Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 56, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

A PARABOLIC BIPOLYNOMIAL FRACTIONAL DIRICHLET-LAPLACE PROBLEM

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ABSTRACT. We derive existence results for a parabolic bipolynomial abstract and classical problems containing fractional powers of the Dirichlet-Laplace operator on a bounded domain, in the sense of the Stone-von Neumann operator calculus. The main tools are theorems on the existence and uniqueness of a weak solutions to an abstract problem, due to Friedman, and a general theorem on the equivalence of weak and strong solutions to some operator equation.

1. Introduction

The basic model of diffusion processes is

$$u_t(t,x) + (-\Delta)u(t,x) = 0$$

where $(-\Delta)$ is the Laplace operator in x. To represent some anomalous diffusions, one considers the so-called fractional model

$$u_t(t,x) + (-\Delta)^{\beta}u(t,x) = 0$$

with $(-\Delta)^{\beta}$ being a fractional Laplacian of order β. Different kinds of such a Laplacian are used, e.g. spectral, restricted, censored (see [4, 5, 15]).

In this article, we consider the non-homogenous parabolic equation

$$u_t(t,x) + \sum_{i,j=0}^{k} \alpha_i\alpha_j(-\Delta)^{\beta_i+\beta_j}u(t,x) = f(t,x), \quad \text{a.e.} x \in \Omega, \tag{1.1}$$

containing the "bipolynomial" term $\sum_{i,j=0}^{k} \alpha_i\alpha_j(-\Delta)^{\beta_i+\beta_j}u(t,x)$, with initial condition

$$u(0,x) = 0, \quad x \in \Omega, \tag{1.2}$$

and boundary condition

$$u(t,\cdot) \in H_0^1(\Omega), \quad t \in (0,T), \tag{1.3}$$

where $\alpha_i > 0$ for $i = 0,\dots,k$, $0 \le \beta_0 < \dots < \beta_k$, $(-\Delta)^{\beta_i+\beta_j}$ is the spectral fractional lapalacian for $i,j = 0,\dots,k$, $f : (0,T)\times\Omega \to \mathbb{R}$, $0 < T < \infty$, Ω is an open and bounded set in $\mathbb{R}^N$ with $N \in \mathbb{N}$, $H_0^1(\Omega)$ is the classical Sobolev space of real-valued functions. Here, the spectral fractional Laplacian is a power of

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2020 *Mathematics Subject Classification.* 35K10, 35R11.
Key words and phrases. Parabolic equation; fractional Dirichlet-Laplace operator; existence of solutions.

Submitted December 8, 2021. Published July 31, 2022.

the Dirichlet-Laplace operator in the Stone-von Neumann operator calculus sense. Particular cases of the above problem are: the classical parabolic Dirichlet-Laplace problem

$$u_t(t,x) + [(-\Delta)_\omega]u(t,x) = f(t,x), \quad \text{a.e } x \in \Omega, \tag{1.4}$$

the biharmonic parabolic problem

$$u_t(t,x) + [(-\Delta)_\omega]^2 u(t,x) = f(t,x), \quad \text{a.e. } x \in \Omega, \tag{1.5}$$

and the standard fractional (in state) parabolic problem

$$u_t(t,x) + [(-\Delta)_\omega]^\beta u(t,x) = f(t,x), \quad \text{a.e. } x \in \Omega. \tag{1.6}$$

In [13], we derived the existence of a strong solution to the nonlinear non-autonomous partial elliptic system

$$\sum_{i,j=0}^{k} \alpha_i\alpha_j [(-\Delta)_\omega]^{\beta_i+\beta_j} u(x) - au(x) = D_u F(x, u(x)), \quad \text{a.e. } x \in \Omega,$$

with $a \in \mathbb{R}$ and Dirichlet boundary conditions. We applied there a direct variational method and some results based on the dual least action principle.

