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INTEGRAL INEQUALITIES SIMILAR TO GRONWALL INEQUALITY

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Abstract. In the present paper, we establish some nonlinear integral inequalities for functions of one variable, with a further generalization functions with n independent variables. We apply our results to a system of nonlinear differential equations for functions of one variable and to the nonlinear hyperbolic partial integrodifferential equation in n-independent variables. These results extend the Gronwall type inequalities obtained by Pachpatte [6] and Oguntuase [5].

1. Introduction

Integral inequalities play a big role in the study of differential integral equation and partial differential equations. They were introduced for by Gronwall in 1919 [2], who gave their applications in the study of some problems concerning ordinary differential equation. One of the most useful inequalities with one variable of Gronwall type is stated as follows.

Lemma 1.1. *Let u, Ψ and g be real continuous functions defined in $[a, b]$, $g(t) \geq 0$ for $t \in [a, b]$. Suppose that on $[a, b]$ we have the inequality*

$$u(t) \leq \Psi(t) + \int_a^t g(s)u(s)ds. \tag{1.1}$$

Then

$$u(t) \leq \Psi(t) + \int_a^t g(s)\Psi(s) \exp\Big[\int_a^s g(\sigma)d\sigma\Big]ds. \tag{1.2}$$

Since that time, the theory of these inequalities knew a fast growth and a great number of monographs were devoted to this subject [1, 3, 4, 7]. The applications of the integral inequalities were developed in a remarkable way in the study of the existence, the uniqueness, the comparison, the stability and continuous dependence of the solution in respect to data. In the last few years, a series of generalizations of those inequalities appeared. Among these generalization, we can quote Pachpatte's work [6].

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In the present paper we establish some new nonlinear integral inequalities for functions of one variable, with a further generalization of these inequalities to function with n independent variables. These results extend the Gronwall type inequalities obtained by Pachpatte [6] and Oguntuase [5].

2. Mains Results

Our main results are given in the following theorems:

Theorem 2.1. *Let $u(t)$, $f(t)$ be nonnegative continuous functions in a real interval $I = [a, b]$. Suppose that $k(t, s)$ and its partial derivatives $k_t(t, s)$ exist and are nonnegative continuous functions for almost every $t, s \in I$. Let $\Phi(u(t))$ be real-valued, positive, continuous, strictly non-decreasing, subadditive, and submultiplicative function for $u(t) \geq 0$ and let $W(u(t))$ be real-valued, positive, continuous, and non-decreasing function defined for $t \in I$. Assume that $a(t)$ is a positive continuous function and nondecreasing for $t \in I$. If*

$$u(t) \leq a(t) + \int_a^t f(s)u(s)ds + \int_a^t f(s)W(\int_a^s k(s,\tau)\Phi(u(\tau))d\tau)ds, \tag{2.1}$$

for $a \leq \tau \leq s \leq t \leq b$, then for $a \leq t \leq t_1$,

$$\begin{aligned} u(t) \leq p(t)\Big\{a(t) + \int_a^t f(s)\Psi^{-1}\Big(\Psi(\zeta) \\ + \int_a^s k(s,\tau)\Phi(p(\tau))\Phi(\int_a^\tau f(\sigma)d\sigma)d\tau\Big)ds\Big\}, \end{aligned} \tag{2.2}$$

where

$$p(t) = 1 + \int_a^t f(s)\exp(\int_a^s f(\sigma)d\sigma)ds, \tag{2.3}$$

$$\zeta = \int_a^b k(b,s)\Phi(p(s)a(s))ds, \tag{2.4}$$

$$\Psi(x) = \int_{x_0}^x \frac{ds}{\Phi(W(s))}, \quad x \geq x_0 > 0. \tag{2.5}$$

Here Ψ^{-1} is the inverse of Ψ and $t_1$is chosen so that

$$\Psi(\zeta) + \int_a^s k(s,\tau)\Phi(p(\tau))\Phi(\int_a^\tau f(\sigma)d\sigma)d\tau \in \mathrm{Dom}(\Psi^{-1}).$$

Proof. Define a function $z(t)$ by

$$z(t) = a(t) + \int_a^t f(s)W\Big(\int_a^s k(s,\tau)\Phi(u(\tau))d\tau\Big)ds, \tag{2.6}$$

then (2.6) can be restated as

$$u(t) \leq z(t) + \int_a^t f(s)u(s)ds. \tag{2.7}$$

Clearly $z(t)$ is nonnegative and continuous in $t \in I$, using lemma 1.1 to (2.7), we get

$$u(t) \leq z(t) + \int_a^t f(s)z(s)\exp\Big(\int_a^s f(\sigma)d\sigma\Big)ds; \tag{2.8}$$

moreover if $z(t)$ is nondecreasing in $t \in I$, we obtain

$$u(t) \le z(t)p(t), \tag{2.9}$$

where $p(t)$ is defined by 2.3. From (2.6), we have

$$z(t) \le a(t) + \int_a^t f(s)W(v(s))ds, \tag{2.10}$$

where

$$v(t) = \int_a^t k(t,s)\Phi(u(s))ds. \tag{2.11}$$

From (2.9) we observe that

$$\begin{aligned}
v(t) &\le \int_a^t k(t,s)\Phi\Big[p(s)\Big(a(s) + \int_a^s f(\tau)W(v(\tau))d\tau\Big)\Big]ds \\
&\le \int_a^t k(t,s)\Phi(p(s)a(s))ds + \int_a^t k(t,s)\Phi\Big(p(s)\int_a^s f(\tau)W(v(\tau))d\tau\Big)ds \\
&\le \int_a^b k(b,s)\Phi(p(s)a(s))ds + \int_a^t k(t,s)\Phi\Big(p(s)\int_a^s f(\tau)d\tau\Big)\Phi(W(v(s)))ds \\
&\le \zeta + \int_a^t k(t,s)\Phi\Big(p(s)\int_a^s f(\tau)d\tau\Big)\Phi(W(v(s)))ds.
\end{aligned} \tag{2.12}$$

Where ζ is defined by (2.4).

