > 0$,

$$\inf_{t\in\mathbb{R}} \frac{1}{l} \int_t^{t+l} e^{\int_0^s a(\tau)d\tau} ds > 0.,. \tag{2.14}$$

Now we show that for any $l > 0$,

$$\inf_{t\in\mathbb{R}} \sup_{s\in[t,t+l]} \int_0^s a(\tau) d\tau > -\infty.$$

Otherwise, we can pick up a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$\sup_{s\in[t_n,t_n+l]} \int_0^s a(\tau) d\tau \le -n, \quad n = 1, 2, \dots.$$

So,

$$\int_0^s a(\tau) d\tau \le -n$$

for all $s \in [t_n, t_n + l], n = 1, 2, \dots$. Hence,

$$\frac{1}{l}\int_{t_n}^{t_n+l} e^{\int_0^s a(\tau)d\tau} ds \le \frac{1}{l}\int_{t_n}^{t_n+l} e^{-n} ds = e^{-n}, \quad n = 1, 2, \dots.$$

This implies that

$$\lim_{n\to\infty} \frac{1}{l}\int_{t_n}^{t_n+l} e^{\int_0^s a(\tau)d\tau} ds = 0$$

This contradicts (2.14) and the proof of this corollary is completed. $\square$

Corollary 2.4. *Let $a(t)$ be an almost periodic function defined on $\mathbb{R}$. If there exists a positive constant M such that $\int_0^t a(s)ds \le M$ and*

$$\inf_{t\in\mathbb{R}} \frac{1}{l}\int_{t}^{t+l} e^{\int_0^s a(\tau)ds} ds = 0$$

then the function $\exp\big(\int_0^t a(s)ds\big)$ *is not Stepanov's almost periodic on $\mathbb{R}$.*

Example. According to Corollary 2.4, we can construct many examples of equations whose solutions are bounded, but not Stepanov's almost periodic on $\mathbb{R}$. To construct such an example, let $n \ge 3$ and define

$$g_n(t) = \begin{cases} 0 & t \in [0, 1] \cup [2^{n-1} - 1, 2^{n-1}] \\ -n/(2^{n-1} - 1) & t \in [2, 2^{n-1} - 2] \\ \text{linear} & t \in [1, 2] \cup [2^{n-1} - 2, 2^{n-1} - 1]. \end{cases}$$

Now, extend $g_n(t)$ to be odd and periodic with period 2^n. Then, $g_n(t)$ satisfies

$$\int_0^t g_n(s)ds \le 0, \quad \text{for all } t \in \mathbb{R}; \tag{2.15}$$

$$\sup_{t\in\mathbb{R}} |g_n(t)| = \frac{n}{2^{n-1} - 1}; \tag{2.16}$$

$$\int_0^t g_n(s)ds = -n\frac{2^{n-1} - 3}{2^{n-1} - 1}, \quad \text{for all } t \in [2^{n-1} - 1, 2^{n-1}] \tag{2.17}$$

for each $n \in Z^+$. Since

$$\sum_{n=3}^{\infty} \sup_{t\in\mathbb{R}} |g_n(t)| = \sum_{n=3}^{\infty} \frac{n}{2^{n-1} - 1} < \infty\,,$$

the function

$$g(t) = \sum_{n=3}^{\infty} g_n(t)$$

is almost periodic on $\mathbb{R}$ and

$$\int_0^t g(s)ds = \sum_{n=3}^{\infty} \int_0^t g_n(s)ds \le 0, \quad t \in \mathbb{R}. \tag{2.18}$$

Now, let $l = 1$, $t_n = 2^{n-1} - 1$, then

$$\begin{aligned} \frac{1}{l}\int_{t_n}^{t_n+l} e^{\int_0^s g(\tau)d\tau} ds &= \int_{2^{n-1}-1}^{2^{n-1}} e^{\int_0^s g(\tau)d\tau} ds \\ &\le \int_{2^{n-1}-1}^{2^{n-1}} e^{\int_0^s g_n(\tau)d\tau} ds \\ &= \int_{2^{n-1}-1}^{2^{n-1}} e^{-n\frac{2^{n-1}-3}{2^{n-1}-1}} ds \\ &= e^{-n\frac{2^{n-1}-3}{2^{n-1}-1}} \end{aligned} \tag{2.19}$$

for each $n \in Z^+$. So,

$$\lim_{n\to\infty} \frac{1}{l}\int_{t_n}^{t_n+l} e^{\int_0^s g(\tau)d\tau} ds = 0.$$

This implies

$$\inf_{t\in\mathbb{R}} \frac{1}{l}\int_{t}^{t+l} e^{\int_0^s g(\tau)d\tau} ds = 0.$$

By Corollary 2.4, the function $\exp\big(\int_0^t g(s)ds\big)$ is not Stepanov,s almost periodic on $\mathbb{R}$. Therefore, all non-trivial solutions of equation

$$x' = g(t)x$$

are not Stepanov's almost periodic on $\mathbb{R}$, but they are all bounded on $\mathbb{R}$ because (2.18) holds.

Acknowledgement. The author thanks the referee for his/her critical review of the original manuscript and for the changes suggested; in particular, for the proof of Lemma 2.1. The author also wants to thank Professor Mingarelli, at Carleton University, for his help and suggestions.

References

[1] A. S. Besicovitch; *Almost Periodic Functions*, Dover, New York, 1954.

[2] C. C. Conley and R. K. Miller; *Asymptotic stability without uniform stability: almost periodic coefficients*, J. Deffrentail Equations, 1(1965) 333–336.

[3] A. M. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics, Vol. 377, Springer-Verlay, New York, 1974.

[4] Z. Hu and A. B. Mingarelli; *On a question in theory of almost periodic differential equations*, Proc. Amer. Math. Soc. Vol. 127 (1999), pp. 2665-2670.

[5] Angelo B. Mingarelli, F. Q. Pu , L. Zheng; *A counter-example in the theory of almost periodic differential equations*, Rocky Mountain J. of Math. 25(1995), 437–440.

School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, K1S 5B6, Canada

E-mail address: zshhu@yahoo.com