The aim of this article is to prove existence of a solution to the parabolic problem (1.1)-(1.2)-(1.3) containing the bipolynomial term. An abstract form of the problem (1.1) -(1.2)-(1.3) is as follows

$$\mathbf{u}'(t)(x) + \sum_{i,j=0}^{k} \alpha_i\alpha_j (-\Delta)^{\beta_i+\beta_j} \mathbf{u}(t)(x) = \mathbf{f}(t)(x), \quad t \in [0,T), \text{ a.e. } x \in \Omega, \tag{1.7}$$

$$\mathbf{u}(0) = 0, \tag{1.8}$$

$$\mathbf{u}(t) \in H_0^1(\Omega), t \in (0,T), \tag{1.9}$$

where $\mathbf{u}$ acts from $[0,T)$ to the domain of the operator $\sum_{i,j=0}^{k} \alpha_i\alpha_j (-\Delta)^{\beta_i+\beta_j}$. First, we prove existence of a solution to problem (1.7)-(1.8)-(1.9) and next apply this result to obtain existence of a solution to problem (1.1)-(1.2)-(1.3). In the proof of the existence of a solution to the abstract problem we shall use an existence result for an abstract parabolic variational problem, due to Friedman (see [9, Theorem 17, Part X.6]), and a general theorem on the equivalence of weak and strong solutions to some operator equation, obtained in [13] as a result of the Stone-von Neumann operator calculus. This equivalence theorem was also used in [13]. Our paper is the second part of a program of research of fundamental partial differential equations (elliptic, parabolic, hyperbolic) containing such a term. In the future, we plan to investigate a bipolynomial hyperbolic problem. Let us add that in the paper [14], using a global implicit function theorem, we studied an elliptic nonlinear problem containing the spectral fractional Laplacian and showed that for each functional parameter there exists a unique solution to the problem and that its dependence on this parameter is continuously differentiable.

This article consists of three parts. In the first part, we recall and complete some facts concerning the different kinds of derivatives of abstract functions. Next, we formulate the Friedman theorem. In the second, main part of the paper, we derive an existence and uniqueness result for the abstract problem (1.7)-(1.8)-(1.9). As a corollary, under some additional assumptions, we obtain existence of a solution to problem (1.1)-(1.2)-(1.3), for $\beta_k \geq 1$. To the best of our knowledge, such bipolynomial problems were not investigated up to now. The last part (Appendix) contains the basics concerning the powers of a weak Dirichlet-Laplace operator and the mentioned theorem on the equivalence of weak and strong solutions to an operator equation.

2. Preliminaries

2.1. **Derivatives of abstract functions.** Details about the results presented in this section can be found in [9] (strong and weak derivatives) and [6] (Sobolev derivative). We recall them for the convenience of the reader. We prove the results that are not proved in the mentioned monographs.

Let $(a,b) \subset \mathbb{R}$ be an interval, X - a real Hilbert space with scalar product $(\cdot,\cdot)_X$ and the corresponding norm $\|\cdot\|_X$.

By $D^k(a,b;X)$ we denote the space of functions $\mathbf{f}:(a,b)\to X$ that are k-times continuously classic differentiable and $\mathbf{f},\mathbf{f}',\dots,\mathbf{f}^k \in L^2(a,b;X)$. The norm in $D^k(a,b;X)$ is

$$\|\mathbf{f}\|_{D^k} = \Big(\sum_{i=0}^{k}\int_a^b \|\mathbf{f}^i(t)\|_X^2\Big)^{1/2}. \tag{2.1}$$

The completion of $D^k(a,b;X)$ in $L^2(a,b;X)$ with respect to the norm $\|\cdot\|_{D^k}$ is denoted by $H^k(a,b;X)$.

We say that a function $\mathbf{f}:(a,b)\to X$ has k strong derivatives provided that $\mathbf{f}\in H^k(\alpha,\beta;X)$ for any $a<\alpha<\beta<b$. If $\mathbf{f}\in H^k(a,b;X)$, then by the strong derivative $\mathbf{f}^j_{s,(a,b)}$of $\mathbf{f}$ on (a,b) of order $j=1,\dots,k$ we mean the limit in $L^2(a,b;X)$ of the sequence $(D^j\mathbf{u}_m)$ of classical derivatives of order j of functions $\mathbf{u}_m$ where $(\mathbf{u}_m)\subset D^k(a,b;X)$ is the Cauchy sequence with respect to the norm (2.1) converging in $L^2(a,b;X)$ to $\mathbf{f}$ (one can show that the strong derivatives do not depend on the choice of the sequence $(\mathbf{u}_m)$). If $\mathbf{f}\notin H^k(a,b;X)$, then by the strong derivative $\mathbf{f}^j_{s,(a,b)}$ of $\mathbf{f}$ on (a,b) of order $j=1,\dots,k$ we mean the function that is the extension on (a,b) of each function $\mathbf{f}^j_{s,(\alpha,\beta)}$.