Since Φ is a subadditive and a submultiplicative function, W and $v(t)$ are nondecreasing. Define $r(t)$ as the right side of (2.12), then $r(a) = \zeta$ and $v(t) \le r(t)$, $r(t)$ is positive nondecreasing in $t \in I$ and

$$\begin{aligned}
r'(t) &= k(t,t)\Phi\Big(p(t)\int_a^t f(\tau)d\tau\Big)\Phi(W(v(t))) \\
&\quad + \int_a^t k_t(t,s)\Phi(p(s)\int_a^s f(\tau)d\tau)\Phi(W(v(s)))ds, \\
&\le \Phi(W(r(t)))\Big[k(t,t)\Phi\Big(p(t)\int_a^t f(\tau)d\tau\Big) \\
&\quad + \int_a^t k_t(t,s)\Phi\Big(p(s)\int_a^s f(\tau)d\tau\Big)ds\Big],
\end{aligned} \tag{2.13}$$

dividing both sides of (2.13) by $\Phi(W(r(t)))$ we obtain

$$\frac{r'(t)}{\Phi(W(r(t)))} \le \Big[\int_a^t k(t,s)\Phi(p(s)\int_a^s f(\tau)d\tau)ds\Big]'. \tag{2.14}$$

Note that for

$$\Psi(x) = \int_{x_0}^x \frac{ds}{\Phi(W(s))}, \quad x \ge x_0 > 0,$$

it follows that

$$[\Psi(r(t))]' = \frac{r'(t)}{\Phi(W(r(t)))}. \tag{2.15}$$

From (2.15) and (2.14), we have

$$[\Psi(r(t))]' \le \Big[\int_a^t k(t,s)\Phi(p(s)\int_a^s f(\tau)d\tau)ds\Big]', \tag{2.16}$$

integrate (2.16) from a to t, leads to

$$\Psi(r(t)) \leq \Psi(\zeta) + \int_a^t k(t,s)\Phi(p(s)\int_a^s f(\tau)d\tau)ds,$$

then

$$r(t) \leq \Psi^{-1}\Big(\Psi(\zeta) + \int_a^t k(t,s)\Phi(p(s))\Phi(\int_a^s f(\tau)d\tau)ds\Big). \tag{2.17}$$

By (2.17), (2.12), (2.10) and (2.9) we have the desired result □

The preceding theorem is a generalization of the result obtained by Pachpatte in [6, Theorem 2.1].

Theorem 2.2. *Let $u(t), f(t), b(t)$, $h(t)$ be nonnegative continuous functions in a real interval $I = [a,b]$. Suppose that $h(t) \in C^1(I, \mathbb{R}^+)$ is nondecreasing. Let $\Phi(u(t)), W(u(t))$ and $a(t)$ be as defined in Theorem 2.1. If*

$$u(t) \leq a(t) + \int_a^t f(s)u(s)ds + \int_a^t f(s)h(s)W\Big(\int_a^s b(\tau)\Phi(u(\tau))d\tau\Big)ds,$$

for $a \leq \tau \leq s \leq t \leq b$, then for $a \leq t \leq t_2$,

$$\begin{aligned} u(t) \leq p(t)\Big\{a(t) + \int_a^t f(s)h(s)\Psi^{-1}\Big(\Psi(\vartheta) \\ + \int_a^s b(\tau)\Phi\Big(p(\tau)\int_a^\tau f(\sigma)h(\sigma)d\sigma\Big)d\tau\Big)ds\Big\}. \end{aligned}$$

Where $p(t)$ is defined by (2.3), Ψ is defined by (2.5) and

$$\vartheta = \int_a^b b(s)\Phi(p(s)a(s))ds,$$

the t_2 is chosen so that $\Psi(\vartheta) + \int_a^s b(\tau)\Phi(p(\tau)\int_a^\tau f(\sigma)h(\sigma)d\sigma)d\tau$ is in $\mathrm{Dom}(\Psi^{-1})$.

The proof of the above theorem follows similar arguments as the proof of Theorem 2.1; So we omit it.

The preceding theorem is a generalization of the result obtained by Oguntuase in [5, Theorem 2.3, 2.9].

In this section we use the following class of function. A function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to the class S if it satisfies the following conditions,

(1) $g(u)$ is positive, nondecreasing and continuous for $u \geq 0$ and
(2) $(1/v)g(u) \leq g(u/v)$, $u > 0$, $v \geq 1$.

