*Electronic Journal of Differential Equations*, Vol. 2016 (2016), No. 19, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# EXPONENTIAL STABILITY OF SOLUTIONS TO NONLINEAR TIME-DELAY SYSTEMS OF NEUTRAL TYPE

GENNADII V. DEMIDENKO, INESSA I. MATVEEVA

Abstract. We consider a nonlinear time-delay system of neutral equations with constant coefficients in the linear terms

$$\frac{d}{dt}\big(y(t) + Dy(t-\tau)\big) = Ay(t) + By(t-\tau) + F(t, y(t), y(t-\tau)),$$

where

$$\|F(t,u,v)\| \le q_1\|u\|^{1+\omega_1} + q_2\|v\|^{1+\omega_2}, \quad q_1, q_2, \omega_1, \omega_2 > 0.$$

We obtain estimates characterizing the exponential decay of solutions at infinity and estimates for attraction sets of the zero solution.

## 1. Introduction

There is large number of works devoted to the study of delay differential equations (see for instance the books [1, 3, 13, 15, 16, 17, 18, 20, 21, 22, 23, 29, 31] and the bibliography therein). The question of asymptotic stability of solutions is very important from the theoretical and practical viewpoints because delay differential equations arise in many applied problems when describing the processes whose speeds are defined by present and previous states (see for example [14, 24, 25] and the bibliography therein).

This article presents a continuation of our works on stability of solutions to delay differential equations [4, 5, 6, 7, 8, 9, 10, 11, 12, 26, 27]. We consider the system of nonlinear delay differential equations

$$\frac{d}{dt}\big(y(t) + Dy(t-\tau)\big) = Ay(t) + By(t-\tau) + F(t, y(t), y(t-\tau)), \quad t > 0, \tag{1.1}$$

where $A$, $B$, $D$ are constant $(n \times n)$ matrices, $\tau > 0$ is the time delay, and $F$ is a continuous vector function mapping $[0,\infty) \times \mathbb{C}^n \times \mathbb{C}^n$ into $\mathbb{C}^n$. We assume that $F(t,u,v)$ satisfies the Lipschitz condition with respect to $u$ on every compact $G \subset [0,\infty) \times \mathbb{C}^n \times \mathbb{C}^n$ and the inequality

$$\|F(t,u,v)\| \le q_1\|u\|^{1+\omega_1} + q_2\|v\|^{1+\omega_2}, \quad t \ge 0,\ u, v \in \mathbb{C}^n, \tag{1.2}$$

2010 *Mathematics Subject Classification.* 34K20.
*Key words and phrases.* Time-delay systems; neutral equation; exponential stability; attraction sets.

Submitted October 6, 2015. Published January 11, 2016.

for some constants $q_1$, $q_2$, $\omega_1$, $\omega_2 > 0$. Here and hereafter we use the following dot product and vector norm

$$\langle x, z\rangle = \sum_{j=1}^{n} x_j \bar{z}_j, \quad \|x\| = \sqrt{\langle x, x\rangle}.$$

Our aim is to study the exponential stability of the zero solution; namely, to obtain estimates characterizing the decay rate of solutions at infinity and estimates for attraction sets of the zero solution. To establish conditions of stability, researchers often use various Lyapunov or Lyapunov–Krasovskii functionals. At present, there is large number of works in this direction; for example, see the bibliographies in the survey [2] and in the book [31] devoted wholly to obtaining conditions of stability by the use of Lyapunov-Krasovskii functionals. However, not every Lyapunov-Krasovskii functional makes it possible to obtain estimates characterizing exponential decay of solutions at infinity. In recent years, the study in this direction has developed rapidly. For constant coefficients, there are a lot of works for linear delay differential equations including equations of neutral type (for example, see [15, 20] and the bibliography therein).

The case of nonlinear equations is of special interest and is more complicated in comparison with the case of linear equations. Along with estimates of exponential decay of solutions, a very important question is deriving estimates of attraction sets for nonlinear equations. The natural problem is to obtain such estimates by means of the Lyapunov–Krasovskii functionals used for exponential stability analysis of equations defined by the linear part. To the best of our knowledge, the first constructive estimates of attraction sets for the system

$$\frac{d}{dt}y(t) = Ay(t) + By(t-\tau) + F(t, y(t), y(t-\tau)), \tag{1.3}$$

using Lyapunov–Krasovskii functionals associated with the exponentially stable linear system

$$\frac{d}{dt}y(t) = Ay(t) + By(t-\tau), \tag{1.4}$$

were obtained in [4, 5, 6, 28].

To study asymptotic stability of solutions to (1.4) the authors in [4] proposed to use the Lyapunov–Krasovskii functional

$$\langle Hy(t), y(t)\rangle + \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle\, ds, \tag{1.5}$$

where the matrices $H$ and $K(s)$ satisfy

$$H = H^* > 0, \quad K(s) = K^*(s) \in C^1[0,\tau], \quad K(s) > 0, \quad \frac{d}{ds}K(s) < 0, \tag{1.6}$$

for $s \in [0,\tau]$. Here $H > 0$ means that $H$ is positive definite. Using (1.5), we obtained estimates of exponential decay of solutions to linear systems of the form (1.4). In [4, 5] the authors considered nonlinear systems of delay differential equations of the form (1.3) with $F(t, u, v)$ satisfying (1.2). Conditions of asymptotic stability of the zero solution were obtained, estimates characterizing the decay rate at infinity were established, and estimates of attraction sets of the zero solution were derived. Using a generalization of the functional in (1.5), analogous results were obtained for linear and nonlinear systems of delay differential equations with periodic coefficients in the linear terms [4, 5, 6, 11, 26, 27].

To study exponential stability of solutions to the system of linear differential equations of neutral type

$$\frac{d}{dt}(y(t) + Dy(t-\tau)) = Ay(t) + By(t-\tau) \tag{1.7}$$

the first author in [7] introduced the Lyapunov-Krasovskii functional

$$\begin{aligned} V(\varphi) &= \langle H(\varphi(0) + D\varphi(-\tau)), (\varphi(0) + D\varphi(-\tau))\rangle \\ &\quad + \int_{-\tau}^{0} \langle K(-s)\varphi(s), \varphi(s)\rangle \, ds, \quad \varphi(s) \in C[-\tau, 0], \end{aligned} \tag{1.8}$$

where the matrices $H$ and $K(s)$ satisfy (1.6). In particular, the following result was obtained.

**Theorem 1.1.** *Suppose that there exist matrices $H$ and $K(s)$ satisfying* (1.6) *and the matrix*

$$C = -\begin{pmatrix} HA + A^*H + K(0) & HB + A^*HD \\ B^*H + D^*HA & D^*HB + B^*HD - K(\tau) \end{pmatrix} \tag{1.9}$$

*is positive definite. Then the zero solution to* (1.7) *is exponentially stable.*

Using the functional (1.8), the study of exponential stability of solutions to time-delay systems of the form (1.1) was conducted in [7, 8, 9, 10, 12, 30]. Note that in [7, 8, 30] the estimates of exponential decay of solutions to (1.1) were obtained in the case $\|D\| < 1$ (here and hereafter we use the spectral norm of matrices). In [9], for the linear case ($F(t, u, v) \equiv 0$), analogous estimates were established when the spectrum of the matrix $D$ belongs to the unit disk $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$. However, in the case $\|D\| < 1$, the estimates are weaker in comparison with the estimates obtained in [7]. More precise exponential estimates for the linear systems were obtained in [10]. If the spectrum of the matrix $D$ belongs to the unit disk, the authors in [12] investigated the time-delay system (1.1) with $F(t, u, v)$ satisfying (1.2) with $\omega_1 = \omega_2 = 0$.

In this article we study a more complicated case; namely, we consider the nonlinear time-delay system (1.1) if $\omega_1$, $\omega_2 > 0$. Supposing that the spectrum of $D$ belongs to the unit disk, we establish estimates characterizing exponential decay of solutions at infinity and estimates for attraction sets of the zero solution. The main results are formulated in Theorems 2.2–2.8 and their proofs are given in the next section. It should be noted that some sufficient conditions for exponential stability of the zero solution to (1.1) are established in [15, Theorem 7.5] in the case $\|D\| < 1$.