Remark 2.1. Of course, if $\mathbf{f}\in H^k(a,b;X)$, then $\mathbf{f}\in H^k(\alpha,\beta;X)$ for any $a<\alpha<\beta<b$. It is easy to see that the function being the extension on (a,b) of each function $\mathbf{f}^j_{s,(\alpha,\beta)}$ coincides with the strong derivative $\mathbf{f}^j_{s,(a,b)}$ for any $j=1,\dots,k$.

We say that a function $\mathbf{f}\in L^2(a,b;X)$ has the weak derivative of order k if there exists a function $\mathbf{v}\in L^2(a,b;X)$ such that

$$(-1)^k\int_a^b(\mathbf{f}(t),\psi^{(k)}(t))_X dt = \int_a^b(\mathbf{v}(t),\psi(t))_X dt \tag{2.2}$$

for any function $\psi:[a,b]\to X$ continuously differentiable k times and such that $\psi^{(j)}(a)=\psi^{(j)}(b)=0$ for $j=0,\dots,k-1$ (if $a=-\infty$ we require ψ to be such that $\psi(t)=0$ for sufficiently small t; similarly in the case of $b=\infty$). By the weak derivative we mean the function $\mathbf{v}$ and denote it as $\mathbf{f}'_\omega$.

We say that a function $\mathbf{f}\in L^2(a,b;X)$ has the Sobolev derivative $\mathbf{f}'_{Sob}$ if there exists a function $\mathbf{g}\in L^2(a,b;X)$ such that

$$\int_a^b \mathbf{f}(t)\varphi'(t)dt = -\int_a^b \mathbf{g}(t)\varphi(t)dt$$

for any $\varphi\in C_c^\infty(a,b;\mathbb{R})$. We put in such a case $\mathbf{f}'_{Sob}=\mathbf{g}$. The set of all functions possessing the Sobolev derivative is denoted by $W^{1,2}(a,b;X)$. From [16, Part I, Thm. 2.1] it follows that, for any function $\mathbf{f} \in W^{1,2}(a,b;X)$, there exists a sequence $(\mathbf{f}_n) \subset C^\infty([a,b];X)$ such that

$$\mathbf{f}_n \to \mathbf{f} \text{ and } \mathbf{f}_n' \to \mathbf{f}_{Sob}' \text{ in } L^2(a,b;X). \tag{2.3}$$

One proves that if $\mathbf{f} \in W^{1,2}(a,b;X)$ then there exists a continuous function $\widetilde{\mathbf{f}} : \overline{(a,b)} \to X$ such that $\mathbf{f} = \widetilde{\mathbf{f}}$ a.e. on I and

$$\widetilde{\mathbf{f}}(t) - \widetilde{\mathbf{f}}(s) = \int_s^t \mathbf{f}_{Sob}'(\tau)d\tau \tag{2.4}$$

for a.e. $t, s \in \overline{(a,b)}$.

In the case of (a,b) bounded, it follows from the results presented in [6]. When (a,b) is unbounded one can repeat the proofs of Corollary 4.24, Lemma 8.1, Lemma 8.2 and Theorem 8.2 from the book [7] in the case of any Hilbert space X (in [7], the proofs are presented for $X = \mathbb{R}$).

In such a case, $\mathbf{f}$ has the classical derivative $\mathbf{f}'$ a.e. on I (in the sense of the representative a.e.) and

$$\mathbf{f}'(t) = \mathbf{f}_{Sob}'(t), \quad \text{a.e. } t \in (a,b). \tag{2.5}$$

We have the following two lemmas containing a comparison of the strong derivatives with the weak and Sobolev ones.

