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MULTIPLICITY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO FRACTIONAL FIGURE-KIRCHHOFF TYPE PROBLEMS WITH CRITICAL SOBOLEV-HARDY EXPONENT

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Abstract. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $0 \in \Omega$. For $0 < s < 1$, $1 \le r < q < p$, $0 \le \alpha < ps < N$ and a positive parameter λ, we consider the fractional (p,q)-Laplacian problems involving a critical Sobolev-Hardy exponent. This model comes from a nonlocal problem of Kirchhoff type

$$\begin{gathered}\big(a + b[u]_{s,p}^{(\theta-1)p}\big)(-\Delta)_p^s u + (-\Delta)_q^s u = \frac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} + \lambda f(x)\frac{|u|^{r-2}u}{|x|^c} \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\end{gathered}$$

where $a, b > 0$, $c < sr + N(1 - r/p)$, $\theta \in (1, p_s^*(\alpha)/p)$ and $p_s^*(\alpha)$ is critical Sobolev-Hardy exponent. For a given suitable $f(x)$, we prove that there are least two nontrivial solutions for small λ, by way of the mountain pass theorem and Ekeland's variational principle. Furthermore, we prove that these two solutions converge to two solutions of the limiting problem as $a \to 0^+$. For the limiting problem, we show the existence of infinitely many solutions, and the sequence tends to zero when λ belongs to a suitable range.

1. Introduction

Let $0 < s < 1$, $q < p < \frac{N}{s}$ and $B_\delta(x) = \{y \in \mathbb{R}^N : |x - y| < \delta\}$. The fractional t-Laplacian $(-\Delta)_t^s$ with $t \in \{p, q\}$ is defined (up to normalization factors) for any $x \in \mathbb{R}^N$ with

$$(-\Delta)_t^s \varphi = 2 \lim_{\delta \to 0^+} \int_{\mathbb{R}^N \setminus B_\delta(x)} \frac{|\varphi(x) - \varphi(y)|^{t-2}\big(\varphi(x) - \varphi(y)\big)}{|x - y|^{N+sp}} dy \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N).$$

For further details on the fractional p-Laplacian, we can refer to [17, 22] and the references therein. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and $0 \in \Omega$. In this paper, we prove the existence of multiple solutions for Kirchhoff type problem of fractional (p,q)-Laplacian, with $0 \le \alpha < sp$ and λ a positive parameter,

$$\begin{gathered}\big(a + b[u]_{s,p}^{(\theta-1)p}\big)(-\Delta)_p^s u + (-\Delta)_q^s u = \frac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} + \lambda f(x)\frac{|u|^{r-2}u}{|x|^c} \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\end{gathered} \tag{1.1}$$

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where $a, b > 0$, $r \in [1, q)$ is a constants, $\theta \in (1, p_s^*(\alpha)/p)$ with $p_s^*(\alpha) = \frac{p(N-\alpha)}{N-sp} \leq p_s^*(0) = p_s^*$ is the so-called critical Hardy-Sobolev exponent.

Nonlocal fractional operators arise in a quite natural way in contexts, such as optimization, continuum mechanics, phase transition phenomena, and game theory; see [4, 5, 13, 17] and the references therein. As we know, for the classical setting of $s = 1$, problem (1.1) reduces to a (p, q)-Laplacian elliptic problem of the form

$$
\begin{gathered}
-\Delta_p u - \Delta_q u = g(x, u) \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where several and interesting results have been obtained by many authors [6, 7, 24, 25, 36]. For $s = 1$ and $p = q = 2$, He and Zou [21] proved the existence of infinitely many solutions to a singular elliptic problem involving critical Hardy-Sobolev exponents. Subsequently, such a result has been extended to that of quasilinear equations in [26]. For the setting of fractional p-Laplacian with $p = q$, Fiscella and Mirzaee [20] established the existence of infinitely many solutions to the problem

$$
\begin{gathered}
(-\Delta)_p^s u - \mu \frac{|u|^{p-2}u}{|x|^{ps}} = \lambda \frac{|u|^{q-2}u}{|x|^a} + \frac{|u|^{p_s^*(b)-2}u}{|x|^b} \quad \text{in } \Omega, \\
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\end{gathered}
$$

In particular, we would like to mention that Ambrosio and Isernia [3] also obtained the existence of infinitely many solutions to the fractional (p, q)-Laplacian problem involving critical Hardy-Sobolev exponents. To this end, the main point in the study of these problems is due to the lack of compactness caused by the presence of the critical Hardy-Sobolev exponent.

On the other hand, great interest recently has been devoted to Kichhoff type equations in the past decades. For example, Xie and Chen [33] presented a multiplicity result on the Kirchhoff-type problems in the bounded domain by using the Nehari manifold, fibering maps and Ljusternik-Schnirelmann category. Xiang et al.[32] recently generalized the above fractional p-Laplacian analysis with the subscritical growth to the Kichhoff type problem

$$
\Big(a + b\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{\theta-1}\Big)(-\Delta)_p^s u = |u|^{p_s^*(\alpha)} u + \lambda f(x) \quad \text{in } \mathbb{R}^N,
$$

and they proved the existence of at least two different solutions to the above problem by way of a combination of mountain pass lemma and Ekeland variational principle. It is a well-known fact that the Kirchhoff equation is related to the following stationary analogue of equation

$$
\rho \frac{\partial^2 u}{\partial t^2} - \Big(\big(\frac{p_0}{h} + \frac{E}{2L}\big) \int_0^L \big|\frac{\partial u}{\partial x}\big|^2 dx\Big) \frac{\partial^2 u}{\partial x^2} = 0,
$$

where ρ, p_0, h, E, L are the constants which represent some physical meanings, respectively. This is an extension of the classical D'Alembert wave equation by considering the effect of changes in the length of strings during the vibrations. The Kichhoff equation received much attention due to Lions' seminal work [27] where he proposed an abstract framework to this kind of problems, see also for example [1, 11] and the references therein.

As a natural extension of the above papers, we are mainly interested in searching multiplicity of solutions to Problem (1.1). Our main point is here a combination of fractional double-phase problems of (p,q)-Kichhoff problems and critical Sobolev-Hardy exponents. To the best of our knowledge, there is only few papers deal with fractional (p,q)-Kichhoff type problems with critical Sobolev-Hardy exponents and Hardy term. Our main aim is in an effort to handle the multiplicity of solutions to Problem (1.1) by comparison with the recent paper [3] regarding the existence of solutions. Inspired by the papers in [34, 35], we additionally prefer to study an asymptotic behavior of solutions to Problem (1.1). More precisely, we are to show that there exists a sequence of many arbitrarily small solutions converging to zero for the limit problem of (1.1) by using a new version of the symmetric mountain-pass lemma due to Kajikiya [23].

Before stating our main results, let us recall some related notations and useful facts. For $0 < s < 1$, $1 < p < \infty$, we first recall some basic conclusions involved in the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$, for more details also see [8]. For $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function, we set

$$[u]_{s,p} = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}.$$

Then the fractional Sobolev space is

$$W^{s,p}(\Omega) := \Big\{u \in L^p(\Omega) : u \text{ is a measurable function and } [u]_{s,p} < \infty\Big\}$$

with the norm

$$\|u\|_{s,p} = \Big([u]_{s,p}^p + |u|_p^p\Big)^{1/p} \quad \text{with } |u|_p := \Big(\int\limits_{\mathbb{R}^N} |u|^p dx\Big)^{1/p}.$$

Note that the fractional Sobolev space $\mathbb{X} := W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\Omega) | u = 0, x \in \mathbb{R}^N \setminus \Omega\}$ is equipped with the norm $\|\cdot\| = [\cdot]_{s,p}$, which is a uniformly convex Banach space. As mentioned in Section 2 below, we know that $W_0^{s,p}(\Omega) \subset W_0^{s,q}(\Omega)$ for $q \le p$, which allows us to consider the problem (1.1) easily in $\mathbb{X}$. We are now to give the definition of weak solution to (1.1).

Definition 1.1. We say that $u \in \mathbb{X}$ is a weak solution of (1.1), if u satisfies

$$\big(a + b\|u\|^{(\theta-1)p}\big)\langle u, v\rangle_{s,p} + \langle u, v\rangle_{s,q} = \langle u, v\rangle_{\mathbb{H}_\alpha} + \lambda \int_\Omega f(x)\frac{|u|^{r-2}uv}{|x|^c}dx,$$

for all $v \in \mathbb{X}$, where

$$\langle u, v\rangle_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+sp}}\,dx\,dy,$$

$$\langle u, v\rangle_{s,q} = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{q-2}\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+sq}}\,dx\,dy,$$

$$\langle u, v\rangle_{\mathbb{H}_\alpha} = \int_\Omega \frac{|u(x)|^{p_s^*(\alpha)-2}u(x)v(x)}{|x|^\alpha}dx.$$

The energy functional $I : \mathbb{X} \to R$ associated with problem (1.1) is

$$I(u) = \frac{a}{p}\|u\|^p + \frac{b}{\theta p}\|u\|^{\theta p} + \frac{1}{q}[u]_{s,q}^q - \frac{1}{p_s^*(\alpha)}\int_\Omega \frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha}dx - \frac{\lambda}{r}\int_\Omega f(x)\frac{|u|^r}{|x|^c}dx.$$

Let us now make a necessary assumption on the function $f(x)$,

(A1) $f \in L^\infty(\Omega)$, and there are two positive constants ω_1 and ω_2 such that $0 < \omega_1 \le f(x) \le \omega_2 < +\infty, \forall x \in \Omega$.