## 2. Main results

Suppose that the conditions of Theorem 1.1 are satisfied. Using the matrices $H$ and $K(s)$, introduce the following notation

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{pmatrix}, \tag{2.1}$$

$$S_{11} = -HA - A^*H - K(0), \quad S_{12} = HAD + K(0)D - HB,$$

$$S_{22} = K(\tau) - D^*K(0)D,$$

$$R = S_{11} - S_{12}S_{22}^{-1}S_{12}^*, \quad P = S_{22}. \tag{2.2}$$

It is not hard to verify that the matrix $C$ in (1.9) is positive definite if and only if the matrix $S$ is positive definite (see the proof of Theorem 2.1 for details). Obviously, $R$ and $P$ are positive definite if and only if the matrix $S$ is positive definite. Denote by $r_{\min} > 0$ and $p_{\min} > 0$ the minimal eigenvalues of $R$ and $P$, respectively. Let $\kappa > 0$ be the maximal number such that

$$\frac{d}{ds}K(s) + \kappa K(s) \le 0, \quad s \in [0, \tau]. \tag{2.3}$$

We consider the initial value problem for (1.1),

$$\begin{gathered}\frac{d}{dt}(y(t) + Dy(t-\tau)) = Ay(t) + By(t-\tau) + F(t, y(t), y(t-\tau)), \quad t > 0,\\ y(t) = \varphi(t), \quad t \in [-\tau, 0],\\ y(+0) = \varphi(0),\end{gathered} \tag{2.4}$$

where $\varphi(t) \in C^1[-\tau, 0]$ is a given vector function. Let $y(t)$ be a noncontinuable solution to the initial value problem (2.4), defined for $t \in [0, t')$. Using the matrices $H$ and $K(s)$ indicated in Theorem 1.1, we consider the Lyapunov-Krasovskii functional (1.8). Introducing the conventional notation

$$y_t : \theta \to y(t+\theta), \quad \theta \in [-\tau, 0],$$

we have

$$\begin{aligned}V(y_t) &= \langle H(y_t(0) + Dy_t(-\tau)), (y_t(0) + Dy_t(-\tau))\rangle + \int_{-\tau}^{0} \langle K(-\theta)y_t(\theta), y_t(\theta)\rangle\, d\theta\\ &= \langle H(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle\\ &\quad + \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle\, ds.\end{aligned} \tag{2.5}$$

**Theorem 2.1.** *Let the conditions of Theorem 1.1 be satisfied. Then*

$$\begin{aligned}\frac{d}{dt}V(y_t) &\le \varepsilon_0 V^{1+\omega_1/2}(y_t) - (r_{\min} - \delta_1(\|y(t-\tau)\|))\|y(t) + Dy(t-\tau)\|^2\\ &\quad - (p_{\min} - \delta_2(\|y(t-\tau)\|))\|z(t)\|^2 - \kappa \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle\, ds,\end{aligned} \tag{2.6}$$

*for $t \in [0, t')$, where*

$$\varepsilon_0 = \frac{2q_1\|H\|(1+\varepsilon_1)^{\omega_1}}{h_{\min}^{1+\omega_1/2}}, \tag{2.7}$$

$$\delta_1(s) = \Big(\|S_{22}^{-1}S_{12}^*\| + \frac{1}{4\varepsilon_2}\Big)\delta_0(s), \quad \delta_2(s) = \varepsilon_2\delta_0(s), \quad s \ge 0, \tag{2.8}$$

$$\begin{gathered}\delta_0(s) = 2\|H\|\Big[q_1\Big(\frac{1+\varepsilon_1}{\varepsilon_1}\Big)^{\omega_1}\|D\|^{1+\omega_1}s^{\omega_1} + q_2 s^{\omega_2}\Big],\\ \varepsilon_1 > 0, \quad \varepsilon_2 > 0,\end{gathered}$$

$$z(t) = S_{22}^{-1}S_{12}^*(y(t) + Dy(t-\tau)) + y(t-\tau), \tag{2.9}$$

*and $h_{\min} > 0$ is the minimal eigenvalue of the matrix $H$.*

*Proof.* We use the proof scheme in [4]. Obviously, the time derivative of the functional $V(y_t)$ is

$$\begin{aligned}\frac{d}{dt}V(y_t) \equiv \;& \langle H(Ay(t)+By(t-\tau)),(y(t)+Dy(t-\tau))\rangle\\ &+\langle H(y(t)+Dy(t-\tau)),(Ay(t)+By(t-\tau))\rangle\\ &+\langle HF(t,y(t),y(t-\tau)),(y(t)+Dy(t-\tau))\rangle\\ &+\langle H(y(t)+Dy(t-\tau)),F(t,y(t),y(t-\tau))\rangle\\ &+\langle K(0)y(t),y(t)\rangle-\langle K(\tau)y(t-\tau),y(t-\tau)\rangle\\ &+\int_{t-\tau}^{t}\langle\frac{d}{dt}K(t-s)y(s),y(s)\rangle ds.\end{aligned}$$

Using the matrix $C$ defined in (1.9), we obtain

$$\begin{aligned}\frac{d}{dt}V(y_t) \equiv \;& -\Big\langle C\begin{pmatrix}y(t)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix}y(t)\\ y(t-\tau)\end{pmatrix}\Big\rangle\\ &+\langle HF(t,y(t),y(t-\tau)),(y(t)+Dy(t-\tau))\rangle\\ &+\langle H(y(t)+Dy(t-\tau)),F(t,y(t),y(t-\tau))\rangle\\ &+\int_{t-\tau}^{t}\langle\frac{d}{dt}K(t-s)y(s),y(s)\rangle ds.\end{aligned}\tag{2.10}$$

We consider the first summand in the right-hand side of (2.10). Obviously,

$$\begin{pmatrix}y(t)\\ y(t-\tau)\end{pmatrix}=\begin{pmatrix}I & -D\\ 0 & I\end{pmatrix}\begin{pmatrix}y(t)+Dy(t-\tau)\\ y(t-\tau)\end{pmatrix}.$$

Then

$$\Big\langle C\begin{pmatrix}y(t)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix}y(t)\\ y(t-\tau)\end{pmatrix}\Big\rangle \equiv \Big\langle S\begin{pmatrix}y(t)+Dy(t-\tau)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix}y(t)+Dy(t-\tau)\\ y(t-\tau)\end{pmatrix}\Big\rangle,$$

where

$$S=\begin{pmatrix}I & 0\\ -D^* & I\end{pmatrix}C\begin{pmatrix}I & -D\\ 0 & I\end{pmatrix}.$$

Taking into account (1.9), the matrix $S$ has the form (2.1). By the conditions of Theorem 1.1, the matrix $C$ is positive definite. Clearly, $S$ is positive definite if and only if $C$ is positive definite. Using the representation

$$S=\begin{pmatrix}I & S_{12}S_{22}^{-1}\\ 0 & I\end{pmatrix}\begin{pmatrix}S_{11}-S_{12}S_{22}^{-1}S_{12}^* & 0\\ 0 & S_{22}\end{pmatrix}\begin{pmatrix}I & 0\\ S_{22}^{-1}S_{12}^* & I\end{pmatrix},$$

we have

$$\begin{aligned}&\Big\langle S\begin{pmatrix}y(t)+Dy(t-\tau)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix}y(t)+Dy(t-\tau)\\ y(t-\tau)\end{pmatrix}\Big\rangle\\ &=\langle R(y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\rangle+\langle Pz(t),z(t)\rangle,\end{aligned}$$

where $R$, $P$ and $z(t)$ are defined by (2.2) and (2.9), respectively. Obviously, the matrix $S$ is positive definite if and only if the matrices $R$ and $P$ are positive definite. Consequently, we derive

$$\Big\langle C\begin{pmatrix}y(t)\\ y(t-\tau)\end{pmatrix},\begin{pmatrix}y(t)\\ y(t-\tau)\end{pmatrix}\Big\rangle \ge r_{\min}\|y(t)+Dy(t-\tau)\|^2+p_{\min}\|z(t)\|^2,\tag{2.11}$$

where $r_{\min}>0$ and $p_{\min}>0$ are the minimal eigenvalues of $R$ and $P$, respectively.