Lemma 2.2. *If $\mathbf{f} \in H^k(a,b;X)$, then the strong derivatives of $\mathbf{f}$ are the weak ones.*

Proof. It is sufficient to consider the case of bounded interval $(a,b) \subset \mathbb{R}$. Moreover, since k-th strong derivative is the first strong derivative of $(k-1)$-th strong derivative therefore it is sufficient to prove the lemma for $k = 1$. The proof consists of three steps.

Step 1. First, let us assume that $\mathbf{f} \in C^\infty([a,b];X)$. We know that

$$\frac{d}{dt}(\mathbf{f}(t), \psi(t))_X = (\mathbf{f}'(t), \psi(t))_X + (\mathbf{f}(t), \psi'(t))_X \tag{2.6}$$

for $t \in [a,b]$ and any function $\psi : [a,b] \to X$ continuously differentiable with $\psi(a) = \psi(b) = 0$. So,

$$\begin{aligned} \int_a^b (\mathbf{f}'(t), \psi(t))_X dt + \int_a^b (\mathbf{f}(t), \psi'(t))_X dt &= \int_a^b \frac{d}{dt}(\mathbf{f}(t), \psi(t))_X dt \\ &= (\mathbf{f}(b), \psi(b))_X - (\mathbf{f}(a), \psi(a))_X = 0. \end{aligned}$$

So, $\mathbf{f}$ has the weak derivative $\mathbf{f}_\omega'$ and it is equal to the classical one $\mathbf{f}'$ which coincides with the strong derivative $\mathbf{f}_{s,(a,b)}'$.

Step 2. Now, let us assume that $\mathbf{f} \in D^1(a,b;X)$. In the same way as in the first case we assert that

$$\frac{d}{dt}(\mathbf{f}(t)\varphi(t)) = \mathbf{f}'(t)\varphi(t) + \mathbf{f}(t)\varphi'(t)$$

for $t \in (a,b)$, and any function $\varphi \in C_c^\infty(a,b;\mathbb{R})$. Clearly, the function $\mathbf{f}(t)\varphi(t)$ has compact support contained in (a,b). Consequently, $\frac{d}{dt}(\mathbf{f}(t)\varphi(t))$ also has the compact support and there exists $\varepsilon > 0$ such that $\mathbf{f}(t)\varphi(t) = 0$ and $\frac{d}{dt}(\mathbf{f}(t)\varphi(t)) = 0$ for $t \in (a, a+\varepsilon) \cup (b-\varepsilon, b)$. Continuity of $\frac{d}{dt}(\mathbf{f}(t)\varphi(t))$ implies the differentiability of the function

$$F : (a,b) \ni t \mapsto \int_a^t \frac{d}{ds}(\mathbf{f}(s)\varphi(s))ds \in X$$

everywhere on (a,b) and

$$F'(t) = \frac{d}{dt}(\mathbf{f}(t)\varphi(t))$$

for all $t \in (a,b)$.

Thus, there exists a constant $c \in X$ such that

$$\int_a^t \frac{d}{ds}(\mathbf{f}(s)\varphi(s))ds = c + \mathbf{f}(t)\varphi(t)$$

for $t \in (a,b)$. So,

$$\begin{aligned}
\int_a^b \mathbf{f}'(t)\varphi(t)dt + \int_a^b \mathbf{f}(t)\varphi'(t)dt &= \int_a^b \frac{d}{dt}(\mathbf{f}(t)\varphi(t))dt \\
&= \int_{a+\varepsilon}^{b-\varepsilon} \frac{d}{dt}(\mathbf{f}(t)\varphi(t))dt \\
&= \mathbf{f}(b-\varepsilon)\varphi(b-\varepsilon) - \mathbf{f}(a+\varepsilon)\varphi(a+\varepsilon) = 0.
\end{aligned}$$

From property (2.3) it follows that there exists a sequence $(f_n) \subset C^\infty([a,b];X)$ such that

$$\mathbf{f}_n \to \mathbf{f} \text{ and } \mathbf{f}_n' \to \mathbf{f}' \text{ in } L^2(a,b;X).$$