Theorem 2.3. *Let $u(t)$, $f(t), a(t), k(t,s), \Phi$ and W be as defined in Theorem 2.1, let $g \in S$. If*

$$u(t) \leq a(t) + \int_a^t f(s)g(u(s))ds + \int_a^t f(s)W\Big(\int_a^s k(s,\tau)\Phi(u(\tau))d\tau\Big)ds, \tag{2.18}$$

for $a \leq \tau \leq s \leq t \leq b$, then for $a \leq t \leq t_3$,

$$\begin{aligned} u(t) \leq \overline{p}(t)\Big\{a(t) + \int_a^t f(s)\Psi^{-1}\Big(\Psi(\overline{\zeta}) \\ + \int_a^s k(s,\tau)\Phi(\overline{p}(\tau))\Phi(\int_a^\tau f(\sigma)d\sigma)d\tau\Big)ds\Big\}, \end{aligned} \tag{2.19}$$

where

$$\overline{p}(t) = \Omega^{-1}\Big(\Omega(1) + \int_a^t f(s)ds\Big), \tag{2.20}$$

$$\overline{\zeta} = \int_a^b k(b,s)\Phi(\overline{p}(s)a(s))ds, \tag{2.21}$$

$$\Omega(\delta) = \int_\varepsilon^\delta \frac{ds}{g(s)}, \quad \delta \geq \varepsilon > 0. \tag{2.22}$$

Here Ω^{-1} is the inverse function of Ω, and Ψ, Ψ^{-1} are defined in theorem 2.1, t_3 is chosen so that $\Omega(1) + \int_a^t f(s)ds$ is in the domain of Ω^{-1}, and

$$\Psi(\overline{\zeta}) + \int_a^s k(s,\tau)\Phi(\overline{p}(\tau))\Phi\big(\int_a^\tau f(\sigma)d\sigma\big)d\tau,$$

is in the domain of Ψ^{-1}.

Proof. Define the function

$$z(t) = a(t) + \int_a^t f(s)W\Big(\int_a^s k(s,\tau)\Phi(u(\tau))d\tau\Big)ds. \tag{2.23}$$

Then (2.18) can be restated as

$$u(t) \leq z(t) + \int_a^t f(s)g(u(s))ds. \tag{2.24}$$

When $z(x)$ is a positive, continuous, nondecreasing in $x \in I$ and $g \in S$, then it can be restated as

$$\frac{u(t)}{z(t)} \leq 1 + \int_a^t f(s)g(\frac{u(s)}{z(s)})ds. \tag{2.25}$$

The inequality (2.25) may be treated as one-dimensional Bihari-La Salle inequality (see [1]), which implies

$$u(t) \leq \overline{p}(t)z(t), \tag{2.26}$$

where $\overline{p}(t)$ is defined by (2.20). By (2.23) and (2.26) we get

$$u(t) \leq \overline{p}(t)\Big[a(t) + \int_a^t f(s)W(v(s))ds\Big],$$

where

$$v(s) = \int_a^s k(s,\tau)\Phi(u(\tau))d\tau.$$

Now, by following the argument as in the proof of Theorem 2.1, we obtain the desired inequality in (2.19). $\square$

Theorem 2.4. *Let $u(t)$, $f(t)$, $b(t)$, $h(t)$, $\Phi(u(t))$, $W(u(t))$ and $a(t)$ be as defined in Theorem 2.2, let $g \in S$. If*

$$u(t) \leq a(t) + \int_a^t f(s)g(u(s))ds + \int_a^t f(s)h(s)W(\int_a^s b(\tau)\Phi(u(\tau))d\tau)ds,$$

for $a \le \tau \le s \le t \le b$, then for $a \le t \le t_4$,

$$u(t) \le \overline{p}(t)\Big\{a(t) + \int_a^t f(s)h(s)\Psi^{-1}\Big(\Psi(\overline{\vartheta}) + \int_a^s b(\tau)\Phi(\overline{p}(\tau)\int_a^\tau f(\sigma)h(\sigma)d\sigma)d\tau\Big)ds\Big\}.$$

Here $\overline{p}(t)$ is defined by (2.20), Ψ is defined by (2.5) and

$$\overline{\vartheta} = \int_a^b b(s)\Phi(\overline{p}(s)a(s))ds,$$

the value t_4 is chosen so that $\Psi(\overline{\vartheta}) + \int_a^s b(\tau)\Phi(\overline{p}(\tau)\int_a^\tau f(\sigma)h(\sigma)d\sigma)d\tau \in \mathrm{Dom}(\Psi^{-1})$.

The proof of the above theorem follows similar arguments as in the proof of Theorem 2.3, we omit it.

3. Integral Inequalities in several variables

In what follows we denote by $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+ = [0, \infty)$. All the functions which appear in the inequalities are assumed to be real valued of n variables which are nonnegative and continuous. All integrals are assumed to exist on their domains of definitions.

Throughout this paper, we assume that $\mathbb{I} = [a; b]$ in any bounded open set in the dimensional Euclidean space $\mathbb{R}^n$ and that our integrals are on $\mathbb{R}^n (n \ge 1)$, where $a = (a_1, a_2, \dots, a_n)$, $b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n_+$. For $x = (x_1, x_2, \dots x_n)$, $t = (t_1, t_2, \dots t_n) \in \mathbb{I}$, we shall denote

$$\int_a^x = \int_{a_1}^{x_1}\int_{a_2}^{x_2}\dots\int_{a_n}^{x_n}\dots dt_n \dots dt_1 .$$

Furthermore, for $x, t \in \mathbb{R}^n$, we shall write $t \le x$ whenever $t_i \le x_i$, $i = 1, 2, \dots, n$ and $0 \le a \le x \le b$, for $x \in \mathbb{I}$, and $D = D_1 D_2 \dots D_n$, where $D_i = \frac{\partial}{\partial x_i}$ for $i = 1, 2, \dots, n$. Let $C(\mathbb{I}, \mathbb{R}_+)$ denote the class of continuous functions from $\mathbb{I}$ to $\mathbb{R}_+$.