We consider the second and the third summands in the right-hand side of (2.10). In view of (1.2) we have

$$\begin{aligned}
&\langle HF(t,y(t),y(t-\tau)),(y(t)+Dy(t-\tau))\rangle\\
&+\langle H(y(t)+Dy(t-\tau)),F(t,y(t),y(t-\tau))\rangle\\
&\le 2\|H\|\big(q_1\|y(t)\|^{1+\omega_1}+q_2\|y(t-\tau)\|^{1+\omega_2}\big)\|y(t)+Dy(t-\tau)\|\\
&\le 2\|H\|\big(q_1(\|y(t)+Dy(t-\tau)\|+\|D\|\|y(t-\tau)\|)^{1+\omega_1}+q_2\|y(t-\tau)\|^{1+\omega_2}\big)\\
&\quad\times\|y(t)+Dy(t-\tau)\|.
\end{aligned}$$

It is not hard to show that

$$(a+b)^{1+\omega}\le(1+\varepsilon_1)^{\omega}a^{1+\omega}+\big(\frac{1+\varepsilon_1}{\varepsilon_1}\big)^{\omega}b^{1+\omega},\quad a,b\ge 0,\ \varepsilon_1>0.$$

Hence,

$$\begin{aligned}
&(\|y(t)+Dy(t-\tau)\|+\|D\|\|y(t-\tau)\|)^{1+\omega_1}\\
&\le(1+\varepsilon_1)^{\omega_1}\|y(t)+Dy(t-\tau)\|^{1+\omega_1}+\big(\frac{1+\varepsilon_1}{\varepsilon_1}\big)^{\omega_1}\|D\|^{1+\omega_1}\|y(t-\tau)\|^{1+\omega_1}.
\end{aligned}$$

For example, choosing $\varepsilon_1=\|D\|$, we have

$$\begin{aligned}
&(\|y(t)+Dy(t-\tau)\|+\|D\|\|y(t-\tau)\|)^{1+\omega_1}\\
&\le(1+\|D\|)^{\omega_1}\|y(t)+Dy(t-\tau)\|^{1+\omega_1}+(1+\|D\|)^{\omega_1}\|D\|\|y(t-\tau)\|^{1+\omega_1}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\langle HF(t,y(t),y(t-\tau)),(y(t)+Dy(t-\tau))\rangle\\
&+\langle H(y(t)+Dy(t-\tau)),F(t,y(t),y(t-\tau))\rangle\\
&\le 2q_1\|H\|(1+\varepsilon_1)^{\omega_1}\|y(t)+Dy(t-\tau)\|^{2+\omega_1}\\
&\quad+2\|H\|\Big[q_1\Big(\frac{1+\varepsilon_1}{\varepsilon_1}\Big)^{\omega_1}\|D\|^{1+\omega_1}\|y(t-\tau)\|^{\omega_1}+q_2\|y(t-\tau)\|^{\omega_2}\Big]\\
&\quad\times\|y(t-\tau)\|\|y(t)+Dy(t-\tau)\|.
\end{aligned}$$

By the definition of $z(t)$,

$$\|y(t-\tau)\|\le\|S_{22}^{-1}S_{12}^*\|\|y(t)+Dy(t-\tau)\|+\|z(t)\|.$$

Hence,

$$\begin{aligned}
&\|y(t-\tau)\|\|y(t)+Dy(t-\tau)\|\\
&\le\|S_{22}^{-1}S_{12}^*\|\|y(t)+Dy(t-\tau)\|^2+\|z(t)\|\|y(t)+Dy(t-\tau)\|\\
&\le\Big(\|S_{22}^{-1}S_{12}^*\|+\frac{1}{4\varepsilon_2}\Big)\|y(t)+Dy(t-\tau)\|^2+\varepsilon_2\|z(t)\|^2,\quad\varepsilon_2>0.
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\langle HF(t,y(t),y(t-\tau)),(y(t)+Dy(t-\tau))\rangle\\
&+\langle H(y(t)+Dy(t-\tau)),F(t,y(t),y(t-\tau))\rangle\\
&\le 2q_1\|H\|(1+\varepsilon_1)^{\omega_1}\|y(t)+Dy(t-\tau)\|^{2+\omega_1}\\
&\quad+\delta_1(\|y(t-\tau)\|)\|y(t)+Dy(t-\tau)\|^2+\delta_2(\|y(t-\tau)\|)\|z(t)\|^2,
\end{aligned}\tag{2.12}$$

where $\delta_1(s)$, $\delta_2(s)$ are defined by (2.8).

By (2.11) and (2.12), we have

$$
\begin{aligned}
&-\Big\langle C\begin{pmatrix} y(t)\\ y(t-\tau)\end{pmatrix}, \begin{pmatrix} y(t)\\ y(t-\tau)\end{pmatrix}\Big\rangle\\
&+\langle HF(t,y(t),y(t-\tau)),(y(t)+Dy(t-\tau))\rangle\\
&+\langle H(y(t)+Dy(t-\tau)),F(t,y(t),y(t-\tau))\rangle\\
&\le 2q_1\|H\|(1+\varepsilon_1)^{\omega_1}\|y(t)+Dy(t-\tau)\|^{2+\omega_1}\\
&\quad -(r_{\min}-\delta_1(\|y(t-\tau)\|))\|y(t)+Dy(t-\tau)\|^2-(p_{\min}-\delta_2(\|y(t-\tau)\|))\|z(t)\|^2.
\end{aligned}
$$

It follows from (2.10) that

$$
\begin{aligned}
\frac{d}{dt}V(y_t)&\le 2q_1\|H\|(1+\varepsilon_1)^{\omega_1}\|y(t)+Dy(t-\tau)\|^{2+\omega_1}\\
&\quad -(r_{\min}-\delta_1(\|y(t-\tau)\|))\|y(t)+Dy(t-\tau)\|^2\\
&\quad -(p_{\min}-\delta_2(\|y(t-\tau)\|))\|z(t)\|^2\\
&\quad +\int_{t-\tau}^{t}\Big\langle\frac{d}{dt}K(t-s)y(s),y(s)\Big\rangle ds.
\end{aligned}
$$

By (2.3), we obtain

$$
\begin{aligned}
\frac{d}{dt}V(y_t)&\le 2q_1\|H\|(1+\varepsilon_1)^{\omega_1}\|y(t)+Dy(t-\tau)\|^{2+\omega_1}\\
&\quad -(r_{\min}-\delta_1(\|y(t-\tau)\|))\|y(t)+Dy(t-\tau)\|^2\\
&\quad -(p_{\min}-\delta_2(\|y(t-\tau)\|))\|z(t)\|^2-\kappa\int_{t-\tau}^{t}\langle K(t-s)y(s),y(s)\rangle\,ds.
\end{aligned}
$$

Using the matrix $H$, we have

$$
\begin{aligned}
&\frac{1}{\|H\|}\big\langle H(y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\big\rangle\\
&\le\|y(t)+Dy(t-\tau)\|^2\\
&\le\frac{1}{h_{\min}}\big\langle H(y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\big\rangle,
\end{aligned}
\tag{2.13}
$$

where $h_{\min}>0$ is the minimal eigenvalue of $H$. Hence,

$$
\begin{aligned}
\frac{d}{dt}V(y_t)&\le\frac{2q_1\|H\|(1+\varepsilon_1)^{\omega_1}}{h_{\min}^{1+\omega_1/2}}\big\langle H(y(t)+Dy(t-\tau)),(y(t)+Dy(t-\tau))\big\rangle^{1+\omega_1/2}\\
&\quad -(r_{\min}-\delta_1(\|y(t-\tau)\|))\|y(t)+Dy(t-\tau)\|^2\\
&\quad -(p_{\min}-\delta_2(\|y(t-\tau)\|))\|z(t)\|^2-\kappa\int_{t-\tau}^{t}\langle K(t-s)y(s),y(s)\rangle\,ds.
\end{aligned}
$$

By the definition of $V(y_t)$, we obtain

$$
\begin{aligned}
\frac{d}{dt}V(y_t)&\le\varepsilon_0V^{1+\omega_1/2}(y_t)-(r_{\min}-\delta_1(\|y(t-\tau)\|))\|y(t)+Dy(t-\tau)\|^2\\
&\quad -(p_{\min}-\delta_2(\|y(t-\tau)\|))\|z(t)\|^2-\kappa\int_{t-\tau}^{t}\langle K(t-s)y(s),y(s)\rangle\,ds,
\end{aligned}
$$

where $\varepsilon_0$ is defined by (2.7). The proof is complete. □

The parameters $\varepsilon_1$, $\varepsilon_2 > 0$ allow us to control $\varepsilon_0$, $\delta_1(s)$, $\delta_2(s)$ in (2.6).