It is clear that we can employ the argument used in [31] to prove that $I(u)$ is well-defined and of the class $C^1(\mathbb{X}, R)$. Moreover, we see that any solution of the problem (1.1) is just a critical point of $I(u)$. Therefore, we are now in a position to state our first main results as follows.

Theorem 1.2. *Assume that $f(x)$ satisfies* (A1). *Then there exists a constant $\lambda^* > 0$ such that problem* (1.1) *has at least two nontrivial solutions u^1 and u^2 satisfying*

$$I(u^2) < 0 < I(u^1), \quad \forall \lambda \in (0, \lambda^*).$$

To show the existence of at least two critical points of the energy functional. We use the mountain pass theorem (cf. [2]) to prove the existence of solution u^1 with $I(u^1) > 0$, and employ Ekeland variational principle (cf.[18]) to show the second solution u^2 with $I(u^2) < 0$. Indeed, the techniques for finding the solutions are partially borrowed from Cao, Li and Zhou's work in [10]. Here, a key point of proving Theorem 1.2 mainly stems from the critical nonlocal terms, where the $(PS)_c$ condition is verified by the concentration-compactness lemma developed by Fiscella [19] and Mosconi [28].

Furthermore, an asymptotic behavior of the solutions of Problem (1.1) obtained by Theorem 1.2 is stated as follows.

Theorem 1.3. *Let $f(x)$ satisfy* (A1). *For $\lambda \in (0, \lambda^*)$ and fixed $b > 0$, if u_a^1 and u_a^2 are two solutions of* (1.1) *obtained in Theorem 1.2. Then $u_a^1 \to u^1$ and $u_a^2 \to u^2$ in $\mathbb{X}$ as $a \to 0^+$, where $u^1 \neq u^2$, respectively, are two nontrivial solutions of the problem*

$$\begin{gathered} b[u]_{s,p}^{(\theta-1)p}(-\Delta)_p^s u + (-\Delta)_q^s u = \frac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} + \lambda f(x)\frac{|u|^{r-2}u}{|x|^c} \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{gathered} \tag{1.2}$$

Our approach of proving asymptotic behavior of the solutions for problem (1.1) comes from the idea of the papers [30, 35]. By analyzing the convergence property of u_a^1 and u_a^2 as $a \to 0^+$, we derive Theorem 1.3. Finally, we state the existence of infinitely many solutions of the problem (1.2).

Theorem 1.4. *Let $f(x)$ satisfy* (A1). *Then there exists a constant $\Lambda > 0$ such that* (1.2) *has infinitely many solutions for any $\lambda \in (0, \Lambda)$.*

The idea to prove Theorem 1.4 is based on this argument developed by He and Zou in [21], where the authors proved the existence of infinitely many solutions by combining a variant of the fractional concentration-compactness lemma (cf. [19, 28]) and the symmetric mountain pass lemma (cf. [23]). Additionally, it is necessary to introduce a truncated functional that allows us to apply the symmetric mountain pass lemma in [23]. As its application of the above consequence, we know that the critical points of the corresponding truncated functional are just the solutions of the original problem (1.2). Finally, it is unavoidable that the presence of fractional (p, q)-Laplacian operators makes our analysis more complicated so that we employ a more delicate technique above to adapt our setting.

The rest of this paper is organized as follows. In Section 2, the variational framework and some preliminaries are recalled. We devote Section 3 to show two distinct nontrivial weak solutions for problem (1.1) by using the mountain pass theorem and Ekeland variational principle. In Section 4, the concentration of the weak solutions is considered. Finally, we focus on the existence of infinitely many solutions of the problem (1.2) based on the symmetric mountain pass theorem in Section 5.

2. Preliminaries

We devote this section to state some related notation and useful facts. Let us begin with recalling a few of elementary embedding inequalities.

Lemma 2.1 ([3]). *For $q \le p$, the embedding $W_0^{s,p}(\Omega) \hookrightarrow W_0^{s,q}(\Omega)$ is continuous, i.e., there exists a positive constant C_q such that*

$$[u]_{s,q} \le C_q [u]_{s,p} \quad \text{for any } u \in W_0^{s,p}(\Omega).$$

Lemma 2.2 (Hardy-Sobolev inequality, [12, 19]). *For $0 \le \alpha < ps$, there exists a positive constant C_α possibly depending only on N, p, s and α such that*

$$\Big(\int_\Omega |u(x)|^{p_s^*(\alpha)} \frac{dx}{|x|^\alpha}\Big)^{1/p_s^*(\alpha)} \le C_\alpha \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy\Big)^{1/p} \tag{2.1}$$

for every $u \in \mathbb{X}$.

Consequently, this fractional Hardy-Sobolev embedding relation $\mathbb{X} \hookrightarrow L^{p_s^*(\alpha)}(\Omega, |x|^{-\alpha})$ is continuous, but not compact. Further, the best Hardy-Sobolev constant H_α is given by

$$H_\alpha = \inf_{u \in W_0^{s,p}(\Omega)\setminus\{0\}} \frac{[u]_{s,p}^p}{\|u\|_{\mathbb{H}_\alpha}^p} \quad \text{with } \|u\|_{\mathbb{H}_\alpha} := \Big(\int_\Omega \frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha} dx\Big)^{\frac{1}{p_s^*(\alpha)}}.$$

We remark that the number H_α is strictly positive, and it coincides with the best fractional Sobolev constant for $\alpha = 0$. The following embedding results has been proved in [12, 19].

Lemma 2.3. *For $0 \le \alpha < ps$, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, and $0 \in \Omega$. Then for any $1 \le r < \frac{p(N-\alpha)}{N-ps}$ and $\mu < sr + N(1-\frac{r}{p})$, there exists a constant $C_{r,c} = C(N,s,\alpha,r,c) > 0$ such that*

$$\int_\Omega \frac{|u|^r}{|x|^\mu}\,dx \le C_{r,c} \|u\|_{\mathbb{H}_\alpha}^r$$

for any $u \in \mathbb{X}$. Moreover, the embedding $\mathbb{X} \hookrightarrow L^r(\Omega, |x|^{-\mu})$ is compact.

In what follows, let us introduce the Brézis-Lieb type Lemma (cf. [3, Lemma 2.1]). We briefly prove it by a usual way due to the lack for the fractional Sobolev version.

Lemma 2.4. *If $\{u_n\}_{n\in N}$ is bounded in $W_0^{s,p}(\Omega)$, then, up to a subsequence, there exists a function u in $W^{s,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W^{s,p}(\Omega)$ with*

$$[u_n - u]_{s,p}^p = [u_n]_{s,p}^p - [u]_{s,p}^p + o(1), \tag{2.2}$$

$$\|u_n - u\|_{\mathbb{H}_\alpha}^{p_s^*(\alpha)} = \|u_n\|_{\mathbb{H}_\alpha}^{p_s^*(\alpha)} - \|u\|_{\mathbb{H}_\alpha}^{p_s^*(\alpha)} + o(1) \tag{2.3}$$

Proof. Thanks to the Brézis-Lieb Lemma [9], we see that if $\{g_n\}_{n\in N} \subset L^p(\mathbb{R}^N)$ for $p \in (1,\infty)$ is a bounded sequence such that $g_n \to g$ a.e. in $\mathbb{R}^N$, then we have

$$|g_n - g|_{L^p(\mathbb{R}^N)}^p = |g_n|_{L^p(\mathbb{R}^N)}^p - |g|_{L^p(\mathbb{R}^N)}^p + o_n(1).$$

By taking

$$g_n = \frac{u_n(x) - u_n(y)}{|x-y|^{\frac{N+sp}{p}}} \quad \text{and} \quad g = \frac{u(x)-u(y)}{|x-y|^{\frac{N+sp}{p}}},$$

we find that

$$[u_n - u]_{s,p}^p = [u_n]_{s,p}^p - [u]_{s,p}^p + o(1),$$

which leads to the desired result (2.2). Similarly, we can obtain formula (2.3). $\square$

Next, we recall the concentration-compactness principle for the version of fractional p-Laplacian. The following definition can be found in [31].

Definition 2.5. Let $\mathcal{M}(\mathbb{R}^N)$ denote the finite nonnegative Borel measure space in $\mathbb{R}^N$. For $\mu \in \mathcal{M}(\mathbb{R}^N)$ with $\mu(\mathbb{R}^N) = \|\mu\|_0$, we say that $\mu_n \rightharpoonup \mu$ weakly $*$ in $\mathcal{M}(\mathbb{R}^N)$, if $(\mu_n, \eta) \to (\mu, \eta)$ holds for all $\eta \in C_0(\mathbb{R}^N)$ as $n \to \infty$.

Let us recall the following fractional concentration-compactness lemma, see [19, 28].