The following theorem deals with n-independent variables versions of the inequalities established in Pachpatte [6, Theorem 2.3].

Theorem 3.1. *Let $u(x), f(x), a(x)$ be in $C(\mathbb{I}, \mathbb{R}_+)$ and let $K(x,t)$, $D_i k(x,t)$ be in $C(\mathbb{I} \times \mathbb{I}, \mathbb{R}_+)$ for all $i = 1, 2, \dots, n$, and let c be a nonnegative constant. (1) If*

$$u(x) \le c + \int_a^x f(s)\Big[u(s) + \int_a^s k(s,\tau)u(\tau)d\tau\Big]ds, \tag{3.1}$$

for $x \in \mathbb{I}$ and $a \le \tau \le s \le b$, then

$$u(x) \le c\Big[1 + \int_a^x f(t)\exp\Big(\int_a^t (f(s) + k(b,s))ds\Big)dt\Big] \tag{3.2}$$

(2) If

$$u(x) \le a(x) + \int_a^x f(s)\Big[u(s) + \int_a^s k(s,\tau)u(\tau)d\tau\Big]ds, \tag{3.3}$$

for $x \in \mathbb{I}$ and $a \le \tau \le s \le b$, then

$$u(x) \le a(x) + e(x)\Big[1 + \int_a^x f(t)\exp\Big(\int_a^t (f(s) + k(b,s))ds\Big)dt\Big], \tag{3.4}$$

where

$$e(x) = \int_a^x f(s)\Big[a(s) + \int_a^s k(s,\tau)a(\tau)d\tau\Big]ds. \tag{3.5}$$

Proof. (1) The inequality (3.1) implies the estimate

$$u(x) \le c + \int_a^x f(s)\Big[[u(s) + \int_a^s k(b,\tau)u(\tau)d\tau\Big]ds.$$

We define the function

$$z(x) = c + \int_a^x f(s)\Big[u(s) + \int_a^s k(b,\tau)u(\tau)d\tau\Big]ds.$$

Then $z(a_1, x_2, \dots, x_n) = c$, $u(x) \le z(x)$ and

$$\begin{aligned} Dz(x) &= f(x)\Big[u(x) + \int_a^x k(b,s)u(s)ds\Big], \\ &\le f(x)\Big[z(x) + \int_{x^0}^x k(b,s)z(s)ds\Big]. \end{aligned}$$

Define the function

$$v(x) = z(x) + \int_a^x k(b,s)z(s)ds,$$

then $z(a_1, x_2, \dots, x_n) = v(a_1, x_2, \dots, x_n) = c$, $Dz(x) \le f(x)v(x)$ and $z(x) \le v(x)$, and we have

$$Dv(x) = Dz(x) + k(b,x)z(x) \le (f(x) + k(b,x))v(x). \tag{3.6}$$

Clearly $v(x)$ is positive for all $x \in \mathbb{I}$, hence the inequality (3.6) implies the estimate

$$\frac{v(x)Dv(x)}{v^2(x)} \le f(x) + k(b,x);$$

that is

$$\frac{v(x)Dv(x)}{v^2(x)} \le f(x) + k(b,x) + \frac{(D_n v(x))(D_1 D_2 \dots D_{n-1} v(x))}{v^2(x)};$$

hence

$$D_n\Big(\frac{D_1 D_2 \dots D_{n-1} v(x)}{v(x)}\Big) \le f(x) + k(b,x).$$

Integrating with respect to x_n from a_n to x_n, we have

$$\frac{D_1 D_2 \dots D_{n-1} v(x)}{v(x)} \le \int_{a_n}^{x_n} [f(x_1, \dots, x_{n-1}, t_n) + k(b, x_1, \dots, x_{n-1}, t_n)]dt_n;$$

thus

$$\begin{aligned} \frac{v(x) D_1 D_2 \dots D_{n-1} v(x)}{v^2(x)} &\le \int_{a_n}^{x_n} [f(x_1, \dots, x_{n-1}, t_n) + k(b, x_1, \dots, x_{n-1}, t_n)]dt_n \\ &\quad + \frac{(D_{n-1} v(x))(D_1 D_2 \dots D_{n-2} v(x))}{v^2(x)}. \end{aligned}$$

That is,

$$D_{n-1}\big(\frac{D_1 D_2 \dots D_{n-2} v(x)}{v(x)}\big) \le \int_{a_n}^{x_n} [f(x_1, \dots, x_{n-1}, t_n) + k(b, x_1, \dots, x_{n-1}, t_n)]dt_n,$$

integrating with respect to x_{n-1} from a_{n-1} to x_{n-1}, we have

$$\begin{aligned}\frac{D_1 D_2 \dots D_{n-2} v(x)}{v(x)} \le \int_{a_{n-1}}^{x_{n-1}} \int_{a_n}^{x_n} \big[& f(x_1, \dots, x_{n-2}, t_{n-1}, t_n) \\ & + k(b, x_1, \dots, x_{n-2}, t_{n-1}, t_n) \big] dt_n dt_{n-1}.\end{aligned}$$