Let $\sigma > 0$ be a number such that

$$\delta_0(\sigma) < \min\Big\{\frac{r_{\min}}{\big(\|S_{22}^{-1}S_{12}^*\| + \frac{1}{4\varepsilon_2}\big)}, \frac{p_{\min}}{\varepsilon_2}\Big\}.$$

Then $r_{\min} - \delta_1(\sigma) > 0$ and $p_{\min} - \delta_2(\sigma) > 0$. We introduce the notation

$$\gamma = \min\{r_{\min} - \delta_1(\sigma), \kappa\|H\|\} > 0, \quad \beta = \frac{\gamma}{2\|H\|}. \tag{2.14}$$

Since the spectrum of the matrix $D$ belongs to the unit disk $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$, it follows that $\|D^j\| \to 0$ as $j \to \infty$. Let $l > 0$ be the minimal integer such that $\|D^l\| < 1$. We distinguish three cases

$$\|D^l\| < e^{-l\beta\tau}, \quad \|D^l\| = e^{-l\beta\tau}, \quad e^{-l\beta\tau} < \|D^l\| < 1.$$

We describe in every case an attraction set for the initial function $\varphi(t)$ and show that the solution to the initial value problem (2.4) with this function is defined for $t > 0$. We establish estimates characterizing exponentially decay of this solution at infinity.

**Theorem 2.2.** *Let the conditions of Theorem 1.1 be satisfied and*

$$\|D^l\| < e^{-l\beta\tau}. \tag{2.15}$$

*Suppose that $\varphi(t) \in \mathcal{E}_1$, where*

$$\begin{gathered}\mathcal{E}_1 = \Big\{\varphi(s) \in C^1[-\tau, 0] : \Phi < \sigma,\ V(\varphi) < \Big(\frac{\gamma}{\varepsilon_0\|H\|}\Big)^{2/\omega_1}, \\ \alpha\big(1 - \|D^l\|e^{l\beta\tau}\big)^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau} + \max\{\|D\|, \dots, \|D^l\|\}\Phi < \sigma\Big\},\end{gathered} \tag{2.16}$$

$$\alpha = \big[1 - \frac{\varepsilon_0\|H\|}{\gamma}V^{\omega_1/2}(\varphi)\big]^{-1/\omega_1}\sqrt{\frac{V(\varphi)}{h_{\min}}}, \quad \Phi = \max_{s\in[-\tau,0]}\|\varphi(s)\|. \tag{2.17}$$

*Then the solution to the initial value problem* (2.4) *is defined for $t > 0$ and*

$$\begin{aligned}\|y(t)\| \le \Big[&\alpha\big(1 - \|D^l\|e^{l\beta\tau}\big)^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau} \\ &+ \max\big\{\|D\|e^{\beta\tau}, \dots, \|D^l\|e^{l\beta\tau}\big\}\,\Phi\Big]e^{-\beta t}, \quad t > 0.\end{aligned} \tag{2.18}$$

The proof of the above theorem is based on the next two lemmas.

**Lemma 2.3.** *Let*

$$\|D^l\| < e^{-l\beta\tau}.$$

*Then*

$$\begin{aligned}&\alpha\sum_{j=0}^{k}\|D^j\|e^{-\beta(t-j\tau)} + \|D^{k+1}\|\Phi \\ &\le \Big[\alpha\big(1 - \|D^l\|e^{l\beta\tau}\big)^{-1}\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau} \\ &\quad + \max\big\{\|D\|e^{\beta\tau}, \dots, \|D^l\|e^{l\beta\tau}\big\}\,\Phi\Big]e^{-\beta t}\end{aligned} \tag{2.19}$$

*for $t \in [k\tau, (k+1)\tau)$, $k = 0, 1, \dots,$ where $\alpha$, $\beta$, and $\Phi$ are defined in* (2.14), (2.17).

*Proof.* Obviously,

$$
\begin{aligned}
&\alpha \sum_{j=0}^{k} \|D^j\| e^{-\beta(t-j\tau)} + \|D^{k+1}\|\Phi \\
&\le \Big[\alpha \sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} + \|D^{k+1}\| e^{(k+1)\beta\tau}\Phi\Big] e^{-\beta t}, \quad t \in [k\tau, (k+1)\tau).
\end{aligned}
$$

In view of the condition on $\|D^l\|$, we obtain the estimate

$$
\begin{aligned}
&\alpha \sum_{j=0}^{k} \|D^j\| e^{-\beta(t-j\tau)} + \|D^{k+1}\|\Phi \\
&\le \Big[\alpha \sum_{j=0}^{\infty} \|D^j\| e^{j\beta\tau} + \max\big\{\|D\| e^{\beta\tau}, \dots, \|D^l\| e^{l\beta\tau}\big\}\,\Phi\Big] e^{-\beta t}.
\end{aligned} \tag{2.20}
$$

We consider the series $\sum_{j=0}^{\infty} \|D^j\| e^{j\beta\tau}$. Obviously,

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \|D^j\| e^{j\beta\tau} \\
&= \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \sum_{j=l}^{2l-1} \|D^j\| e^{j\beta\tau} + \sum_{j=2l}^{3l-1} \|D^j\| e^{j\beta\tau} + \dots \\
&\le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \|D^l\| e^{l\beta\tau} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \big(\|D^l\| e^{l\beta\tau}\big)^2 \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \dots \\
&= \Big(1 + \|D^l\| e^{l\beta\tau} + \big(\|D^l\| e^{l\beta\tau}\big)^2 + \dots\Big) \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.
\end{aligned}
$$

By (2.15), we have

$$
\sum_{j=0}^{\infty} \|D^j\| e^{j\beta\tau} \le \big(1 - \|D^l\| e^{l\beta\tau}\big)^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.
$$

Using this inequality, we derive (2.19) from (2.20). The proof is complete. □

**Lemma 2.4.** *Let the conditions of Theorem 2.2 be satisfied. Then the solution to the initial value problem* (2.4) *is defined for $t > 0$; moreover, on each segment $t \in [k\tau, (k+1)\tau)$, $k = 0, 1, \dots$, it satisfies the estimate*

$$
\|y(t)\| \le \alpha \sum_{j=0}^{k} \|D^j\| e^{-\beta(t-j\tau)} + \|D^{k+1}\|\Phi, \tag{2.21}
$$

*where $\alpha$, $\beta$, and $\Phi$ are defined in* (2.14), (2.17).

*Proof.* Let $t \in [0, \tau)$. If the initial function $\varphi(t)$ belongs to the attraction set $\mathcal{E}_1$ defined by (2.16), then

$$
\max_{s \in [-\tau, 0]} \|\varphi(s)\| < \sigma\,.
$$

Consequently,

$$
r_{\min} - \delta_1(\|y(t-\tau)\|) = r_{\min} - \delta_1(\|\varphi(t-\tau)\|) > r_{\min} - \delta_1(\sigma) > 0,
$$

$$p_{\min} - \delta_2(\|y(t-\tau)\|) = p_{\min} - \delta_2(\|\varphi(t-\tau)\|) > p_{\min} - \delta_2(\sigma) > 0.$$

By Theorem 2.1,

$$\begin{aligned}\frac{d}{dt}V(y_t) &\le \varepsilon_0 V^{1+\omega_1/2}(y_t) - (r_{\min} - \delta_1(\sigma))\|y(t) + Dy(t-\tau)\|^2 \\ &\quad - \kappa \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle\, ds\end{aligned}$$

for $t \in [0, t_1)$, where $t_1 = \min\{\tau, t'\}$. Using (2.13), we have

$$\begin{aligned}&\frac{d}{dt}V(y_t) \\ &\le \varepsilon_0 V^{1+\omega_1/2}(y_t) - \frac{r_{\min} - \delta_1(\sigma)}{\|H\|}\langle H(y(t) + Dy(t-\tau)), (y(t) + Dy(t-\tau))\rangle \\ &\quad - \kappa \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle ds.\end{aligned}$$

Taking into account (2.5), we obtain

$$\frac{d}{dt}V(y_t) \le \varepsilon_0 V^{1+\omega_1/2}(y_t) - \frac{\gamma}{\|H\|}V(y_t),$$

where $\gamma$ is defined in (2.14). If $\varphi(t) \in \mathcal{E}_1$ then

$$\frac{\varepsilon_0\|H\|}{\gamma}V^{\omega_1/2}(\varphi) < 1.$$

By a Gronwall-like inequality (for example, see [19]), we have the estimate

$$V(y_t) \le \big[1 - \frac{\varepsilon_0\|H\|}{\gamma}V^{\omega_1/2}(\varphi)\big]^{-2/\omega_1} V(\varphi)\exp\big(-\frac{\gamma t}{\|H\|}\big).$$