In consequence, using the first step we obtain

$$\begin{aligned}
\int_a^b (\mathbf{f}(t), \psi'(t))_X dt &= \big(\mathbf{f}(\cdot), \psi'(\cdot)\big)_{L^2(a,b;X)} = \Big(\lim \mathbf{f}_n(\cdot), \psi'(\cdot)\Big)_{L^2(a,b;X)} \\
&= \lim \big(\mathbf{f}_n(\cdot), \psi'(\cdot)\big)_{L^2(a,b;X)} = -\lim \big(\mathbf{f}_n'(\cdot), \psi(\cdot)\big)_{L^2(a,b;X)} \\
&= -\big(\lim \mathbf{f}_n'(\cdot), \psi(\cdot)\big)_{L^2(a,b;X)} = -\big(\mathbf{f}'(\cdot), \psi(\cdot)\big)_{L^2(a,b;X)} \\
&= -\int_a^b (\mathbf{f}'(t), \psi(t))_X dt
\end{aligned} \tag{2.7}$$

for any function $\psi : [a,b] \to X$ continuously differentiable with $\psi(a) = \psi(b) = 0$.

Step 3. To finish the proof assume that $\mathbf{f} \in H^1(a,b;X)$. So, there exists a sequence $(\mathbf{u}_n) \subset D^1(a,b;X)$ such that

$$\mathbf{u}_n \to \mathbf{f} \text{ and } \mathbf{u}_n' \to \mathbf{f}'_{s,(a,b)} \text{ in } L^2(a,b;X).$$

In the same way as in (2.7), we show that

$$\int_a^b (\mathbf{f}(t), \psi'(t))dt = -\int_a^b (\mathbf{f}'_{s,(a,b)}(t), \psi(t))dt$$

for any function $\psi : [a,b] \to X$ continuously differentiable with $\psi(a) = \psi(b) = 0$. This means that $\mathbf{f}$ has the weak derivative $\mathbf{f}'_\omega$ which is equal to the strong one $\mathbf{f}'_{s,(a,b)}$ and the proof is completed. ☐

Lemma 2.3. *If $\mathbf{f} \in H^1(a,b;X)$, then the strong derivative $\mathbf{f}'_{s,(a,b)}$ is the Sobolev one.*

Proof. First, let us assume that $(a,b) \subset \mathbb{R}$ is bounded. In the second step of the proof of Lemma 2.2, it is proved that if $\mathbf{f} \in D^1(a,b;X)$, then the classical derivative $\mathbf{f}'$ is the Sobolev one. Now, let $\mathbf{f} \in H^1(a,b;X)$ and $(u_n) \subset D^1(a,b;X)$ be such that

$$\mathbf{u}_n \to \mathbf{f} \text{ and } \mathbf{u}_n' \to \mathbf{f}'_{s,(a,b)} \text{ in } L^2(a,b;X).$$

We have

$$\begin{aligned}\int_a^b \mathbf{f}(t)\varphi'(t)dt &= \lim \int_a^b \mathbf{u}_n(t)\varphi'(t)dt \\ &= -\lim \int_a^b \mathbf{u}_n'(t)\varphi(t)dt \\ &= -\int_a^b \mathbf{f}'_{s,(a,b)}(t)\varphi(t)dt\end{aligned}$$

for any $\varphi \in C_c^\infty(a,b;\mathbb{R})$ (the first and third equalities follow from the linearity and continuity of the operator $L^2(a,b;X) \ni \mathbf{v} \mapsto \int_a^b \mathbf{v}(t)\omega(t)dt \in X$ where $\omega \in C_c^\infty(a,b;\mathbb{R})$). It means that the strong derivative $\mathbf{f}'_{s,(a,b)}$ of the function $\mathbf{f} \in H^1(a,b;X)$ is the Sobolev one.