Continuing this process, we obtain

$$\frac{D_1 v(x)}{v(x)} \le \int_{a_2}^{x_2} \dots \int_{a_n}^{x_n} [f(x_1, t_2, t_3, \dots, t_n) + k(b, x_1, t_2, t_3, \dots, t_n)] dt_n \dots dt_2 .$$

Integrating with respect to x_1 from a_1 to x_1, we have

$$\log \frac{v(x)}{v(a_1, x_2, \dots, x_n)} \le \int_a^x [f(t) + k(b,t)] dt;$$

that is,

$$v(x) \le c \exp\Big(\int_a^x [f(t) + k(b,t)] dt \Big). \tag{3.7}$$

Substituting (3.7) into $Dz(x) \le f(x)v(x)$, we have

$$Dz(x) \le c f(x) \exp\Big(\int_a^x [f(t) + k(b,t)] dt \Big), \tag{3.8}$$

integrating (3.8) with respect to the x_n component from a_n to x_n, then with respect to the a_{n-1} to x_{n-1}, and continuing until finally a_1 to x_1, and noting that $z(a_1, x_2, \dots, x_n) = c$, we have

$$z(x) \le c\Big[1 + \int_a^x f(t) \exp\Big(\int_a^t [f(s) + k(b,s)] ds \Big) dt \Big].$$

This completes the proof of the first part.

(2) Define a function $z(x)$ by

$$z(x) = \int_a^x f(s) \Big[u(s) + \int_a^s k(s,\tau) u(\tau) d\tau \Big] ds. \tag{3.9}$$

Then from (3.3), $u(x) \le a(x) + z(x)$ and using this in (3.9), we get

$$\begin{aligned} z(x) &\le \int_a^x f(s) \Big[a(s) + z(s) + \int_a^s k(s,\tau)[a(\tau) + z(\tau)] d\tau \Big] ds, \\ &\le e(x) + \int_a^x f(s) \Big[z(s) + \int_a^s k(s,\tau) z(\tau) d\tau \Big] ds, \end{aligned} \tag{3.10}$$

where $e(x)$ is defined by (3.5). Clearly $e(x)$ is positive, continuous an nondecreasing for all $x \in \mathbb{I}$. From (3.10) it is easy to observe that

$$\frac{z(x)}{e(x)} \le 1 + \int_a^x f(s) \Big[\frac{z(s)}{e(s)} + \int_a^s k(s,\tau) \frac{z(\tau)}{e(\tau)} d\tau \Big] ds.$$

Now, by applying the inequality in part 1, we have

$$z(x) \le e(x) \Big[1 + \int_a^x f(t) \exp\Big(\int_a^t (f(s) + k(b,s)) ds \Big) dt \Big]. \tag{3.11}$$

The desired inequality in (3.4) follows from (3.11) and the fact that $u(x) \le a(x) + z(x)$. $\square$

The following theorem deals with n-independent variables versions of the inequalities established in Theorem 2.3. We need the inequalities in the following lemma (see [4]).

Lemma 3.2. *Let $u(x)$ and $b(x)$ be nonnegative continuous functions, defined for $x \in \mathbb{I}$, and let $g \in S$. Assume that $a(x)$ is positive, continuous function, nondecreasing in each of the variables $x \in \mathbb{I}$. Suppose that*

$$u(x) \le c + \int_a^x b(t)g(u(t))dt, \tag{3.12}$$

holds for all $x \in \mathbb{I}$ with $x \ge a$, then

$$u(x) \le G^{-1}\Big[G(c) + \int_a^x b(t)dt\Big], \tag{3.13}$$

for all $x \in \mathbb{I}$ such that $G(c) + \int_a^x b(t)dt \in \mathrm{Dom}(G^{-1})$, where $G(u) = \int_{u_0}^u dz/g(z)$, $u > 0 (u_0 > 0)$.

Theorem 3.3. *Let $u(x), f(x), a(x)$ and $k(x,t)$ be as defined in Theorem 3.1. Let $\Phi(u(x))$ be real-valued, positive, continuous, strictly non-decreasing, subadditive and submultiplicative function for $u(x) \ge 0$ and let $W(u(x))$ be real-valued, positive, continuous and non-decreasing function defined for $x \in \mathbb{I}$. Assume that $a(x)$ is positive continuous function and nondecreasing for $x \in \mathbb{I}$. If*

$$u(x) \le a(x) + \int_a^x f(t)g(u(t))dt + \int_a^x f(t)W\Big(\int_a^t k(t,s)\Phi(u(s))ds\Big)dt, \tag{3.14}$$

for $a \le s \le t \le x \le b$, then for $a \le x \le x^*$,

$$\begin{aligned} u(x) \le \beta(x)\Big\{a(x) + \int_a^x f(t)W\Big[\Psi^{-1}\Big(\Psi(\eta) \\ + \int_a^t k(b,s)\Phi[\beta(s)\int_a^s f(\tau)d\tau]ds\Big)\Big]dt\Big\}, \end{aligned} \tag{3.15}$$

where

$$\beta(x) = G^{-1}(G(1) + \int_a^x f(s)ds), \tag{3.16}$$

$$\eta = \int_a^b k(b,s)\Phi(\beta(s)a(s))ds, \tag{3.17}$$

$$G(u) = \int_{u_0}^u 1/g(z)\, dz, \quad u > 0 (u_0 > 0), \tag{3.18}$$

$$\Psi(x) = \int_{x_0}^x \frac{ds}{\Phi(W(s))}, \quad x \ge x_0 > 0. \tag{3.19}$$

Here G^{-1} is the inverse function of G, and Ψ is the inverse function of Ψ^{-1}, $x^*$is chosen so that $G(1) + \int_a^x f(s)ds$ is in the domain of G^{-1}, and

$$\Psi(\eta) + \int_a^t k(b,s)\Phi\big[\beta(s)\int_a^s f(\tau)d\tau\big]ds,$$

is in the domain of Ψ^{-1}.