Using the definition of the functional (2.5), we obtain

$$\|y(t) + Dy(t-\tau)\| \le \sqrt{\frac{V(y_t)}{h_{\min}}}.$$

Consequently,

$$\|y(t) + Dy(t-\tau)\| \le \Big[1 - \frac{\varepsilon_0\|H\|}{\gamma}V^{\omega_1/2}(\varphi)\Big]^{-1/\omega_1}\sqrt{\frac{V(\varphi)}{h_{\min}}}\exp\Big(-\frac{\gamma t}{2\|H\|}\Big).$$

Hence,

$$\begin{aligned}\|y(t)\| &\le \|y(t) + Dy(t-\tau)\| + \|Dy(t-\tau)\| \\ &\le \alpha e^{-\beta t} + \|D\|\,\|\varphi(t-\tau)\| \le \alpha e^{-\beta t} + \|D\|\Phi, \quad t \in [0, t_1),\end{aligned} \tag{2.22}$$

where $\alpha$, $\beta$, and $\Phi$ are defined in (2.14) and (2.17). The function in the right-hand side of (2.22) is continuous and bounded for $t > 0$. Then $t_1 = \tau$ and the noncontinuable solution $y(t)$ to (2.4) is defined for $t \in [0, \tau]$. Consequently, $t' > \tau$. It follows from (2.22) that $y(t)$ satisfies (2.21) for $k = 0$. Obviously, if the initial function $\varphi(t)$ belongs to the attraction set $\mathcal{E}_1$ then

$$\|y(t)\| < \sigma, \quad t \in [0, \tau].$$

Let $t \in [\tau, 2\tau)$. Consequently,

$$r_{\min} - \delta_1(\|y(t-\tau)\|) > r_{\min} - \delta_1(\sigma) > 0,$$

$$p_{\min} - \delta_2(\|y(t-\tau)\|) > p_{\min} - \delta_2(\sigma) > 0.$$

By Theorem 2.1,

$$\begin{aligned}\frac{d}{dt}V(y_t) \le \varepsilon_0 V^{1+\omega_1/2}(y_t) - (r_{\min} - \delta_1(\sigma))\|y(t) + Dy(t-\tau)\|^2 \\ - \kappa \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle\, ds\end{aligned}$$

for $t \in [0, t_2)$, where $t_2 = \min\{2\tau, t'\}$. Repeating the same reasoning as above, we have

$$\|y(t) + Dy(t-\tau)\| \le \alpha e^{-\beta t}, \quad t \in [0, t_2),$$

where $\alpha$ and $\beta$ are defined in (2.17) and (2.14), respectively. Hence,

$$\begin{aligned}\|y(t)\| &\le \|y(t) + Dy(t-\tau)\| + \|Dy(t-\tau)\| \\ &\le \alpha e^{-\beta t} + \|Dy(t-\tau) - D^2 y(t-2\tau)\| + \|D^2 y(t-2\tau)\| \\ &\le \alpha e^{-\beta t} + \alpha\|D\| e^{-\beta(t-\tau)} + \|D^2\|\Phi, \quad t \in [\tau, t_2).\end{aligned} \tag{2.23}$$

The function in the right-hand side of (2.23) is continuous and bounded for $t > 0$. Then $t_2 = 2\tau$ and the noncontinuable solution $y(t)$ to (2.4) is defined for $t \in [0, 2\tau]$. Consequently, $t' > 2\tau$. It follows from (2.23) that $y(t)$ satisfies (2.21) for $k = 1$. By Lemma 2.3,

$$\begin{aligned}&\alpha e^{-\beta t} + \alpha\|D\| e^{-\beta(t-\tau)} + \|D^2\|\Phi \\ &\le \Big[\alpha\big(1 - \|D^l\| e^{l\beta\tau}\big)^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}\Big] e^{-\beta t} + \|D^2\|\Phi, \quad t \in [\tau, 2\tau].\end{aligned}$$

Consequently, if the initial function $\varphi(t)$ belongs to the attraction set $\mathcal{E}_1$, then

$$\|y(t)\| < \sigma, \quad t \in [0, 2\tau].$$

Repeating the same reasoning, we obtain that the solution to (2.4) is defined for $t > 0$, and it satisfies (2.21) on each segment $t \in [k\tau, (k+1)\tau)$, $k \in \mathbb{N}$. By Lemma 2.3 and the condition $\|D^l\| < 1$, we have

$$\|y(t)\| \le \Big[\alpha\big(1 - \|D^l\| e^{l\beta\tau}\big)^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}\Big] e^{-\beta t} + \max\{\|D\|, \dots, \|D^l\|\}\Phi, \quad t > 0.$$

Consequently, if the initial function $\varphi(t)$ belongs to the attraction set $\mathcal{E}_1$, then

$$\|y(t)\| < \sigma, \quad t > 0.$$

The proof is complete. □

By Lemmas 2.3 and 2.4, we obtain that the solution to (2.4) satisfies (2.18). Therefore, Theorem 2.2 is proved.

**Theorem 2.5.** *Let the conditions of Theorem 1.1 be satisfied and*

$$\|D^l\| = e^{-l\beta\tau}. \tag{2.24}$$

*Suppose that* $\varphi(t) \in \mathcal{E}_2$*, where*

$$\begin{aligned}\mathcal{E}_2 = \Big\{&\varphi(s) \in C^1[-\tau, 0] : \Phi < \sigma,\ V(\varphi) < \big(\frac{\gamma}{\varepsilon_0 \|H\|}\big)^{2/\omega_1},\\ &\frac{\alpha}{l\tau\beta} e^{l\tau\beta - 1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \max\{\|D\|, \dots, \|D^l\|\}\Phi < \sigma\Big\}.\end{aligned} \tag{2.25}$$

*Then the solution to the initial value problem* (2.4) *is defined for* $t > 0$ *and the estimate holds*

$$\begin{aligned}\|y(t)\| \le \Big[&\alpha\big(1 + \frac{t}{l\tau}\big) \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}\\ &+ \max\Big\{1, \|D\| e^{\beta\tau}, \dots, \|D^{l-1}\| e^{(l-1)\beta\tau}\Big\}\,\Phi\Big] e^{-\beta t}, \quad t > 0,\end{aligned} \tag{2.26}$$

*where* $\alpha$*,* $\beta$*, and* $\Phi$ *are defined in* (2.14)*,* (2.17)*.*

The proof of the above theorem is based on the next two lemmas.

**Lemma 2.6.** *Let*

$$\|D^l\| = e^{-l\beta\tau}.$$

*Then*

$$\begin{aligned}&\alpha \sum_{j=0}^{k} \|D^j\| e^{-\beta(t - j\tau)} + \|D^{k+1}\|\Phi\\ &\le \Big[\alpha\big(1 + \frac{t}{l\tau}\big) \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}\\ &\quad + \max\Big\{1, \|D\| e^{\beta\tau}, \dots, \|D^{l-1}\| e^{(l-1)\beta\tau}\Big\}\,\Phi\Big] e^{-\beta t}\end{aligned} \tag{2.27}$$

*for* $t \in [k\tau, (k+1)\tau)$*,* $k = 0, 1, \dots$*. where* $\alpha$*,* $\beta$*, and* $\Phi$ *are defined in* (2.14)*,* (2.17)*.*

*Proof.* Obviously,

$$\begin{aligned}&\alpha \sum_{j=0}^{k} \|D^j\| e^{-\beta(t - j\tau)} + \|D^{k+1}\|\Phi\\ &\le \Big[\alpha \sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} + \|D^{k+1}\| e^{(k+1)\beta\tau}\Phi\Big] e^{-\beta t}, \quad t \in [k\tau, (k+1)\tau).\end{aligned}$$

In view of the condition (2.24) on $\|D^l\|$, we obtain the estimate

$$\begin{aligned}&\alpha \sum_{j=0}^{k} \|D^j\| e^{-\beta(t - j\tau)} + \|D^{k+1}\|\Phi\\ &\le \Big[\alpha \sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} + \max\Big\{1, \|D\| e^{\beta\tau}, \dots, \|D^{l-1}\| e^{(l-1)\beta\tau}\Big\}\,\Phi\Big] e^{-\beta t}.\end{aligned} \tag{2.28}$$

If $k \le l - 1$ then (2.27) follows from (2.28).

Let $l \le k \le 2l-1$; i.e., $1 \le \frac{t}{l\tau} < 2$. We consider the sum $\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau}$. Clearly,

$$\begin{aligned}
\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} &= \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \sum_{j=l}^{k} \|D^j\| e^{j\beta\tau} \\
&\le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \|D^l\| e^{l\beta\tau} \sum_{j=0}^{k-l} \|D^j\| e^{j\beta\tau} \\
&= \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \sum_{j=0}^{k-l} \|D^j\| e^{j\beta\tau}.
\end{aligned}$$

Then we have

$$\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \frac{t}{l\tau} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.$$

Using this inequality, (2.27) follows from (2.28).