Now, let assume that (a,b) is unbounded. Fix a function $\varphi \in C_c^\infty(a,b;\mathbb{R})$ and let $(\alpha,\beta) \subset (a,b)$ be such a bounded interval that $\operatorname{supp}\varphi \subset (\alpha,\beta)$. Of course, $\mathbf{f} \in H^1(\alpha,\beta;X)$. Consequently, from the first part of the proof and Remark 2.1 we obtain

$$\int_a^b \mathbf{f}(t)\varphi'(t)dt = \int_\alpha^\beta \mathbf{f}(t)\varphi'(t)dt = -\int_\alpha^\beta \mathbf{f}'_{s,(\alpha,\beta)}(t)\varphi(t)dt = -\int_a^b \mathbf{f}'_{s,(a,b)}(t)\varphi(t)dt.$$

The proof is complete. □

Theorem 2.4. *Let $(a,b) \subset \mathbb{R}$ be a bounded interval and $\mathbf{f} : (a,b) \to X$. The following conditions are equivalent:*

(a) $\mathbf{f} \in H^1(a,b;X)$*;*
(b) $\mathbf{f}$ *has the weak derivative of order* 1*;*
(c) $\mathbf{f} \in W^{1,2}(a,b;X)$*;*
(d) $\mathbf{f} \in L^2(a,b;X)$ *and*

$$\Big|\int_a^b \langle \mathbf{f}(t), \varphi'(t)\rangle_{X\times X^*} dt\Big| \le c\|\varphi\|_{L^2(a,b;X^*)} \tag{2.8}$$

for any function $\varphi : (a,b) \to X^*$ infinitely differentiable with compact $\operatorname{supp}\varphi$ *contained in (a,b).*

Proof. (a)⇒(b). This implication follows from Lemma 2.2.

(b)⇒(d). Let us fix a function $\varphi : (a,b) \to X^*$ infinitely differentiable with the compact $\operatorname{supp}\varphi \subset (a,b)$. Consider the function $\psi : [a,b] \to X$ given by

$$\psi(t) = \begin{cases} 0, & t \in \{a,b\}, \\ {}^{\#}\varphi(t), & t \in (a,b), \end{cases}$$

where ${}^{\#}\varphi(t)$ is the unique element from X determining $\varphi(t)$ according to Riesz theorem. It is easy to see that, for any $t \in (a,b)$, the condition

$$\lim_{h\to 0} \Big\|\frac{\varphi(t+h)-\varphi(t)}{h} - \varphi'(t)\Big\|_{X^*} = 0$$

implies

$$\lim_{h\to 0} \Big\|\frac{\psi(t+h)-\psi(t)}{h} -{}^{\#} \varphi'(t)\Big\|_X = 0.$$

Indeed,

$$\begin{aligned}\|\frac{\psi(t+h)-\psi(t)}{h} -^{\#} \varphi'(t)\|_X &= \|\frac{^{\#}\varphi(t+h) -^{\#} \varphi(t)}{h} -^{\#} \varphi'(t)\|_X \\ &= \|^{\#}(\frac{\varphi(t+h)-\varphi(t)}{h} - \varphi'(t))\|_X \\ &= \|\frac{\varphi(t+h)-\varphi(t)}{h} - \varphi'(t)\|_{X^*} \to 0\end{aligned}$$

as $h \to 0$. So, function ψ is differentiable at t and $\psi'(t) =^{\#} \varphi'(t)$. Moreover, if $t_n \to t$, then

$$\begin{aligned}\|\psi'(t_n) - \psi'(t)\|_X &= \|^{\#}(\varphi'(t_n)) -^{\#} (\varphi'(t))\|_X \\ &= \|^{\#}(\varphi'(t_n) - \varphi'(t))\|_X \\ &= \|\varphi'(t_n) - \varphi'(t)\|_{X^*} \to 0.\end{aligned}$$