Proof. Define the function

$$z(x) = a(x) + \int_a^x f(t)W\Big(\int_a^t k(t,s)\Phi(u(s))ds\Big)dt. \tag{3.20}$$

Then 3.14 can be restated as

$$u(x) \le z(x) + \int_a^x f(t)g(u(t))dt.$$

We have $z(x)$ is a positive, continuous, nondecreasing in $x \in \mathbb{I}$ and $g \in S$. Then the above inequality can be restated as

$$\frac{u(x)}{z(x)} \le 1 + \int_a^x f(t)g\big(\frac{u(t)}{z(t)}\big)dt. \tag{3.21}$$

By Lemma 3.2 we have

$$u(x) \le z(x)\beta(x), \tag{3.22}$$

where $\beta(x)$ is defined by (3.16). By (3.20), we have

$$z(x) = a(x) + \int_a^x f(t)W(v(t))dt, \tag{3.23}$$

where

$$v(x) = \int_a^x k(x,t)\Phi(u(t))dt. \tag{3.24}$$

By (3.24) and (3.22) , we observe that

$$\begin{aligned} v(x) &\le \int_a^x k(b,t)\Phi\Big[\beta(t)\Big(a(t) + \int_a^t f(s)W(v(s))ds\Big)\Big]dt \\ &\le \int_a^x k(b,s)\Phi(\beta(s)a(s))ds \\ &\quad + \int_a^t k(b,s)\Phi(\beta(s)\int_a^s f(\tau)W(v(\tau))d\tau)ds, \\ &\le \eta + \int_a^x k(b,s)\Phi[\beta(s)\int_a^s f(\tau)d\tau]\Phi(W(v(s)))ds. \end{aligned} \tag{3.25}$$

Where η is defined by (3.17). Since Φ is subadditive and submultiplicative function, W and $v(x)$ are nondecreasing for all $x \in \mathbb{I}$. Define $r(x)$ as the right side of (3.25), then $r(a_1, x_2, \dots, x_n) = \eta$ and $v(x) \le r(x)$, $r(x)$ is positive and nondecreasing in each of the variables $x_1, x_2, x_3, \dots x_n$. Hence

$$\frac{Dr(x)}{\Phi(W(r(x)))} \le k(b,x)\Phi[\beta(x)\int_a^x f(s)ds].$$

Since

$$D_n\big(\frac{D_1\dots D_{n-1}r(x)}{\Phi(W(r(x)))}\big) = \frac{Dr(x)}{\Phi(W(r(x)))} - \frac{D_n\Phi(W(r(x)))D_1\dots D_{n-1}r(x)}{\Phi^2(W(r(x)))},$$

the above inequality implies

$$D_n\big(\frac{D_1\dots D_{n-1}r(x)}{\Phi(W(r(x)))}\big) \le \frac{Dr(x)}{\Phi(W(r(x)))},$$

and

$$D_n\big(\frac{D_1\dots D_{n-1}r(x)}{\Phi(W(r(x)))}\big) \le k(b,x)\Phi[\theta(x)],$$

where $\theta(x) = \beta(x) \int_a^x f(s)ds$. Integrating with respect to x_n from a_n to x_n, we have

$$\frac{D_1 \dots D_{n-1} r(x)}{\Phi(W(r(x)))} \le \int_{a_n}^{x_n} k(b, x_1, x_2, \dots, x_{n-1}, s_n) \Phi[\theta(x_1, x_2, \dots, x_{n-1}, s_n)] ds_n.$$

Repeating this argument, we find that

$$\begin{aligned} &\frac{D_1 r(x)}{\Phi(W(r(x)))} \\ &\le \int_{a_2}^{x_2} \dots \int_{a_{n-1}}^{x_{n-1}} \int_{a_n}^{x_n} k(b, x_1, s_2, \dots, s_n) \Phi[\theta(x_1, s_2, \dots, s_n)] ds_n ds_{n-1} \dots ds_2. \end{aligned}$$

Integrating both sides of the above inequality with respect to x_1 from a_1 to x_1, we have

$$\Psi(r(x)) - \Psi(\eta) \le \int_a^x k(b, s) \Phi[\theta(s)] ds,$$

and

$$r(x) \le \Psi^{-1}\Big(\Psi(\eta) + \int_a^x k(b, s) \Phi\Big[\beta(s) \int_a^s f(\tau) d\tau\Big] ds\Big).$$

From this we obtain

$$v(x) \le r(x) \le \Psi^{-1}\Big(\Psi(\eta) + \int_a^x k(b, s) \Phi[\beta(s) \int_a^s f(\tau) d\tau] ds\Big). \tag{3.26}$$

By (3.22), (3.23) and (3.26) we obtain the desired inequality in (3.15). □

4. Some applications

In this section, our results are applied to the qualitative analysis of two application. The first is the system of nonlinear differential equations for one variable functions. The second is a nonlinear hyperbolic partial integrodifferential equation of n-independent variables.