Lemma 2.6. *For $0 \le \alpha < sp$, let $\{u_n\}_{n\in N} \subset D^{s,p}(\mathbb{R}^N)$ be a bounded sequence satisfying*

$$\begin{gathered}
u_n \rightharpoonup u \in D^{s,p}(\mathbb{R}^N);\\
\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}} dy \rightharpoonup \mu \quad \textit{weakly* in } \mathcal{M}(\mathbb{R}^N);\\
|u_n|^{p_s^*(\alpha)} |x|^{-\alpha} \rightharpoonup \nu \quad \textit{weakly* in } \mathcal{M}(\mathbb{R}^N).
\end{gathered}$$

Then there exist a countable sequence of points $\{x_j\}_{j\in J} \subset \mathbb{R}^N$, the families of positive numbers $\{\mu_j\}_{j\in J}$ and $\{\nu_j\}_{j\in J}$ such that

$$\nu = \frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha} + \sum_{j\in J} \nu_j \delta_{x_j}, \quad \mu \ge \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dy + \sum_{j \in J} \mu_j \delta_{x_j}.$$

Moreover,

$$\mu_j \ge H_\alpha \nu_j^{p/p_s^*(\alpha)} \quad \textit{for all } j \in J,$$

where δ_{x_j} is the Dirac mass centered at x_j.

Finally, the following proposition, which can be found in [32], is useful to our main proofs.

Proposition 2.7. *Assume that $\{u_n\} \subset D^{s,p}(\mathbb{R}^N)$ is the sequence given by Lemma 2.6. Let $x_0 \in \mathbb{R}^N$ be fixed point, and ϕ be a smooth cut-off function such that $0 \le \phi \le 1, \phi \equiv 0$ for $x \in B_2^c(0)$, $\phi \equiv 1$ for $x \in B_1(0)$ and $|\nabla \phi| \le 2$. Then for any $\varepsilon > 0$, we have*

$$\lim_{\varepsilon \to 0} \limsup_{n\to\infty} \Big(\iint_{\mathbb{R}^{2N}} \frac{\big|\big(\phi_{\varepsilon,j}(x) - \phi_{\varepsilon,j}(y)\big) u_n(x)\big|^p}{|x-y|^{N+ps}}\, dx\, dy\Big)^{1/p} = 0,$$

where $\phi_{\varepsilon,j}(x) = \phi(\frac{x-x_j}{\varepsilon})$ for any $x \in \mathbb{R}^N$.

3. Proof of Theorem 1.2

To show the existence of solutions for (1.1), let us recall the following general mountain pass theorem (cf. [2]), which allows us to find a $(PS)_c$ sequence.

Theorem 3.1. *Let E be a real Banach space, and $J \in C^1(E,R)$ with $J(0)=0$. Suppose that*

(i) *there exists $\rho,\delta>0$ such that $J(u)\ge\delta$ for $u\in E$ with $\|u\|_E=\rho$;*
(ii) *there exists $e\in E$ satisfying $\|u\|_E>\rho$ such that $J(e)<0$.*

Then, for $\Gamma=\{\gamma\in C^1([0,1];E):\gamma(0)=0,\gamma(1)=e\}$ we have

$$c=\inf_{\gamma\in\Gamma}\max_{0\le t\le 1} J\big(\gamma(t)\big)\ge\delta,$$

and there exists a $(PS)_c$ sequence $\{u_n\}_n\subset E$.

Before employing the mountain pass theorem to prove Theorem 1.2, we first verify that the functional I possesses the mountain pass geometry (i) and (ii).

Lemma 3.2. *Let $f(\cdot)$ satisfy Condition (A1). Then there exist $\lambda_0>0$ and two positive constants δ_λ and ρ such that $I(u)\ge\delta_\lambda>0$ (independent of a), for any $u\in\mathbb{X}$ with $\|u\|=\rho$ and $\lambda\in(0,\lambda_0)$.*

Proof. By (A1) and Lemma 2.3, for all $u\in\mathbb{X}$ we have

$$\int_\Omega f(x)|x|^{-c}|u|^r dx\le\omega_2 C_{r,c}\Big(\int_\Omega\frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha}dx\Big)^{r/p_s^*(\alpha)}. \tag{3.1}$$

Therefore,

$$I(u)\ge\frac{b}{\theta p}\|u\|^{\theta p}-\frac{1}{p_s^*(\alpha)}\int_\Omega\frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha}dx-\lambda\frac{\omega_2C_{r,c}}{r}\Big(\int_\Omega\frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha}dx\Big)^{r/p_s^*(\alpha)}.$$

It follows from the definition of H_α that

$$\begin{aligned} I(u)&\ge\frac{b}{\theta p}\|u\|^{\theta p}-\frac{1}{p_s^*(\alpha)}H_\alpha^{-p_s^*(\alpha)/p}[u]_{s,p}^{p_s^*(\alpha)}-\lambda\frac{\omega_2C_{r,c}}{r}H_\alpha^{-r/p}[u]_{s,p}^r\\ &\ge\Big(\frac{b}{\theta p}\|u\|^{\theta p-r}-\frac{1}{p_s^*(\alpha)}H_\alpha^{-p_s^*(\alpha)/p}\|u\|^{p_s^*(\alpha)-r}-\lambda\frac{\omega_2C_{r,c}}{r}H_\alpha^{-r/p}\Big)\|u\|^r.\end{aligned}$$

Let us define

$$g(t):=\frac{b}{\theta p}t^{\theta p-r}-\frac{1}{p_s^*(\alpha)}H_\alpha^{-p_s^*(\alpha)/p}t^{p_s^*(\alpha)-r}-\lambda\frac{\omega_2C_{r,c}}{r}H_\alpha^{-r/p}\quad\text{for all } t\ge0.$$

It is easy to check that for $t=t^*=\Big(\frac{bp_s^*(\alpha)(\theta p-r)}{\theta pH_\alpha^{-p_s^*(\alpha)/p}(p_s^*(\alpha)-r)}\Big)^{\frac{1}{p_s^*(\alpha)-\theta p}}$ one has

$$\max_{t\ge0}g(t)=\frac{b\big(p_s^*(\alpha)-\theta p\big)}{\theta p\big(p_s^*(\alpha)-r\big)}\Big(\frac{bp_s^*(\alpha)(\theta p-r)}{\theta pH_\alpha^{-p_s^*(\alpha)/p}\big(p_s^*(\alpha)-r\big)}\Big)^{\frac{\theta p-r}{p_s^*(\alpha)-\theta p}}-\lambda\frac{\omega_2C_{r,c}}{r}H_\alpha^{-r/p}>0,$$

provided that

$$0<\lambda<\lambda_0=\frac{bH_\alpha^{r/p}r\big(p_s^*(\alpha)-\theta p\big)}{\omega_2C_{r,c}\theta p\big(p_s^*(\alpha)-r\big)}\Big(\frac{bp_s^*(\alpha)(\theta p-r)}{\theta pH_\alpha^{-p_s^*(\alpha)/p}\big(p_s^*(\alpha)-r\big)}\Big)^{\frac{\theta p-r}{p_s^*(\alpha)-\theta p}}.$$

Then the conclusion follows only by letting $\rho=t^*>0$ and $\delta_\lambda=g(\rho)\rho^r>0$. The proof is complete. $\square$

Lemma 3.3. *Let $f(\cdot)$ satisfy* (A1). *Then there exists $a_* > 0$ such that for each $a \in (0, a_*)$, we have $I(e) < 0$ for some $e \in \mathbb{X}$ with $\|e\| > \rho$, where $\rho > 0$ is shown as in Lemma 3.2.*

Proof. Firstly, we notice that $f(x) > 0$ for a.e. $x \in \Omega$ due to Condition (A1). Let us choose a function $u_0 \in \mathbb{X}$ such that

$$\|u_0\| = 1 \quad \text{and} \quad \frac{1}{p_s^*(\alpha)} \int_\Omega \frac{|u_0|^{p_s^*(\alpha)}}{|x|^\alpha} dx > 0.$$

Then

$$I(tu_0) \le \frac{a}{p} t^p \|u_0\|^p + \frac{b}{\theta p} t^{\theta p} \|u_0\|^{\theta p} + \frac{1}{q} t^q [u_0]_{s,q}^q - \frac{1}{p_s^*(\alpha)} t^{p_s^*(\alpha)} \int_\Omega \frac{|u_0|^{p_s^*(\alpha)}}{|x|^\alpha} dx.$$

By considering $q < p < \theta p < p_s^*(\alpha)$ we see that there exists $t \ge 1$ large enough that $\|tu_0\| > \rho$ and $I(tu_0) < 0$. The proof is proved by letting $e = tu_0$. $\square$

With Lemmas 3.2–3.3 and Theorem 3.1 in hand, the $(PS)_c$ sequence of the functional $I(u)$ at the level

$$c := \inf_{\gamma \in \Gamma} \max_{0 \le t \le 1} I\big(\gamma(t)\big) \ge \delta_\lambda > 0$$

can be constructed, where the set of paths is defined by $\Gamma = \{\gamma \in C^1([0,1]; \mathbb{X}) : \gamma(0) = 0, \gamma(1) = e\}$. In other words, there exists a sequence $\{u_n\} \subset \mathbb{X}$ such that

$$I(u_n) \to c \quad I'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Definition 3.4. A sequence $\{u_n\}_n \subset \mathbb{X}$ is called a $(PS)_c$ sequence, if $I(u_n) \to c$ and $I'(u_n) \to 0$. We say I satisfies $(PS)_c$ condition if any $(PS)_c$ sequence admits a converging subsequence.