So, ψ' is continuous at t. Case of $t \in \{a, b\}$ is obvious because of the compactness of $\operatorname{supp}\varphi \subset (a, b)$. Thus, condition (2.2) implies

$$\int_a^b \big(\mathbf{f}(t), \psi'(t)\big)_X dt = -\int_a^b (\mathbf{v}(t), \psi(t))_X dt$$

for some $\mathbf{v} \in L^2(a, b; X)$ and, consequently,

$$\begin{aligned}\Big|\int_a^b \langle \mathbf{f}(t), \varphi'(t)\rangle_{X\times X^*} dt\Big| &= \Big|\int_a^b (\mathbf{f}(t),^{\#} \varphi'(t))_X dt\Big| = \Big|\int_a^b (\mathbf{f}(t), \psi'(t))_X dt\Big| \\ &= \Big|\int_a^b \big(\mathbf{v}(t), \psi(t)\big)_X dt\Big| \\ &\le \|\mathbf{v}\|_{L^2(a,b;X)} \|\psi\|_{L^2(a,b;X)} \\ &= \|\mathbf{v}\|_{L^2(a,b;X)} \Big(\int_a^b \|\psi(t)\|_X^2 dt\Big)^{1/2} \\ &= \|\mathbf{v}\|_{L^2(a,b;X)} \Big(\int_a^b \|^{\#}\varphi(t)\|_X^2 dt\Big)^{1/2} \\ &= \|\mathbf{v}\|_{L^2(a,b;X)} \Big(\int_a^b \|\varphi(t)\|_{X^*}^2 dt\Big)^{1/2} \\ &= \|\mathbf{v}\|_{L^2(a,b;X)} \|\varphi\|_{L^2(a,b;X^*)}.\end{aligned}$$

(d)⇒(a). From condition (2.8) it follows (see [6, Propositions A6, A7 and Corollaire A2]) that $\mathbf{f} \in W^{1,2}(a, b; X)$. So, from property (2.3) it follows that there exists a sequence $(\varphi_n) \subset C^\infty([a, b], X)$ such that

$$\varphi_n \to \mathbf{f} \text{ and } \varphi_n' \to \mathbf{f}'_{Sob} \text{ in } L^2(a, b; X).$$

It means that $\mathbf{f} \in H^1(a, b; X)$.

(d)⇔(c). This part of the theorem follows from [6, Propositions A6, A7 and Corollary A2]. □

From the Theorem 2.4, Lemmas 2.2, 2.3 and from the uniqueness of weak, strong and Sobolev derivatives as well as from (2.5) the following corollary follows.

Corollary 2.5. *If $(a, b) \subset \mathbb{R}$ is bounded, then the sets $W^{1,2}(a, b; X)$, $H^1(a, b; X)$ and the set of functions possessing weak derivative coincide and the notions of*

Sobolev, strong, weak and classical (in the sense of the representative a.e.) derivatives of the first order coincide in this set.

3. Fractional parabolic problem

3.1. **Friedmann theorem.** Let us assume that V, H are real Hilbert spaces with the scalar products $(\cdot,\cdot)_V$, $(\cdot,\cdot)_H$ and the corresponding norms $\|\cdot\|_V$, $\|\cdot\|_H$, respectively, $\overline{V}=H$ and the embedding $V\to H$ is continuous. Let k be a fixed positive integer and

$$a=a(t,u,v):(-\infty,T)\times V\times V\to\mathbb{R}$$

with $T>0$, be a function such that

- (i) for any $t\in(-\infty,T)$, the form $a(t,\cdot,\cdot)$ is bilinear;
- (ii) for any $u,v\in V$, the function $a(\cdot,u,v)$ is k times continuously differentiable in $t\in(-\infty,T)$ and
$$\Big|\frac{d^j}{dt^j}a(t,u,v)\Big|\le M\|u\|_V\|v\|_V$$
for some constant M and any $t\in(-\infty,T)$, $0\le j\le k$;
- (iii) there exist constants $\alpha>0$, λ such that
$$a(t,v,v)+\lambda\|v\|_H^2\ge\alpha\|v\|_V^2\quad\forall v\in V.$$

The following theorem is proved in [9, Part X.6, Theorem 17 and formula (6.30)].