First we consider the system of nonlinear differential equations

$$\frac{du}{dt} = F_1\Big(t, u(t), \int_{x_0}^t K_1(t, u(s)) ds\Big), \tag{4.1}$$

for $t \in I = [t_0, t_\infty] \subset R_+$, where $u \in C(I, \mathbb{R}^n)$, $F_1 \in C(I \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $K_1 \in C(I \times \mathbb{R}^n, \mathbb{R}^n)$.

In what follows, we shall assume that the Cauchy problem

$$\begin{gathered} \frac{du}{dt} = F_1(t, u(t), \int_{t_0}^t K_1(t, u(s)) ds), \quad x \in I, \\ u(t_0) = u_0 \in \mathbb{R}^n, \end{gathered} \tag{4.2}$$

has a unique solution, for every $t_0 \in I$ and $u_0 \in \mathbb{R}^n$. We shall denote this solution by $u(., t_0, u_0)$. The following theorem deals the estimate on the solution of the nonlinear Cauchy problem (4.2).

Theorem 4.1. *Assume that the functions F_1 and K_1 in (4.2) satisfy the conditions*

$$\|K_1(t, u)\| \le h(t) \Phi(\|u\|), \quad t \in I, \tag{4.3}$$

$$\|F_1(t, u, v)\| \le \|u\| + \|v\|, \quad u, v \in \mathbb{R}^n, \tag{4.4}$$

where h and Φ are as defined in Theorem 2.2. Then we have the estimate, for $t_0 \le t \le t_2$,

$$\|u(t,t_0,u_0)\| \le e^{t-t_0}\Big(\|u_0\| + \int_{t_0}^{t} h(s)E_1(s,\|u_0\|)ds\Big), \tag{4.5}$$

where

$$E_1(t,\|u_0\|) = \Psi^{-1}\Big(\Psi(\vartheta) + \int_{t_0}^{t} \Phi\big(e^{\tau-x_0}\int_{t_0}^{\tau} h(\sigma)d\sigma\big)d\tau\Big), \tag{4.6}$$

$$\Psi(t) = \int_a^t \frac{ds}{\Phi(s)}, \quad t \ge a > 0, \tag{4.7}$$

$$\vartheta = \int_{t_0}^{t_\infty} \|u_0\|\Phi(e^{s-t_0})ds, \tag{4.8}$$

and t_2 is chosen so that $\Psi(\vartheta) + \int_{x_0}^{s} \Phi(e^{\tau-t_0}\int_{t_0}^{\tau} h(\sigma)d\sigma)d\tau$ is in $\mathrm{Dom}(\Psi^{-1})$

Proof. Let $t_0 \in I$, $u_0 \in \mathbb{R}^n$ and $u(.,t_0,u_0)$ be the solution of the Cauchy problem (4.2). Then we have

$$u(t,t_0,u_0) = u_0 + \int_{t_0}^{t} F_1\Big(s, u(s,t_0,u_0), \int_{t_0}^{s} K_1(s,u(\tau,t_0,u_0))d\tau\Big)ds. \tag{4.9}$$

Using (4.3) and (4.4) in (4.9), we have

$$\begin{aligned}\|u(t,t_0,u_0)\| &\le \|u_0\| + \int_{t_0}^{t} f(s)\Big[\|u(s,t_0,u_0)\| + \int_{t_0}^{s} \|K_1(s,u(\tau,t_0,u_0))\|d\tau\Big]ds,\\ &\le \|u_0\| + \int_{t_0}^{t} f(s)\Big(\|u(s,t_0,u_0)\| + h(s)\int_{t_0}^{s} \Phi(\|u(\tau,t_0,u_0)\|)d\tau\Big)ds.\end{aligned} \tag{4.10}$$

Now, a suitable application of Theorem 2.2 with $a(t) = \|u_0\|$, $f(t) = b(t) = 1$ and $W(u) = u$ to (4.10) yields (4.5). $\square$

If, in addition, we assume that the function F_1 satisfies the general condition

$$\|F_1(t,u,v)\| \le f(t)(g(\|u\|) + W(\|v\|)), \tag{4.11}$$

where f , g and W are as defined in Theorem 2.4, we obtain an estimate for $u(.,t_0,u_0)$, and from any particular conditions of (4.11) and (4.3), we can get some useful results similar to Theorem 4.1.

Secondly, we shall demonstrate the usefulness of the inequality established in Theorem 3.3 by obtaining pointwise bounds on the solutions of a certain class of nonlinear equation in n-independent variables. We consider the nonlinear hyperbolic partial integrodifferential equation

$$\frac{\partial^n u(x)}{\partial x_1 \partial x_2 \dots \partial x_n} = F\Big(x, u(x), \int_{x^0}^{x} K(x,s,u(s))ds\Big) + G(x,u(x)) \tag{4.12}$$

for all $x \in \mathbb{I} = [x^0; x^\infty] \subset \mathbb{R}_+^n$, where $x = (x_1, x_2, \dots, x_n)$, $x^0 = (x_1^0, x_2^0, \dots, x_n^0)$, $x^\infty = (x_1^\infty, x_2^\infty, \dots, x_n^\infty)$ are in $\mathbb{R}_+^n$ and $u \in C(\mathbb{I},\mathbb{R})$, $F \in C(\mathbb{I}\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $K \in C(\mathbb{I}\times\mathbb{I}\times\mathbb{R},\mathbb{R})$ and $G \in C(\mathbb{I}\times\mathbb{R},\mathbb{R})$. With suitable boundary conditions, the solution of (4.12) is of the form

$$u(x) = l(x) + \int_{x^0}^{x} F\Big(s, u(s), \int_{x^0}^{s} K(s,t,u(t))dt\Big)ds + \int_{x^0}^{x} G(s,u(s))ds. \tag{4.13}$$

The following theorem gives the bound of the solution of (4.12).