Lemma 3.5. *Let $f(\cdot)$ satisfy* (A1). *If $\{u_n\}_n \subset \mathbb{X}$ is a (PS) sequence, then there exists $C > 0$ (independent of a and n) such that $\|u_n\| \le C$ for every $a \in (0, a_*)$.*

Proof. Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathbb{X}$ be a Palais-Smale sequence of I, that is to say,

$$I(u_n) = c + o(1) \quad \text{and} \quad \langle I'(u_n), u_n \rangle = o(1)\|u_n\| \quad \text{as } n \to \infty. \tag{3.2}$$

Taking into account (A1), $1 \le r < q < p$ and $\theta \in (1, p_s^*(\alpha)/p)$, we obtain

$$\begin{aligned}
& c + o(1)\|u_n\| \\
& = I(u_n) - \frac{1}{p_s^*(\alpha)} \langle I'(u_n), u_n \rangle \\
& = \Big(\frac{a}{p} - \frac{a}{p_s^*(\alpha)}\Big)\|u_n\|^p + \Big(\frac{b}{\theta p} - \frac{b}{p_s^*(\alpha)}\Big)\|u_n\|^{\theta p} - \Big(\frac{1}{r} - \frac{1}{p_s^*(\alpha)}\Big)\lambda \int_\Omega f(x) \frac{|u|^r}{|x|^c} dx \\
& \ge \Big(\frac{b}{\theta p} - \frac{b}{p_s^*(\alpha)}\Big)\|u_n\|^{\theta p} - \Big(\frac{1}{r} - \frac{1}{p_s^*(\alpha)}\Big)\lambda \omega_2 C_{r,c} H_\alpha^{-r/p} \|u_n\|^r,
\end{aligned}$$

which implies that $\|u_n\| \le C$ (independent of a) for all $\lambda > 0$ because $\theta p > r$. This completes the proof. $\square$

Lemma 3.6. *Let $f(\cdot)$ satisfy* (A1) *and $\lambda > 0$. Then there exists $a_* > 0$ such that, for each $a \in (0, a_*)$, $I(\cdot)$ satisfies the $(PS)_c$ condition in $\mathbb{X}$ for all*

$$c < \big(\frac{1}{\theta p} - \frac{1}{p_s^*(\alpha)}\big)\big(b H_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha) - p\theta}} - \lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha) - r}} C_0$$

with

$$C_0=\frac{(p_s^*(\alpha)-r)}{p_s^*(\alpha)}\Big(\omega_2 C_{r,c}\big(\frac{1}{r}-\frac{1}{\theta p}\big)\big(\frac{1}{\theta p}-\frac{1}{p_s^*(\alpha)}\big)^{-r}\Big)^{1/(p_s^*(\alpha)-r)}.$$

Proof. Since $\{u_n\}_n\subset\mathbb{X}$ is bounded, up to a subsequence, there exists a function $u\in\mathbb{X}$ such that $u_n\rightharpoonup u$ in $\mathbb{X}$. Hence, in view of Lemma 2.6, there exist a countable sequence of points $\{x_j\}_{j\in J}\subset\mathbb{R}^N$ and the families of positive numbers $\{\mu_j\}_{j\in J}$, $\{\nu_j\}_{j\in J}$ such that as $n\to\infty$ we have

$$\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+sp}}dy\rightharpoonup\mu\ge\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dy+\sum_{j\in J}\mu_j\delta_{x_j}\tag{3.3}$$

and

$$|u_n|^{p_s^*(\alpha)}|x|^{-\alpha}\rightharpoonup\nu=|u|^{p_s^*(\alpha)}|x|^{-\alpha}+\sum_{j\in J}\nu_j\delta_{x_j}\tag{3.4}$$

in the sense of measure, where δ_{x_j} is the Dirac measure concentrated at x_j. Moreover,

$$\mu_j\ge H_\alpha\nu_j^{\frac{p}{p_s^*(\alpha)}}\quad\text{for all } j\in J.\tag{3.5}$$

Next, we prove that $\nu_j=0$ for all $j\in J$. To this end, let x_j be a singular point of the measures μ, ν, and $m(\|u_n\|):=\big(a+b\|u_n\|^{(\theta-1)p}\big)$. We define a cut-off function $\phi_{\varepsilon,j}(x):=\phi\big(\frac{x-x_j}{\varepsilon}\big)$, where $\phi\in C_0^\infty(\Omega)$ is such that $0\le\phi(x)\le1$, $\phi(x)=1$ in $B_1(0)$, $\phi(x)=0$ in $\mathbb{R}^N\setminus B_2(0)$ and $|\nabla\phi(x)|\le\frac{2}{\varepsilon}$. Obviously, $\{\phi_{\varepsilon,j}u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}$. It follows from $\langle I'(u_n),\phi_{\varepsilon,j}u_n\rangle\to0$ that

$$\begin{aligned}
&m(\|u_n\|)\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}\big(u_n(x)-u_n(y)\big)\big(\phi_{\varepsilon,j}(x)-\phi_{\varepsilon,j}(y)\big)u_n(x)}{|x-y|^{N+ps}}\,dx\,dy\\
&+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{q-2}\big(u_n(x)-u_n(y)\big)\big(\phi_{\varepsilon,j}(x)-\phi_{\varepsilon,j}(y)\big)u_n(x)}{|x-y|^{N+qs}}\,dx\,dy\\
&+m(\|u_n\|)\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\phi_{\varepsilon,j}(x)\,dx\,dy\\
&+\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^q}{|x-y|^{N+qs}}\phi_{\varepsilon,j}(x)\,dx\,dy\\
&=\int_\Omega\frac{|u_n(x)|^{p_s^*(\alpha)}\phi_{\varepsilon,j}(x)}{|x|^\alpha}dx+\lambda\int_\Omega f(x)\frac{|u_n(x)|^r\phi_{\varepsilon,j}(x)}{|x|^c}\,dx+o(1).
\end{aligned}\tag{3.6}$$

To the first term on the left hand side of the above formula (3.6), according to Proposition 2.7 we have

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}\Big(\iint_{\mathbb{R}^{2N}}\frac{\big|\big(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y)\big)u_n(x)\big|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}=0.$$

By employing Hölder's inequality we obtain

$$\begin{aligned}
&\Big|m(\|u_n\|)\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}\big(u_n(x)-u_n(y)\big)\big(\phi_{\varepsilon,j}(x)-\phi_{\varepsilon,j}(y)\big)u_n(x)}{|x-y|^{N+ps}}\,dx\,dy\Big|\\
&\le C\Big(\iint_{\mathbb{R}^{2N}}\frac{\big|u_n(x)-u_n(y)\big|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1-\frac1p}\\
&\quad\times\Big(\iint_{\mathbb{R}^{2N}}\frac{\big|\big(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y)\big)u_n(x)\big|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}
\end{aligned}\tag{3.7}$$

$$\le C\Big(\iint_{\mathbb{R}^{2N}} \frac{\big|\big(\varphi_{\varepsilon,j}(x)-\varphi_{\varepsilon,j}(y)\big)u_n(x)\big|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p} \to 0 \quad \text{as } \varepsilon\to 0, n\to\infty.$$

From the second term on the left-hand side of (3.6), similarly we obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{q-2}\big(u_n(x)-u_n(y)\big)\big(\phi_{\varepsilon,j}(x)-\phi_{\varepsilon,j}(y)\big)u_n(x)}{|x-y|^{N+qs}}\,dx\,dy = 0. \tag{3.8}$$

For the third term on the left hand side of (3.6), it follows from $a>0$ and (3.3) that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{n\to\infty} m(\|u_n\|)\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\phi_{\varepsilon,j}(x)\,dx\,dy &(3.9)\\
&\ge \lim_{\varepsilon\to 0}\limsup_{n\to\infty} b\Big(\iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\phi_{\varepsilon,j}(x)\,dx\,dy\Big)^{\theta} &(3.10)\\
&\ge \lim_{\varepsilon\to 0} b\Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\phi_{\varepsilon,j}(x)\,dx\,dy+\mu_j\Big)^{\theta} = b\mu_j^{\theta}. &(3.11)
\end{aligned}$$

In addition, by (3.4) we obtain that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega \frac{|u_n(x)|^{p_s^*(\alpha)}}{|x|^\alpha}\phi_{\varepsilon,j}(x)\,dx = \lim_{\varepsilon\to 0}\int_\Omega \frac{|u(x)|^{p_s^*(\alpha)}}{|x|^\alpha}\phi_{\varepsilon,j}(x)\,dx+\nu_j = \nu_j \tag{3.12}$$

and

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\int_\Omega f(x)\frac{|u_n(x)|^r\phi_{\varepsilon,j}(x)}{|x|^c}\,dx = \lim_{\varepsilon\to 0}\int_\Omega f(x)\frac{|u(x)|^r\phi_{\varepsilon,j}(x)}{|x|^c}\,dx = 0, \tag{3.13}$$

where we used the fact that $\mathbb{X}\hookrightarrow L^r(\mathbb{R}^N,|x|^{-c})$ is a compact embedding due to Lemma 2.3.

Now let us put (3.7)–(3.13) into (3.6) to obtain that

$$\nu_j \ge b\mu_j^{\theta}.$$

Therefore, $\nu_j \ge b\mu_j^{\theta} \ge b\big(H_\alpha \nu_j^{p/p_s^*(\alpha)}\big)^{\theta}$ in accordance with (3.5). This gives $\nu_j=0$ or $\nu_j \ge \big(bH_\alpha^{\theta}\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-\theta p}}$.