Let $ml \le k \le (m+1)l-1$, $m = 2, 3, \dots$; i.e., $m \le \frac{t}{l\tau} < m+1$. We consider the sum $\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau}$. It is not difficult to see that

$$\begin{aligned}
&\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \\
&= \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \sum_{j=l}^{2l-1} \|D^j\| e^{j\beta\tau} + \dots + \sum_{j=ml}^{k} \|D^j\| e^{j\beta\tau} \\
&\le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \|D^l\| e^{l\beta\tau} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \dots + \|D^{ml}\| e^{ml\beta\tau} \sum_{j=0}^{k-ml} \|D^j\| e^{j\beta\tau} \\
&\le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \dots + \sum_{j=0}^{k-ml} \|D^j\| e^{j\beta\tau} \\
&\le (1+m) \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.
\end{aligned}$$

Consequently,

$$\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \le \big(1 + \frac{t}{l\tau}\big) \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.$$

In view of this estimate, (2.27) follows from (2.28). Owing to the arbitrariness of $m$, the proof is complete. □

**Lemma 2.7.** *Let the conditions of Theorem 2.5 be satisfied. Then the solution to the initial value problem* (2.4) *is defined for $t > 0$; moreover, it satisfies* (2.21) *on each segment $t \in [k\tau, (k+1)\tau)$, $k = 0, 1, \dots$.*

*Proof.* Recall that the noncontinuable solution to (2.4) is defined for $t \in [0, t')$. Let $t \in [0, \tau)$. Repeating the same reasoning as in the proof of Lemma 2.4, we obtain that $y(t)$ satisfies (2.22) and is defined for $t \in [0, \tau]$. Consequently, $t' > \tau$. It follows from (2.22) that $y(t)$ satisfies (2.21) for $k = 0$. Obviously, if the initial function $\varphi(t)$ belongs to the attraction set $\mathcal{E}_2$ defined by (2.25), then

$$\|y(t)\| < \sigma, \quad t \in [0, \tau].$$

Let $t \in [\tau, 2\tau)$. Consequently,

$$\begin{aligned}
r_{\min} - \delta_1(\|y(t-\tau)\|) > r_{\min} - \delta_1(\sigma) > 0,\\
p_{\min} - \delta_2(\|y(t-\tau)\|) > p_{\min} - \delta_2(\sigma) > 0.
\end{aligned}$$

By Theorem 2.1,

$$\begin{aligned}
\frac{d}{dt}V(y_t) &\le \varepsilon_0 V^{1+\omega_1/2}(y_t) - (r_{\min} - \delta_1(\sigma))\|y(t) + Dy(t-\tau)\|^2\\
&\quad - \kappa \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle\, ds
\end{aligned}$$

for $t \in [0, t_2)$, where $t_2 = \min\{2\tau, t'\}$. By a similar way as in the proof of Lemma 2.4, we have

$$\|y(t) + Dy(t-\tau)\| \le \alpha e^{-\beta t}, \quad t \in [0, t_2),$$

where $\alpha$ and $\beta$ are defined in (2.17) and (2.14), respectively. Hence,

$$\begin{aligned}
\|y(t)\| &\le \|y(t) + Dy(t-\tau)\| + \|Dy(t-\tau)\|\\
&\le \alpha e^{-\beta t} + \|Dy(t-\tau) - D^2y(t-2\tau)\| + \|D^2y(t-2\tau)\|\\
&\le \alpha e^{-\beta t} + \alpha\|D\|e^{-\beta(t-\tau)} + \|D^2\|\Phi, \quad t \in [\tau, t_2).
\end{aligned} \tag{2.29}$$

The function in the right-hand side of (2.29) is continuous and bounded for $t > 0$. Then $t_2 = 2\tau$ and the noncontinuable solution $y(t)$ to (2.4) is defined for $t \in [0, 2\tau]$. Consequently, $t' > 2\tau$. It follows from (2.29) that $y(t)$ satisfies (2.21) for $k = 1$. By Lemma 2.6,

$$\begin{aligned}
&\alpha e^{-\beta t} + \alpha\|D\|e^{-\beta(t-\tau)} + \|D^2\|\Phi\\
&\le \Big[\alpha\left(1 + \frac{t}{l\tau}\right)\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau}\Big]e^{-\beta t} + \|D^2\|\Phi, \quad t \in [\tau, 2\tau].
\end{aligned} \tag{2.30}$$

We consider the function

$$f(t) = \big(1 + \frac{t}{l\tau}\big)e^{-\beta t}, \quad t \ge 0.$$

It is not difficult to show that

$$f(t) \le \begin{cases} \frac{1}{l\tau\beta}e^{l\tau\beta - 1}, & l\tau\beta \le 1,\\ 1, & l\tau\beta \ge 1.\end{cases}$$

Obviously, $\frac{1}{l\tau\beta}e^{l\tau\beta-1} \ge 1$ for $\tau$, $\beta > 0$. Then

$$f(t) \le \frac{1}{l\tau\beta}e^{l\tau\beta-1}, \quad t \ge 0,$$

for any $\tau$, $\beta > 0$. Taking into account the last inequality, from (2.30) we have

$$\begin{aligned}
&\alpha e^{-\beta t} + \alpha\|D\|e^{-\beta(t-\tau)} + \|D^2\|\Phi\\
&\le \frac{\alpha}{l\tau\beta}e^{l\tau\beta-1}\sum_{j=0}^{l-1}\|D^j\|e^{j\beta\tau} + \|D^2\|\Phi, \quad t \in [\tau, 2\tau].
\end{aligned}$$

Consequently, if the initial function $\varphi(t)$ belongs to the attraction set $\mathcal{E}_2$ then

$$\|y(t)\| < \sigma, \quad t \in [0, 2\tau].$$

Repeating the same reasoning, we obtain that the solution to (2.4) is defined for $t > 0$, and it satisfies (2.21) on each segment $t \in [k\tau, (k+1)\tau)$, $k \in \mathbb{N}$. By Lemma 2.6 and the condition $\|D^l\| < 1$, we have

$$\begin{aligned}
\|y(t)\| &\leq \Big[\alpha\big(1 + \frac{t}{l\tau}\big) \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}\Big] e^{-\beta t} + \max\{\|D\|, \dots, \|D^l\|\}\Phi \\
&\leq \frac{\alpha}{l\tau\beta} e^{l\tau\beta - 1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \max\{\|D\|, \dots, \|D^l\|\}\Phi, \quad t > 0.
\end{aligned}$$

Consequently, if the initial function $\varphi(t)$ belongs to the attraction set $\mathcal{E}_2$ then

$$\|y(t)\| < \sigma, \quad t > 0.$$

The proof is complete. □

By Lemmas 2.6 and 2.7, we obtain that the solution to (2.4) satisfies (2.26). Therefore, Theorem 2.5 is proved.

**Theorem 2.8.** *Let the conditions of Theorem 1.1 be satisfied and*

$$e^{-l\beta\tau} < \|D^l\| < 1. \tag{2.31}$$

*Suppose that $\varphi(t) \in \mathcal{E}_3$, where*

$$\begin{aligned}
\mathcal{E}_3 = \Big\{&\varphi(s) \in C^1[-\tau, 0] : \Phi < \sigma,\ V(\varphi) < \big(\frac{\gamma}{\varepsilon_0\|H\|}\big)^{2/\omega_1}, \\
&\alpha\Big(1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\Big)^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \max\{\|D\|, \dots, \|D^l\|\}\Phi < \sigma\Big\}.
\end{aligned} \tag{2.32}$$

*Then the solution to the initial value problem* (2.4) *is defined for $t > 0$ and*

$$\begin{aligned}
\|y(t)\| \leq &\Big[\alpha\Big(1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\Big)^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} \\
&+ \|D^l\|^{\frac{1}{l}-1} \max\big\{1, \|D\|, \dots, \|D^{l-1}\|\big\}\,\Phi\Big] \exp\Big(\frac{t}{l\tau} \ln \|D^l\|\Big),
\end{aligned} \tag{2.33}$$

*for $t > 0$, where $\alpha$, $\beta$, and $\Phi$ are defined in* (2.14), (2.17).

The proof of the above theorem is based on the next two lemmas.