Theorem 4.2. *Assume that the functions l, F, K and G in* (4.12) *satisfy the conditions*

$$|K(s,t,u(t))| \le k(s,t)\Phi(|u(t)|), \quad t,s \in \mathbb{I},\ u \in \mathbb{R}, \tag{4.14}$$

$$|F(t,u,v)| \le \frac{1}{2}|u| + |v|, \quad u,v \in \mathbb{R},\ t \in \mathbb{I}, \tag{4.15}$$

$$|G(s,u)| \le \frac{1}{2}|u|, \quad s \in \mathbb{I},\ u \in \mathbb{R}, \tag{4.16}$$

$$|l(x)| \le a(x), \quad x \in \mathbb{I}, \tag{4.17}$$

where a, f, k and Φ are as defined in Theorem 2.2, with $f(x) = b(x) + e(x)$ for all $x \in \mathbb{I}$ where $b, e \in C(\mathbb{I}, \mathbb{R}_+)$, then we have the estimate, for $x^0 \le x \le x^*$,

$$|u(x)| \le \exp\Big(\prod_{i=1}^{n}(x_i - x_i^0)\Big)\Big(a(x) + \int_a^x E(t)dt\Big). \tag{4.18}$$

Here

$$E(t) = \Psi^{-1}\Big(\Psi(\eta) + \int_a^t k(x^\infty, s)\Phi\big[\exp\big(\prod_{i=1}^{n}(s_i - x_i^0)\big)\int_a^s f(\tau)d\tau\big]ds\Big), \tag{4.19}$$

$$\eta = \int_{x^0}^{x^\infty} k(x^\infty, s)\Phi\big(a(s)\exp\big(\prod_{i=1}^{n}(s_i - x_i^0)\big)\big)ds, \tag{4.20}$$

$$\Psi(x) = \int_{x^0}^{x}\frac{ds}{\Phi(s)}, \quad x \ge x^0 > 0, \tag{4.21}$$

where $x^*$is chosen so that $\Psi(\eta) + \int_a^t k(x^\infty, s)\Phi[\exp(\prod_{i=1}^{n}(s_i - x_i^0))\int_a^s f(\tau)d\tau]ds$, is in the domain of Ψ^{-1}.

Proof. Using the conditions (4.14), (4.17) in (4.13), we have

$$\begin{aligned} |u(x)| &\le a(x) + \int_{x^0}^{x}|G(s,u(s))|ds + \int_{x^0}^{x} f(s)\big[|u(s)| + \int_{x^0}^{s}|K(s,t,u(t))|dt\big]ds, \\ &\le a(x) + \int_{x^0}^{x}\Big(|u(s)| + \int_{x^0}^{s} k(s,t)\Phi(|u(t)|)dt\Big)ds. \end{aligned} \tag{4.22}$$

Now, a suitable application of Theorem 3.3 with $f(s) = 1$, $g(u) = u$ and $W(u) = u$ to (4.22) yields (4.18). □

Remarks. If we assume that the functions F and G satisfy the general conditions

$$|F(t,u,v)| \le f(t)(g(|u|) + W(|v|)), \tag{4.23}$$

$$|G(t,u)| \le f(t)g(|u|), \quad \text{for } t \in \mathbb{I},\ u \in \mathbb{R}, \tag{4.24}$$

we can obtain an estimation of $u(x)$.

From the particular conditions of (4.14), (4.23) and (4.24), we can obtain some results similar to Theorem 3.3. To save space, we omit the details here.

Under some suitable conditions, the uniqueness and continuous dependence of the solutions of (4.1) and (4.12) can also be discussed using our results.

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References

[1] D. Bainov and P. Simeonov; *Integral Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, (1992).

[2] T. H. Gronwall; *Note on the derivatives with respect to a parameter of solutions of a system of differential equations*, Ann. of Math. ,0 (1919) 4, pp. 292-296.

[3] I. Gyori; *A Generalization of Bellman's Inequality for Stieltjes Integrals and a Uniqueness Theorem*, Studia Sci. Math. Hungar, 6 (1971), pp. 137-145.

[4] H. Khellaf; *On integral inequalities for functions of several independent variables*, Electron. J. Diff. Eqns. , Vol. 2003 (2003), No.123, pp. 1–12.

[5] J. A. Oguntuase; *On an inequality of Gronwall*, J. Ineq. Pure and Appl. Math., 2(1) (2001), No. 9.

[6] B. G. Pachpatte; *Bounds on Certain Integral Inequalities*, J. Ineq. Pure and Appl. Math., 3(3) (2002), Article No. 47.

[7] C. C. Yeh and M-H. Shim; *The Gronwall-Bellman Inequality in Several Variables*, J. Math. Anal. Appl., 86 (1982), No. 1, pp. 157-167.

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