Next we prove by contradiction that it is impossible for $\nu_j \ge \big(bH_\alpha^{\theta}\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-\theta p}}$ for $j\in J$. Applying (A1), Lemma 2.6, (3.5) and Young's inequality we obtain

$$\begin{aligned}
c &= \lim_{n\to\infty}\big(I(u_n)-\frac{1}{\theta p}\langle I'(u_n),u_n\rangle\big)\\
&\ge \lim_{n\to\infty}\big(\frac{a}{p}-\frac{a}{\theta p}\big)\|u_n\|^p+\big(\frac{1}{\theta p}-\frac{1}{p_s^*(\alpha)}\big)\int_\Omega \frac{|u_n|^{p_s^*(\alpha)}}{|x|^\alpha}dx-\lambda\big(\frac{1}{r}\\
&\quad -\frac{1}{\theta p}\big)\int_\Omega f(x)\frac{|u_n|^r}{|x|^c}dx\\
&\ge \Big(\frac{1}{\theta p}-\frac{1}{p_s^*(\alpha)}\Big)\big(\int_\Omega \frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha}dx+\nu_j\big)-\lambda\Big(\frac{1}{r}-\frac{1}{\theta p}\Big)\omega_2 C_{r,c}\Big(\int_\Omega \frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha}dx\Big)^{r/p_s^*(\alpha)}\\
&\ge \Big(\frac{1}{\theta p}-\frac{1}{p_s^*(\alpha)}\Big)\nu_j-\lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}\frac{(p_s^*(\alpha)-r)}{p_s^*(\alpha)}\\
&\quad\times\Big(\omega_2 C_{r,c}\big(\frac{1}{r}-\frac{1}{\theta p}\big)\big(\frac{1}{\theta p}-\frac{1}{p_s^*(\alpha)}\big)^{-r}\Big)^{\frac{1}{(p_s^*(\alpha)-r)}}
\end{aligned}$$

$$
\geq \big(\frac{1}{\theta p}-\frac{1}{p_s^*(\alpha)}\big)\big(bH_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p\theta}}-\lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}C_0,
$$

which contradicts $c<\big(\frac{1}{\theta p}-\frac{1}{p_s^*(\alpha)}\big)\big(bH_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p\theta}}-\lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}C_0$. Therefore $\nu_j=0$ for any $j\in J$, and then

$$
\lim_{n\to\infty}\int_\Omega \frac{|u_n|^{p_s^*(\alpha)}}{|x|^\alpha}dx=\int_\Omega \frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha}dx.
$$

Moreover, using the Proposition 2.4 (Brezis-Lieb Lemma), we have

$$
\lim_{n\to\infty}\int_\Omega \frac{|u_n-u|^{p_s^*(\alpha)}}{|x|^\alpha}dx=0. \tag{3.14}
$$

Finally, we show that $u_n\to u$ in $\mathbb{X}$. Let $\{u_n\}$ be a $(PS)_c$ sequence, then we obtain

$$
\begin{aligned}
o_n(1)&=\langle I'(u_n)-I'(u),u_n-u\rangle\\
&=m(\|u_n\|)\langle u_n,u_n-u\rangle_{s,p}-m(\|u_n\|)\langle u,u_n-u\rangle_{s,p}\\
&\quad+\big(\langle u_n,u_n-u\rangle_{s,q}-\langle u,u_n-u\rangle_{s,q}\big)\\
&\quad+\int_\Omega \frac{\big(|u_n|^{p_s^*(\alpha)-2}u_n-|u|^{p_s^*(\alpha)-2}u\big)\big(u_n-u\big)}{|x|^\alpha}dx\\
&\quad+\lambda\int_\Omega f(x)\frac{\big(|u_n|^{r-2}u_n-|u|^{r-2}u\big)\big(u_n-u\big)}{|x|^c}dx.
\end{aligned} \tag{3.15}
$$

For the fourth term on the right-hand side of (3.15), we claim that

$$
\lim_{n\to\infty}\int_\Omega \frac{\big(|u_n|^{p_s^*(\alpha)-2}u_n-|u|^{p_s^*(\alpha)-2}u\big)\big(u_n-u\big)}{|x|^\alpha}dx=0.
$$

Indeed, since $\{u_n\}$ is uniformly bounded in $\mathbb{X}$, this means that there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u\in\mathbb{X}$ such that

$$
\begin{gathered}
u_n\rightharpoonup u\quad\text{in } \mathbb{X} \text{ and in } L^{p_s^*(\alpha)}(\Omega,|x|^{-\alpha}),\\
|u_n|^{p_s^*(\alpha)-2}u_n\rightharpoonup |u|^{p_s^*(\alpha)-2}u\quad\text{in } L^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-1}}(\Omega,|x|^{-\alpha}),\\
u_n\to u\quad\text{a.e. in } \Omega,\\
|u_n|^{r-2}u_n\to |u|^{r-2}u\quad\text{in } L^{\frac{r}{r-1}}(\Omega,|x|^{-c})
\end{gathered} \tag{3.16}
$$

as $n\to\infty$. This yields

$$
\begin{aligned}
&\lim_{n\to\infty}\int_\Omega \frac{\big(|u_n|^{p_s^*(\alpha)-2}u_n-|u|^{p_s^*(\alpha)-2}u\big)\big(u_n-u\big)}{|x|^\alpha}dx\\
&=\int_\Omega \frac{|u_n-u|^{p_s^*(\alpha)}}{|x|^\alpha}dx+o(1),
\end{aligned} \tag{3.17}
$$

which together with (3.14) implies

$$
\lim_{n\to\infty}\int_\Omega \frac{\big(|u_n|^{p_s^*(\alpha)-2}u_n-|u|^{p_s^*(\alpha)-2}u\big)\big(u_n-u\big)}{|x|^\alpha}dx=0. \tag{3.18}
$$

For the last term on the right-hand side of (3.15), by (3.16) we have

$$
\lim_{n\to\infty}\int_\Omega f(x)\frac{\big(|u_n|^{r-2}u_n-|u|^{r-2}u\big)\big(u_n-u\big)}{|x|^c}dx=0. \tag{3.19}
$$

To estimate the third term on the right-hand side, let us recall the well-known Simon inequalities:

$$
\begin{aligned}
&|\xi-\eta|^p \\
&\leq \begin{cases} C_p'\big(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\big)(\xi-\eta) & \text{for } p\geq 2 \\ C_p''\big[\big(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\big)(\xi-\eta)\big]^{p/2}\big(|\xi|^p+|\eta|^p\big)^{(2-p)/2} & \text{for } 1<p<2, \end{cases}
\end{aligned}
\tag{3.20}
$$

for all $\xi,\eta\in\mathbb{R}^N$, where C_p' and C_p'' are positive constants depending only on p. Therefore, to the third term on the right hand side of (3.15), we obtain

$$
\langle u_n, u_n-u\rangle_{s,q}-\langle u, u_n-u\rangle_{s,q}\geq 0. \tag{3.21}
$$

Let us now put (3.18), (3.19) and (3.21) into (3.15), which yields the inequality

$$
\begin{aligned}
o(1)\geq\ & m(\|u_n\|)\big(\langle u_n,u_n-u\rangle_{s,p}-\langle u,u_n-u\rangle_{s,p}\big)\\
&+m(\|u_n\|)\langle u,u_n-u\rangle_{s,p}-m(\|u_n\|)\langle u,u_n-u\rangle_{s,p}.
\end{aligned}
\tag{3.22}
$$

Note that the $\{u_n\}_n$ is uniformly bounded which lead to that $u_n\rightharpoonup u$ in $\mathbb{X}$, we deduce that

$$
\lim_{n\to\infty} m(\|u_n\|)\langle u,u_n-u\rangle_{s,p}=0,\quad \lim_{n\to\infty} m(\|u_n\|)\langle u,u_n-u\rangle_{s,p}=0.
$$

Hence

$$
\lim_{n\to\infty} m(\|u_n\|)\big(\langle u_n,u_n-u\rangle_{s,p}-\langle u,u_n-u\rangle_{s,p}\big)\leq 0.
$$

This together with $d:=\inf_{n\geq 1}\|u_n\|>0$ and $b>0$ yields

$$
\lim_{n\to\infty}\big(\langle u_n,u_n-u\rangle_{s,p}-\langle u,u_n-u\rangle_{s,p}\big)\leq 0.
$$

It remains to prove the strong convergence of $\{u_n\}$ in $\mathbb{X}$. To this end, we part it in the settings of $p>2$ and $1<p<2$. For $p>2$, it follows from (3.20) that

$$
\begin{aligned}
0&\leq\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{\big|\big(u_n(x)-u_n(y)\big)-\big(u(x)-u(y)\big)\big|^p}{|x-y|^{N+ps}}\,dx\,dy\\
&\leq C_p'\lim_{n\to\infty}\big(\langle u_n,u_n-u\rangle_{s,p}-\langle u,u_n-u\rangle_{s,p}\big)\leq 0
\end{aligned}
$$

as $n\to\infty$. Hence $u_n\to u$ in $\mathbb{X}$. For $1<p<2$, by (3.20) we have

$$
\begin{aligned}
0&\leq\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{\big|\big(u_n(x)-u_n(y)\big)-\big(u(x)-u(y)\big)\big|^p}{|x-y|^{N+ps}}\,dx\,dy\\
&\leq C_p''\lim_{n\to\infty}\big(\langle u_n,u_n-u\rangle_{s,p}-\langle u,u_n-u\rangle_{s,p}\big)^{p/2}\\
&\quad\times\Big(\iint_{\mathbb{R}^{2N}}\frac{\big|u_n(x)-u_n(y)\big|^p+\big|u(x)-u(y)\big|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{(2-p)/2}\\
&\leq C\lim_{n\to\infty}\big(\langle u_n,u_n-u\rangle_{s,p}-\langle u,u_n-u\rangle_{s,p}\big)^{p/2}\leq 0
\end{aligned}
\tag{3.23}
$$

as $n\to\infty$. Hence $u_n\to u$ in $\mathbb{X}$. In conclusion, we obtain $u_n\to u$ strongly in $\mathbb{X}$ as $n\to\infty$.