**Lemma 2.9.** *Let*

$$e^{-l\beta\tau} < \|D^l\| < 1.$$

*Then*

$$\begin{aligned}
&\alpha \sum_{j=0}^{k} \|D^j\| e^{-\beta(t - j\tau)} + \|D^{k+1}\|\Phi \\
&\leq \Big[\alpha\Big(1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\Big)^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} \\
&\quad + \|D^l\|^{\frac{1}{l}-1} \max\big\{1, \|D\|, \dots, \|D^{l-1}\|\big\}\,\Phi\Big] \exp\Big(\frac{t}{l\tau} \ln \|D^l\|\Big)
\end{aligned} \tag{2.34}$$

*for* $t \in [k\tau, (k+1)\tau)$, $k = 0, 1, \dots$. *where* $\alpha$, $\beta$, *and* $\Phi$ *are defined in* (2.14), (2.17).

*Proof.* First we consider the first summand in the left-hand side of (2.34). For $k \le l-1$ we obviously have

$$\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.$$

Let $ml \le k \le (m+1)l - 1$, $m = 1, 2, 3, \dots$. Clearly,

$$\begin{aligned}
&\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \\
&= \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \sum_{j=l}^{2l-1} \|D^j\| e^{j\beta\tau} + \dots + \sum_{j=ml}^{k} \|D^j\| e^{j\beta\tau} \\
&\le \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \|D^l\| e^{l\beta\tau} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} + \dots + \|D^{ml}\| e^{ml\beta\tau} \sum_{j=0}^{k-ml} \|D^j\| e^{j\beta\tau} \\
&\le \big[1 + \|D^l\| e^{l\beta\tau} + \dots + \|D^l\|^m e^{ml\beta\tau}\big] \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \\
&\le \|D^l\|^m e^{ml\beta\tau} \Big[1 + \big(\|D^l\| e^{l\beta\tau}\big)^{-1} + \dots + \big(\|D^l\| e^{l\beta\tau}\big)^{-m}\Big] \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} \\
&\le \|D^l\|^m e^{ml\beta\tau} \Big[1 + \big(\|D^l\| e^{l\beta\tau}\big)^{-1} + \dots + \big(\|D^l\| e^{l\beta\tau}\big)^{-m} + \dots\Big] \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.
\end{aligned}$$

Since $\|D^l\| e^{l\beta\tau} > 1$ owing to (2.31), we have

$$\sum_{j=0}^{k} \|D^j\| e^{j\beta\tau} \le \|D^l\|^m e^{ml\beta\tau} \Big[1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\Big]^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.$$

Taking into account that $ml\tau \le t < (m+1)l\tau$, we obtain

$$\begin{aligned}
\sum_{j=0}^{k} \|D^j\| e^{-\beta(t-j\tau)} &\le \|D^l\|^m e^{-\beta(t-ml\tau)} \Big[1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\Big]^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau} \\
&\le \|D^l\|^{\frac{t}{l\tau}} \Big[1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\Big]^{-1} \sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}.
\end{aligned}$$

As a result, we derive the estimate for the first summand in (2.34) for every $k$,

$$\begin{aligned} &\alpha \sum_{j=0}^{k} \|D^j\| e^{-\beta(t-j\tau)} \\ &\le \alpha \Big[1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\Big]^{-1} \Big(\sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}\Big) \exp\big(\frac{t}{l\tau} \ln \|D^l\|\big). \end{aligned} \tag{2.35}$$

We now we consider the second summand in the left-hand side of (2.34). Obviously, for $0 \le k \le l-2$ we have

$$\|D^{k+1}\| \le \max\big\{\|D\|, \dots, \|D^{l-1}\|\big\}.$$

Let $ml - 1 \le k \le (m+1)l - 2$, $m = 1, 2, \dots$. Hence,

$$\|D^{k+1}\| \le \|D^l\|^m \|D^{k+1-ml}\| \le \|D^l\|^m \max\big\{1, \|D\|, \dots, \|D^{l-1}\|\big\}.$$

Since $\|D^l\| < 1$ and $t < ((m+1)l - 1)\tau$, it follows that

$$\|D^l\|^m \le \|D^l\|^{\frac{t-(l-1)\tau}{l\tau}} = \|D^l\|^{\frac{1}{l}-1} \exp\Big(\frac{t}{l\tau} \ln \|D^l\|\Big).$$

Owing to arbitrariness of $m$, we infer that

$$\|D^{k+1}\| \le \|D^l\|^{\frac{1}{l}-1} \max\big\{1, \|D\|, \dots, \|D^{l-1}\|\big\} \exp\big(\frac{t}{l\tau} \ln \|D^l\|\big)$$

for every $k$. Taking into account the estimate (2.35), we derive (2.34). The proof is complete. □

**Lemma 2.10.** *Let the conditions of Theorem 2.8 be satisfied. Then the solution to the initial value problem* (2.4) *is defined for $t > 0$; moreover, it satisfies* (2.21) *on each segment $t \in [k\tau, (k+1)\tau)$, $k = 0, 1, \dots$.*

*Proof.* Recall that the noncontinuable solution to (2.4) is defined for $t \in [0, t')$. Let $t \in [0, \tau)$. Repeating the same reasoning as in the proof of Lemma 2.4, we obtain that $y(t)$ satisfies (2.22) and is defined for $t \in [0, \tau]$. Consequently, $t' > \tau$. It follows from (2.22) that $y(t)$ satisfies (2.21) for $k = 0$. Obviously, if the initial function $\varphi(t)$ belongs to the attraction set $\mathcal{E}_3$ defined by (2.32), then

$$\|y(t)\| < \sigma, \quad t \in [0, \tau].$$

Let $t \in [\tau, 2\tau)$. Consequently,

$$\begin{gathered} r_{\min} - \delta_1(\|y(t-\tau)\|) > r_{\min} - \delta_1(\sigma) > 0, \\ p_{\min} - \delta_2(\|y(t-\tau)\|) > p_{\min} - \delta_2(\sigma) > 0. \end{gathered}$$

By Theorem 2.1,

$$\begin{aligned} \frac{d}{dt} V(y_t) \le \varepsilon_0 V^{1+\omega_1/2}(y_t) &- (r_{\min} - \delta_1(\sigma)) \|y(t) + Dy(t-\tau)\|^2 \\ &- \kappa \int_{t-\tau}^{t} \langle K(t-s) y(s), y(s) \rangle \, ds \end{aligned}$$

for $t \in [0, t_2)$, where $t_2 = \min\{2\tau, t'\}$. Repeating the same reasoning as above, we have

$$\|y(t) + Dy(t-\tau)\| \le \alpha e^{-\beta t}, \quad t \in [0, t_2),$$

where $\alpha$ and $\beta$ are defined in (2.17) and (2.14), respectively. Hence,

$$\begin{aligned} \|y(t)\| &\le \|y(t) + Dy(t-\tau)\| + \|Dy(t-\tau)\| \\ &\le \alpha e^{-\beta t} + \|Dy(t-\tau) - D^2 y(t-2\tau)\| + \|D^2 y(t-2\tau)\| \\ &\le \alpha e^{-\beta t} + \alpha\|D\| e^{-\beta(t-\tau)} + \|D^2\|\Phi, \quad t \in [\tau, t_2). \end{aligned} \tag{2.36}$$

The function in the right-hand side of (2.36) is continuous and bounded for $t > 0$. Then $t_2 = 2\tau$ and the noncontinuable solution $y(t)$ to (2.4) is defined for $t \in [0, 2\tau]$. Consequently, $t' > 2\tau$. It follows from (2.36) that $y(t)$ satisfies (2.21) for $k = 1$. By Lemma 2.9,

$$\begin{aligned} &\alpha e^{-\beta t} + \alpha\|D\| e^{-\beta(t-\tau)} + \|D^2\|\Phi \\ &\le \alpha\Big[1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\Big]^{-1} \Big(\sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}\Big) \exp\Big(\frac{t}{l\tau} \ln\|D^l\|\Big) + \|D^2\|\Phi, \end{aligned}$$

for $t \in [\tau, 2\tau]$. Consequently, if the initial function $\varphi(t)$ belongs to the attraction set $\mathcal{E}_3$, then

$$\|y(t)\| < \sigma, \quad t \in [0, 2\tau].$$

Repeating the same reasoning, we obtain that the solution to (2.4) is defined for $t > 0$, and it satisfies (2.21) on each segment $t \in [k\tau, (k+1)\tau)$, $k \in \mathbb{N}$. By Lemma 2.9 and the condition $\|D^l\| < 1$, we have

$$\begin{aligned} \|y(t)\| \le \alpha\Big[1 - \big(\|D^l\| e^{l\beta\tau}\big)^{-1}\Big]^{-1} \Big(\sum_{j=0}^{l-1} \|D^j\| e^{j\beta\tau}\Big) \exp\big(\frac{t}{l\tau} \ln\|D^l\|\big) \\ + \max\{\|D\|, \dots, \|D^l\|\}\Phi, \quad t > 0. \end{aligned}$$

Consequently, if the initial function $\varphi(t)$ belongs to the attraction set $\mathcal{E}_3$ then

$$\|y(t)\| < \sigma, \quad t > 0.$$

The proof is complete. $\square$

By Lemmas 2.9 and 2.10, we obtain that the solution to (2.4) satisfies (2.33). Therefore, Theorem 2.8 is proved.