Finally, we consider $\inf_{n\in N}\|u_n\|=0$. If 0 is an accumulation point of the sequence $\{u_n\}_n$, then there exists a subsequence of $\{u_n\}_n$ strongly converging to $u=0$, which leads to the desired result. If 0 is an isolated point of the sequence $\{u_n\}_n$, then there exists a subsequence, still denoted by $\{u_n\}_n$, such that $\inf_{n\in N}\|u_n\|>0$, which was proved as above. This completes the proof. □

Next, we show that the corresponding energy functional satisfies the Palais-Smale condition at the levels less than

$$\Big(\frac{1}{\theta p}-\frac{1}{p_s^*(\alpha)}\Big)\big(bH_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p\theta}}-\lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}C_0$$

by constructing sufficiently small mini-max levels, which is mainly inspired by the reference [16]. By Lemma 2.1 and Condition (A1), we have

$$\begin{aligned}I(u)&\le \frac{a}{p}\|u\|^p+\frac{b}{\theta p}\|u\|^{\theta p}+\frac{1}{q}[u]_{s,q}^q-\frac{1}{p_s^*(\alpha)}\int_\Omega |x|^{-\alpha}|u|^{p_s^*(\alpha)}dx\\
&\le \frac{a}{p}\|u\|^p+\frac{b}{\theta p}\|u\|^{\theta p}+\frac{C}{q}\|u\|^q-\frac{1}{p_s^*(\alpha)}\int_\Omega |x|^{-\alpha}|u|^{p_s^*(\alpha)}dx\end{aligned}$$

for all $u\in\mathbb{X}$. Define the functional $J(u):\mathbb{X}\to R$ by

$$J(u)=\frac{a}{p}\|u\|^p+\frac{b}{\theta p}\|u\|^{\theta p}+\frac{C}{q}\|u\|^q-\frac{1}{p_s^*(\alpha)}\int_\Omega |x|^{-\alpha}|u|^{p_s^*(\alpha)}dx.$$

Then $I(u)\le J(u)$ for all $u\in\mathbb{X}$. Hence it suffices to construct small mini-max levels for $J(u)$.

For any $\delta>0$, one can choose $\phi_\delta\in C_0^\infty(\mathbb{R}^N)$ with $\int_\Omega |x|^{-\alpha}|\phi_\delta|^{p_s^*(\alpha)}dx=1$ and $\operatorname{supp}\phi_\delta\subset\Omega$ such that $\|\phi_\delta\|<\delta$. Thus, for $t\ge 0$ we have

$$J(t\phi_\delta)=\frac{at^p}{p}\delta^p+\frac{bt^{\theta p}}{\theta p}\delta^{\theta p}+\frac{Ct^q}{q}\delta^q-\frac{t^{p_s^*(\alpha)}}{p_s^*(\alpha)}.$$

Then there exists $t^*>0$ such that

$$\begin{aligned}\max_{t\ge 0}J(t\phi_\delta)=J(t_*\phi_\delta)&=\frac{at_*^p}{p}\delta^p+\frac{bt_*^{\theta p}}{\theta p}\delta^{\theta p}+\frac{Ct_*^q}{q}\delta^q-\frac{t_*^{p_s^*(\alpha)}}{p_s^*(\alpha)}\\
&\le \frac{a_*t_*^p}{p}\delta^p+\frac{bt_*^{\theta p}}{\theta p}\delta^{\theta p}+\frac{Ct_*^q}{q}\delta^q-\frac{t_*^{p_s^*(\alpha)}}{p_s^*(\alpha)}.\end{aligned}$$

Let us take $\delta>0$ small enough such that

$$\frac{a_*t_*^p}{p}\delta^p+\frac{bt_*^{\theta p}}{\theta p}\delta^{\theta p}+\frac{Ct_*^q}{q}\delta^q-\frac{t^{p_s^*(\alpha)}}{p_s^*(\alpha)}<\Big(\frac{1}{\theta p}-\frac{1}{p_s^*(\alpha)}\Big)\big(bH_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p\theta}}-\lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}C_0.$$

This leads to the following result.

Lemma 3.7. *Under the assumption of Lemma 3.2, there exist $a_*>0$ and $\lambda_*>0$ such that for each $a\in(0,a_*)$ and $\lambda\in(0,\lambda_*)$, we have that $\hat{\phi_\delta}\in\mathbb{X}$ with $\|\hat{\phi_\delta}\|>\rho$, $I(\hat{\phi_\delta})<0$ and*

$$\max_{t\in[0,1]}I(t\hat{\phi_\delta})\le\Big(\frac{1}{\theta p}-\frac{1}{p_s^*(\alpha)}\Big)\big(bH_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p\theta}}-\lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}C_0$$

Proof. It is obvious that there exists $\lambda_*\in(0,\lambda_0)$ independent of a such that

$$\Big(\frac{1}{\theta p}-\frac{1}{p_s^*(\alpha)}\Big)\big(bH_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p\theta}}-\lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}C_0>0\quad\text{for any }\lambda\in(0,\lambda_*).$$

Let $\phi_\delta\in\mathbb{X}$ be the function defined as above and choosing $\hat{t}>0$ be such that $\hat{t}\|\phi_\delta\|>\rho$ and $I(t\phi_\delta)<0$ for all $t\ge\hat{t}$. The result follows by letting $\hat{\phi_\delta}=\hat{t}\phi_\delta$. $\square$

Theorem 3.8. *Let $f(\cdot)$ satisfy* (A1). *Then there exist $a_*>0$ and $\lambda_*>0$ such that for each $a\in(0,a_*)$ and $\lambda\in(0,\lambda_*)$, Problem* (1.1) *has a nontrivial solution u^1 in $\mathbb{X}$ with $I(u^1)>0$.*

Proof. According to Lemma 3.7, we define
$$c = \inf_{y\in\Gamma}\max_{t\in[0,1]} I(t\hat{\phi_\delta}),$$
where $\Gamma = \{y \in C\big([0,1],\mathbb{X}\big) : y(0) = 0 \text{ and } y(1) = \hat{\phi_\delta}\}$.

By Lemma 3.2, we have $0 < \delta_\lambda \le c < \big(\frac{1}{\theta p} - \frac{1}{p_s^*(\alpha)}\big)\big(bH_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p\theta}} - \lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}C_0$. In view of Lemma 3.6, we know that I satisfies the $(PS)_c$ condition, and there exists $u^1 \in \mathbb{X}$ such that $I'(u^1) = 0$ and $I(u^1) = c$ for all $\lambda \in (0,\lambda_*)$. Thus, u^1 is a solution of (1.1). □

Before give the second solution, we need to introduce the following important proposition.

Proposition 3.9 (Ekeland variational principle, [18, Theorem 1.1])**.** *Let V be a complete metric space and $F : V \to \mathbb{R}\cup\{+\infty\}$ be lower semicontiuous, bounded from below. Then, for any $\varepsilon > 0$, there exists some point $\nu \in V$ with*
$$F(\nu) \le \inf_V + \varepsilon, \quad F(w) \ge F(\nu) - \varepsilon d(\nu, w) \quad \textit{for all } w \in V.$$

In the following, we set $B_\rho = \{u \in \mathbb{X} : \|u\| < \rho\}$, where $\rho > 0$ is given by Lemma 3.2.

Theorem 3.10. *Let $f(\cdot)$ satisfy* (A1)*. Then there exist $a_* > 0$ and $\lambda^* > 0$ such that for each $a \in (0, a_*)$ and $\lambda \in (0,\lambda^*]$, Problem* (1.1) *has another nontrivial solution u^2 in $\mathbb{X}$ with $I(u^2) < 0$.*

Proof. Define $\widetilde{c} = \inf\{I(u) : u \in \overline{B_\rho}\}$, we first claim that $\widetilde{c} < 0$. Indeed, by choosing a nonnegative function $\omega_0 \in C_0^\infty(\mathbb{R}^N)$ we have
$$\lim_{\tau\to 0}\frac{I(\tau\omega_0)}{\tau^r} = -\frac{\lambda}{r}\int_\Omega f(x)|\omega_0|^r dx < 0.$$
Therefore there exists a sufficiently small $\tau > 0$ such that $\|\tau\omega_0\| \le \rho$ and $I(\tau\omega_0) < 0$, which yields that $\widetilde{c} < 0$.

Considering Lemma 3.2 and the Ekeland variational principle yields that there exists a sequence $\{u_n\}_n$ such that
$$\widetilde{c} \le I(u_n) \le \widetilde{c} + \frac{1}{n}, \tag{3.24}$$
$$I(\nu) \ge I(u_n) - \frac{\|u_n - \nu\|}{n} \tag{3.25}$$
for all $\nu \in \overline{B_\rho}$.

Now we show that $\|u_n\| < \rho$ for n sufficiently large. Arguing by contradiction, we assume that $\|u_n\| = \rho$ for any $n \in \mathbb{N}$. By Lemma 3.2 we deduce that
$$I(u_n) \ge \delta_\lambda > 0.$$
This and (3.24) imply that $\widetilde{c} \ge \delta_\lambda > 0$, which contradicts $\widetilde{c} < 0$.

Next we prove that $I'(u_n) \to 0$ in $\mathbb{X}^*$. Set
$$\omega_n = u_n + \tau\nu, \quad \forall \nu \in B_1 := \{\nu \in \mathbb{X} : \|\nu\| = 1\},$$
where $\tau > 0$ small enough that $0 < \tau \le \rho - \|u_n\|$ for fixed n large. Then
$$\|\omega_n\| = \|u_n + \tau\nu\| \le \|u_n\| + \tau \le \rho,$$

which means that $\omega_n \in \overline{B_\rho}$. Thus, it follows from (3.25) that

$$I(\omega_n) \geq I(u_n) - \frac{1}{n}\|u_n - \omega_n\|,$$

or

$$\frac{I(u_n + \tau\nu) - I(u_n)}{\tau} \geq -\frac{1}{n}.$$

By letting $\tau \to 0^+$, we obtain $\langle I'(u_n), \nu\rangle \geq -\frac{1}{n}$ for any fixed n large. Similarly, by choosing $\tau < 0$ such that $|\tau|$ small enough, let us repeat the process as above to obtain

$$\langle I'(u_n), \nu\rangle \leq \frac{1}{n} \quad \text{for any fixed } n \text{ large.}$$

We immediately conclude that

$$\lim_{n\to\infty} \sup_{\nu \in B_1} |\langle I'(u_n), \nu\rangle| = 0,$$

which yields that $I'(u_n) \to 0$ in $\mathbb{X}^*$ as $n \to \infty$. Hence, $\{u_n\}_n$ is a $(PS)_{\widetilde{c}}$ sequence for the functional I with $\widetilde{c} < 0$.

Taking $\lambda^* \in (0, \lambda_*]$ such that $0 < \big(\frac{1}{\theta p} - \frac{1}{p_s^*(\alpha)}\big)\big(bH_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p\theta}} - \lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}} C_0$ for all $\lambda \in (0, \lambda^*)$. We deduce from $\widetilde{c} < 0$ and Lemma 2.6 that there exists u^2 such that $u_n \to u^2$ in $\mathbb{X}$. Then, we obtain a nontrivial solution u^2 of (1.1) satisfying

$$I(u^2) = \widetilde{c} < 0 \quad \text{and } \|u^2\| < \rho,$$

which completes the proof. $\square$

Proof of Theorem 1.2. This proof follows immediately by the combination of Theorem 3.8 and Theorem 3.10. $\square$

4. Asymptotic behavior of solutions

We devote this section to proving the concentration of solutions for Problem (1.1), which is stated by Theorem 1.2. Our main idea is motivated by the recent papers [30, 35].

Proof of Theorem 1.3. For the sequence $\{a_n\}$ with $a_n \to 0$ as $n \to \infty$, let $u_n^i := u_{a_n}^i$ be the critical points of the energy functional I obtained in Theorem 1.2 for $i = 1, 2$, that is to say,

$$\begin{gathered} I'(u_n^1) = 0, \quad I(u_n^1) = c_n, \\ I'(u_n^2) = 0, \quad I(u_n^2) = \widetilde{c_n}. \end{gathered}$$

It is clear that by Lemma 3.5 and $a_n \in (0, a_*)$ there exists a constant $C > 0$ independent of a_n and n such that

$$\|u_n^i\| \leq C \quad \text{for all } n,$$

which shows that $\{u_n^i\}_n$ are uniformly bounded in $\mathbb{X}$. Passing to a subsequence if necessary, we may assume that $u_n^i \rightharpoonup u^i$ weakly in $\mathbb{X}$. Thanks to Lemma 3.6 we immediately obtain that the sequence $\{u_n^i\}(i = 1, 2)$ contain strongly convergent subsequences with

$$\{c_n, \widetilde{c_n}\} < \Big(\frac{1}{\theta p} - \frac{1}{p_s^*(\alpha)}\Big)\big(bH_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p\theta}} - \lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}} C_0.$$

We employ a similar proof as in Lemmas 3.2 and 3.7 to deduce that

$$0 < \delta_\lambda \le c_n \le \big(\frac{1}{\theta p} - \frac{1}{p_s^*(\alpha)}\big)\big(bH_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p\theta}} - \lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}} C_0,$$

and obtain that $\widetilde{c_n} < 0$ with the same proof as Theorem 3.10. Hence there exists subsequences still denoted by themselves, and $u^i \in \mathbb{X}$ such that $u_n^i \to u^i$ in $\mathbb{X}$ as $a \to 0^+$ for $i = 1, 2$. Therefore, for all $\phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
0 &= (a + b\|u_n^i\|^{(\theta-1)p}) \iint_{\mathbb{R}^{2N}} \frac{|u_n^i(x) - u_n^i(y)|^{p-2}(u_n^i(x) - u_n^i(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}}\,dx\,dy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_n^i(x) - u_n^i(y)|^{q-2}(u_n^i(x) - u_n^i(y))(\phi(x) - \phi(y))}{|x-y|^{N+qs}}\,dx\,dy \\
&\quad - \int_\Omega \frac{|u_n(x)^i|^{p_s^*(\alpha)-2}u_n^i(x)\phi(x)}{|x|^\alpha} dx - \lambda \int_\Omega f(x) \frac{|u_n^i(x)|^{r-2}u_n^i(x)\phi(x)}{|x|^c} dx \\
&\to b\|u^i\|^{(\theta-1)p} \iint_{\mathbb{R}^{2N}} \frac{|u^i(x) - u^i(y)|^{p-2}(u^i(x) - u^i(y))(\phi(x) - \phi(y))}{|x-y|^{N+ps}}\,dx\,dy \\
&\quad + \iint_{\mathbb{R}^{2N}} \frac{|u_n^i(x) - u^i(y)|^{q-2}(u^i(x) - u^i(y))(\phi(x) - \phi(y))}{|x-y|^{N+qs}}\,dx\,dy \\
&\quad - \int_\Omega \frac{|u^i(x)|^{p_s^*(\alpha)-2}u^i(x)\phi(x)}{|x|^\alpha} dx - \lambda \int_\Omega f(x) \frac{|u^i(x)|^{r-2}u^i(x)\phi(x)}{|x|^c} dx \quad \text{as } b \to 0^+.
\end{aligned}$$

This makes clear that $u^i \in \mathbb{X}$ for $i = 1, 2$ are solutions of Problem 1.2. Moreover, it follows from the constant δ_λ independent of a that

$$I(u^2) < 0 < \delta_\lambda \le I(u^1),$$

which means that $u^i \neq 0$ and $u^1 \neq u^2$. The proof is complete. □

5. A sequence of arbitrarily small solutions

In this section we prove that Problem (1.2) admits a sequence of nontrivial solutions $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}$ such that $u_n \to 0$ as $n \to \infty$ provided that λ belongs to a suitable range. Let us recall some basic facts involved in the so-called Krasnoselskii genus, which can be found in [14, 29]. For a symmetric group $\mathbb{Z}_2 = \{id, -id\}$ and E being a Banach space, we set

$$\Gamma := \big\{A \subset E \setminus \{0\} : A \text{ is closed and } A = -A\big\}.$$

Definition 5.1. For any $A \in \Gamma$, the Krasnoselskii genus of A is defined by

$$\gamma(A) := \inf \big\{\kappa : \exists \phi \in C(A, \mathbb{R}^\kappa \setminus \{0\}) \text{ and } \phi \textit{ is odd}\big\}.$$

If such a κ does not exist, then we set $\gamma(A) = \infty$.

By definition, it is obvious that $\gamma(\emptyset) = 0$. Let Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and $\gamma(A) \ge k$. First of all, let us list the following main properties of Krasnoselskii genus, see [14] or [23].

Proposition 5.2. *Let A and B be closed symmetric subsets of E which do not contain the origin. Then the following statements hold:*

1. *If there exists an odd continuous mapping from A to B, then $\gamma(A) \le \gamma(B)$.*
2. *If $A \subset B$, then $\gamma(A) \le \gamma(B)$.*
3. *If there exists an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$.*

(4) *The n-dimensional sphere S^n has a genus of $n+1$ by the Borsuk-Ulam Theorem.*
(5) *If $\gamma(B)<\infty$, then $\gamma(A\setminus B)\geq\gamma(A)-\gamma(B)$.*
(6) *If A is compact, then $\gamma(A)<\infty$ and there exists $\delta>0$ and a closed and symmetric neighborhood $N_\delta(A)=\{x\in E:\|x-A\|\leq\delta\}$ of A such that $\gamma(N_\delta(A))=\gamma(A)$.*

The following version of the symmetric mountain pass lemma is form Kajikiya's work in [23].