**Conclusion.** In this article, we investigated the nonlinear time-delay system (1.1) of neutral type with constant coefficients in the linear terms. Supposing that the spectrum of $D$ belongs to the unit disk, we indicated sufficient conditions under which the zero solution to (1.1) is exponentially stable. Depending on the norms of the powers of $D$, we established the constructive estimates for solutions to (1.1) and attraction sets of the zero solution to (1.1) (see Theorems 2.2, 2.5, 2.8). All the values characterizing the exponential decay rate of the solutions at infinity and the attraction sets are written out in explicit form.

**Acknowledgments.** The authors are grateful to the anonymous referee for the helpful comments and suggestions. This research was supported by the Russian Foundation for Basic Research (project no. 16-01-00592).

## References


[1] R. P. Agarwal, L. Berezansky, E. Braverman, A. Domoshnitsky; *Nonoscillation Theory of Functional Differential Equations with Applications*, Springer, New York, 2012.

[2] A. S. Andreev; *The method of Lyapunov functionals in the problem of the stability of functional-differential equations*, Autom. Remote Control **70** (2009), 1438–1486.

[3] N. V. Azbelev, V. P. Maksimov, L. F. Rakhmatullina; *Introduction to the Theory of Functional Differential Equations: Methods and Applications*, Contemporary Mathematics and Its Applications, **3**, Hindawi Publishing Corporation, Cairo, 2007.

[4] G. V. Demidenko, I. I. Matveeva; *Asymptotic properties of solutions to delay differential equations*, Vestnik Novosib. Gos. Univ. Ser. Mat. Mekh. Inform., **5** (2005), 20–28 (Russian).

[5] G. V. Demidenko, I. I. Matveeva; *Matrix process modelling: Asymptotic properties of solutions of differential-difference equations with periodic coefficients in linear terms*, Proceedings of the Fifth International Conference on Bioinformatics of Genome Regulation and Structure (Novosibirsk, Russia, July 16–22, 2006). Novosibirsk: Institute of Cytology and Genetics, **3** (2006), 43–46.

[6] G. V. Demidenko, I. I. Matveeva; *Stability of solutions to delay differential equations with periodic coefficients of linear terms*, Siberian Math. J., **48** (2007), 824–836.

[7] G. V. Demidenko; *Stability of solutions to linear differential equations of neutral type*, J. Anal. Appl., **7** (2009), 119–130.

[8] G. V. Demidenko, T. V. Kotova, M. A. Skvortsova; *Stability of solutions to differential equations of neutral type*, Vestnik Novosib. Gos. Univ. Ser. Mat. Mekh. Inform., **10** (2010), 17–29 (Russian); English transl. in: J. Math. Sci., **186** (2012), 394–406.

[9] G. V. Demidenko, E. S. Vodop'yanov, M. A. Skvortsova; *Estimates of solutions to the linear differential equations of neutral type with several delays of the argument*, J. Appl. Indust. Math., **7** (2013), 472–479.

[10] G. V. Demidenko, I. I. Matveeva; *On exponential stability of solutions to one class of systems of differential equations of neutral type*, J. Appl. Indust. Math., **8** (2014), 510–520.

[11] G. V. Demidenko, I. I. Matveeva; *On estimates of solutions to systems of differential equations of neutral type with periodic coefficients*, Siberian Math. J., **55** (2014), 866–881.

[12] G. V. Demidenko, I. I. Matveeva; *Estimates for solutions to a class of nonlinear time-delay systems of neutral type*, Electron. J. Diff. Equ. **2015** (2015), No. 34, 1–14.

[13] L. E. El′sgol′ts, S. B. Norkin; *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, New York, London, 1973.

[14] T. Erneux; *Applied Delay Differential Equations*, Surveys and Tutorials in the Applied Mathematical Sciences, **3**, Springer, New York, 2009.

[15] M. I. Gil'; *Stability of Neutral Functional Differential Equations*, Atlantis Studies in Differential Equations, **3**, Atlantis Press, Paris, 2014.

[16] K. Gu, V. L. Kharitonov, J. Chen; *Stability of Time-Delay Systems*, Control Engineering, Boston, Birkhäuser, 2003.

[17] I. Györi, G. Ladas; *Oscillation Theory of Delay Differential Equations. With Applications*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991.

[18] J. K. Hale; *Theory of Functional Differential Equations*, Springer-Verlag, New York, Heidelberg, Berlin, 1977.

[19] P. Hartman; *Ordinary Differential Equations*, SIAM, Philadelphia, 2002.

[20] V. L. Kharitonov; *Time-Delay Systems. Lyapunov Functionals and Matrices*, Control Engineering, Birkhauser, Springer, New York, 2013.

[21] D. Ya. Khusainov, A. V. Shatyrko; *The Method of Lyapunov Functions in the Investigation of the Stability of Functional Differential Systems*, Kiev University Press, Kiev, 1997 (Russian).

[22] V. B. Kolmanovskii, A. D. Myshkis; *Introduction to the Theory and Applications of Functional Differential Equations*, Mathematics and its Applications, **463**, Kluwer Academic Publishers, Dordrecht, 1999.

[23] D. G. Korenevskii; *Stability of Dynamical Systems under Random Perturbations of Parameters. Algebraic Criteria*, Naukova Dumka, Kiev, 1989 (Russian).

[24] Y. Kuang; *Delay Differential Equations with Applications in Population Dynamics*, Mathematics in Science and Engineering, **191**, Academic Press, Boston, 1993.

[25] N. MacDonald; *Biological Delay Systems: Linear Stability Theory*, Cambridge Studies in Mathematical Biology, **8**, Cambridge University Press, Cambridge, 1989.
[26] I. I. Matveeva, A. A. Shcheglova; *Some estimates of the solutions to time-delay differential equations with parameters*, J. Appl. Indust. Math., **5** (2011), 391–399.
[27] I. I. Matveeva; *Estimates of solutions to a class of systems of nonlinear delay differential equations*, J. Appl. Indust. Math., **7** (2013), 557–566.
[28] D. Melchor-Aguilar, S.-I. Niculescu; *Estimates of the attraction region for a class of nonlinear time-delay systems*, IMA J. Math. Control Inform., **24** (2007), 523–550.
[29] W. Michiels, S.-I. Niculescu; *Stability and Stabilization of Time-Delay Systems. An Eigenvalue-Based Approach*, Advances in Design and Control, **12**, Philadelphia, Society for Industrial and Applied Mathematics, 2007.
[30] M. A. Skvortsova; *Estimates for solutions and attraction domains of the zero solution to systems of quasi-linear equations of neutral type*, Bulletin of South Ural State University. Series: Mathematical Modelling, Programming & Computer Software, **37** (2011), 30–39 (Russian).
[31] M. Wu, Y. He, J-H. She; *Stability Analysis and Robust Control of Time-Delay Systems*, Science Press Beijing, Beijing; Springer-Verlag, Berlin, 2010.

Gennadii V. Demidenko
Laboratory of Differential and Difference Equations, Sobolev Institute of Mathematics, 4, Acad. Koptyug avenue, Novosibirsk 630090, Russia.
Department of Mechanics and Mathematics, Novosibirsk State University, 2, Pirogov street, Novosibirsk 630090, Russia
*E-mail address*: `demidenk@math.nsc.ru`

Inessa I. Matveeva
Laboratory of Differential and Difference Equations, Sobolev Institute of Mathematics, 4, Acad. Koptyug avenue, Novosibirsk 630090, Russia.
Department of Mechanics and Mathematics, Novosibirsk State University, 2, Pirogov street, Novosibirsk 630090, Russia
*E-mail address*: `matveeva@math.nsc.ru`