Lemma 5.3. *Let E be an infinite-dimensional Banach space. Suppose $I\in C^1(E,R)$ satisfies the following conditions:*

(1) *$I(u)$ is even, bounded from below with $I(0)=0$, and $I(u)$ satisfies the local Palais-Smale condition, i.e. for some $c_*>0$, every sequence $\{u_k\}$ in $\mathbb{X}$ satisfying $\lim_{k\to\infty}I(u_k)=c<c_*$ and $\lim_{k\to\infty}\|I'(u_k)\|_{E^*}=0$ has a convergent subsequence.*
(2) *For each $k\in\mathbb{N}$, there exists an $A_k\in\Gamma_k$ such that $\sup_{u\in A_k}I(u)<0$.*

Then either (i) or (ii) below holds.

(i) *There exists a sequence $\{u_k\}$ such that $I'(u_k)=0$, $I(u_k)<0$ and $\{u_k\}$ converges to zero.*
(ii) *There exist two sequences $\{u_k\}$ and $\{\nu_k\}$ such that $I'(u_k)=0$, $I(u_k)=0$, $u_k\neq 0$, $\lim_{k\to\infty}u_k=0$; $I'(\nu_k)=0$, $I(\nu_k)<0$, $\lim_{k\to\infty}I(\nu_k)=0$ and $\{\nu_k\}$ converges to a non-zero limit.*

We denote by λ_1 the first eigenvalue of $(-\Delta)^s_p$, that is,

$$\lambda_1:=\inf_{u\in\mathbb{X}\setminus\{0\}}\frac{\|u\|^p}{|u|^p_{L^p(\Omega)}}.$$

By using Young's inequality with $\varepsilon=\frac{1}{\lambda_1}$, Condition (A1) and the definition of H_α, we obtain that for any $\lambda\in(0,\lambda_1)$ it holds

$$\begin{aligned}
I(u)&\geq\frac{b}{\theta p}\|u\|^{\theta p}-\frac{1+\lambda\varepsilon p_s^*(\alpha)}{p_s^*(\alpha)}H_\alpha^{-p_s^*(\alpha)/p}\|u\|^{p_s^*(\alpha)}-\lambda b(\varepsilon)\Big(\frac{\omega_2C_{r,c}}{r}\Big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}\\
&\geq\frac{b}{\theta p}\|u\|^{\theta p}-\frac{1+p_s^*(\alpha)}{p_s^*(\alpha)}H_\alpha^{-p_s^*(\alpha)/p}\|u\|^{p_s^*(\alpha)}-\lambda b\Big(\frac{1}{\lambda_1}\Big)\Big(\frac{\omega_2C_{r,c}}{r}\Big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}\\
&=A\|u\|^{\theta p}-B\|u\|^{p_s^*(\alpha)}-\lambda C
\end{aligned}$$

with

$$A:=\frac{b}{\theta p},\quad B:=\frac{1+p_s^*(\alpha)}{p_s^*(\alpha)}H_\alpha^{-p_s^*(\alpha)/p},\quad C:=b\Big(\frac{1}{\lambda_1}\Big)\Big(\frac{\omega_2C_{r,c}}{r}\Big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}.$$

Therefore, we let $g(t):=At^{\theta p}-Bt^{p_s^*(\alpha)}-\lambda C$ which leads to $I(u)\geq g(\|u\|)$. If we select

$$\lambda_1^*:=\min\Big\{\lambda_1,\frac{A(p_s^*(\alpha)-\theta p)}{Cp_s^*(\alpha)}\Big(\frac{A\theta p}{Bp_s^*(\alpha)}\Big)^{\frac{\theta p}{p_s^*(\alpha)-\theta p}}\Big\}>0,$$

we see that for any $\lambda\in(0,\lambda_1^*)$, the function $g(t)$ achieves its positive maximum at $t_1=\big(\frac{A\theta p}{Bp_s^*(\alpha)}\big)^{\frac{1}{p_s^*(\alpha)-\theta p}}$, which means that

$$M_1=g(t_1)=\max_{t\geq 0}g(t)>0.$$

Hence, it is clear that for any $M_0\in(0,M_1)$ we can find $t_0<t_1$ such that $g(t_0)=M_0$.

To our aim, it necessary to introduce a suitable truncated functional related to $I(u)$ so that it satisfies the assumptions of Lemma 5.3. Let us first introduce the function

$$\beta(t) := \begin{cases} 1 & \text{for } 0 \le t \le t_0; \\ \frac{At^{\theta p-\lambda C-M_1}}{Bt^{p_s^*(\alpha)}} & \text{for } t \ge t_1; \\ C^\infty \ \& \ \beta(t) \in [0,1] & \text{for } t_0 \le t \le t_1. \end{cases}$$

Then, it is easy to check that $\beta(t) \in [0,1]$ and $\beta(t) \in C^\infty$. Let $\phi(u) := \beta(\|u\|)$ and we consider the truncated functional $\Phi : \mathbb{X} \to R$ defined as

$$\Phi(u) = \frac{b}{\theta p}\|u\|^p + \frac{1}{q}[u]_{s,q}^q - \frac{\phi(u)}{p_s^*(\alpha)}\int_\Omega \frac{|u|^{p_s^*(\alpha)}}{|x|^\alpha}dx - \frac{\lambda\phi(u)}{r}\int_\Omega f(x)\frac{|u|^r}{|x|^c}dx.$$

In the sequel, we check that $\Phi(u)$ satisfies the assumptions of Lemma 5.3. Obviously,

$$\Phi(u) \ge A\|u\|^{\theta p} - B\phi(u)\|u\|^{p_s^*(\alpha)} - \lambda C := \bar{g}(\|u\|),$$

where $\bar{g}(t) = At^{\theta p} - B\beta(t)t^{p_s^*(\alpha)} - \lambda C$ and

$$\bar{g}(t) = \begin{cases} g(t) & \text{if } 0 \le t \le t_0, \\ M_1 & \text{if } t \ge t_1. \end{cases}$$

By the construction of Φ, the definition of H_α and Lemma 3.6, we verify that Φ enjoys the following properties.

Lemma 5.4. (i) *$\Phi \in C^1(\mathbb{X}, R)$, Φ is even, and bounded from below.*

(ii) *If $\Phi(u) < M_0$, then $\bar{g}(\|u\|) < M_0$, and $\Phi(u) = I(u)$ for $\|u\| < t_0$.*

(iii) *There exists Λ such that for any $\lambda \in (0, \Lambda)$, Φ satisfies a local Palais-Smale condition for $c < M_0 \in (0, M_2)$, where*

$$M_2 = \min\Big\{M_1, \big(\frac{1}{\theta p} - \frac{1}{p_s^*(\alpha)}\big)\big(bH_\alpha^\theta\big)^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-p\theta}} - \lambda^{\frac{p_s^*(\alpha)}{p_s^*(\alpha)-r}}C_0\Big\}$$

with C_0 as in Lemma 3.6.

Lemma 5.5. *Assume that* (A1) *holds. Then, for any $k \in \mathbb{N}$, there exist $\delta = \delta(k) > 0$ such that $\gamma\big(\{u \in \mathbb{X} : \Phi(u) \le -\delta(k)\} \setminus \{0\}\big) \ge k$.*

Proof. Let E_k be a k-dimensional subspace of $\mathbb{X}$. Note that all norms in the finite dimensional space E_k are equivalent, which yields that there exists $\alpha_k > 0$ such that

$$\int_\Omega |x|^{-c}|u|^r dx \ge \alpha_k\|u\|^r \quad \forall u \in E_k.$$

Therefore, for any $u \in E_k$ with $\|u\| = 1$ and sufficiently small d_k we have

$$\begin{aligned} \Phi(d_k u) &\le \frac{b}{\theta p}d_k^{\theta p} + \frac{C}{q}d_k^q - \lambda\frac{\omega_1}{r}\int_\Omega |x|^{-c}|u|^r dx \\ &\le \frac{b}{\theta p}d_k^{\theta p} + \frac{C}{q}d_k^q - \lambda\frac{\omega_1\alpha_k}{r}d_k^r \\ &:= -\delta(k) < 0, \end{aligned}$$

which means that $\{u \in E_k : \|u\| = d_k\} \subset \{u \in \mathbb{X} : \Phi(u) \le -\delta(k)\} \setminus \{0\}$. By Proposition 5.2 (2) then we obtain that $\gamma\big(\{u \in \mathbb{X} : \Phi(u) \le \delta(k)\} \setminus \{0\}\big) \ge \gamma\big(\{u \in \mathbb{X} : \|u\| = d_k\}\big) = \gamma(A)$. Since $A = \{u \in \mathbb{X} : \|u\| = d_k\}$ is a sphere with radius d_k in E_k that is as a k-dimensional subspace of $\mathbb{X}$, it leads to $\gamma(A) = k$ because of Proposition 5.2 (4). This completes the proof. ◻

Finally, we are in the position to prove Theorem 1.4 by way of Lemma 5.3.

Proof of Theorem 1.4. Recall that $\Gamma_k = \{A \in \mathbb{X} \setminus \{0\} : A \text{ is closed and } A = -A, \gamma(A) \geq k\}$ and define

$$c_k = \inf_{A \in \Gamma_k} \sup_{u \in A} \Phi(u).$$

By Lemma 5.4 (i) and Lemma 5.5, we know that $-\infty < c_k < 0$. Therefore, the assumptions (1) and (2) of Lemma 5.3 are satisfied. This means that Φ has a sequence of solutions $\{u_n\}$ converging to zero. Hence, Theorem 1.4 follows from Lemma 5.4(ii). $\square$

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