Theorem 3.1. *If $a=a(t,u,v)$ satisfies* (i)–(iii)*, a function $\mathbf{f}:(-\infty,T)\to V$ has k strong derivatives on $(-\infty,T)$ and*

$$\begin{gathered}\mathbf{f},\mathbf{f}'_{s,(-\infty,T)},\dots,\mathbf{f}^k_{s,(-\infty,T)}\in L^2(-\infty,T;V),\\ \mathbf{f}(t)=0,\quad t<0,\end{gathered}$$

then there exists a unique function $\mathbf{u}:(-\infty,T)\to V$ which has k strong derivatives on $(-\infty,T)$, such that

$$\begin{gathered}\mathbf{u},\mathbf{u}'_{s,(-\infty,T)},\dots,\mathbf{u}^k_{s,(-\infty,T)}\in L^2(-\infty,T;V),\\ \mathbf{u}(t)=0,\quad t<0,\\ a(t,\mathbf{u}(t),v)+\big(\mathbf{u}'_{s,(-\infty,T)}(t),v\big)_H=\big(\mathbf{f}(t),v\big)_H,\qquad a.e.\ t\in(0,T)\end{gathered}$$

for each $v\in V$.

3.2. **Abstract fractional parabolic problem.** Now, let $A:D(A)\subset L^2(\Omega;\mathbb{R})\to L^2(\Omega;\mathbb{R})$ be a self-adjoint operator with non-empty resolvent set $\rho(A)$.

Theorem 3.2. *If $\mathbf{f}:(-\infty,T)\to D(A)$ is continuous, has all strong derivatives and*

$$\begin{gathered}\mathbf{f},\mathbf{f}'_{s,(-\infty,T)},\mathbf{f}''_{s,(-\infty,T)},\dots\in L^2(-\infty,T;D(A)),\\ \mathbf{f}(t)=0,\quad t<0,\end{gathered}$$

then there exists a unique function $\widetilde{\mathbf{u}}:[0,T]\to D(A)$ infinitely differentiable such that

$$\begin{gathered}\widetilde{\mathbf{u}}^{(k)}(0)=0,\quad k=0,1,\dots,\\ \widetilde{\mathbf{u}}(t)\in D(A^2),\quad t\in(0,T),\end{gathered}\tag{3.1}$$

and, for $t\in[0,T)$,

$$\widetilde{\mathbf{u}}'(t)(x)+A^2\widetilde{\mathbf{u}}(t)(x)=\mathbf{f}(t)(x),\quad a.e.\ x\in\Omega\tag{3.2}$$

Proof. Existence. We denote $V = D(A)$, $H = L^2(\Omega;\mathbb{R})$ and

$$(u,v)_V = \int_\Omega u(x)v(x)dx + \int_\Omega Au(x)Av(x)dx$$

for $u, v \in V$, and

$$(u,v)_H = \int_\Omega u(x)v(x)dx$$

for u, $v \in H$. We define $a : (-\infty,T)\times V\times V \to \mathbb{R}$, by

$$a(t,u,v) = \int_\Omega Au(x)Av(x)dx.$$

It is clear that V is continuously embedded in H and a satisfies conditions (i)–(iii) with $M = 1$, $\lambda = 1$, $\alpha = 1$. Using Theorem 3.1 we assert that there exists a unique function $\mathbf{u} : (-\infty,T) \to D(A)$ which has all strong derivatives on $(-\infty,T)$,

$$\begin{gathered}\mathbf{u}, \mathbf{u}'_{s,(-\infty,T)}, \mathbf{u}''_{s,(-\infty,T)}, \dots \in L^2(-\infty,T;D(A)),\\ \mathbf{u}(t) = 0, \quad t<0,\end{gathered}$$

and for $v \in D(A)$,

$$a(t,\mathbf{u}(t),v) + \big(\mathbf{u}'_{s,(-\infty,T)}(t),v\big)_{L^2(\Omega;\mathbb{R})} = \big(\mathbf{f}(t),v\big)_{L^2(\Omega;\mathbb{R})}, \quad \text{a.e. } t\in(0,T)$$

i.e.,

$$\int_\Omega A\mathbf{u}(t)(x)Av(x)dx = \int_\Omega (\mathbf{f}(t) - \mathbf{u}'_{s,(-\infty,T)}(t))(x)v(x)dx \tag{3.3}$$

for a.e. $t \in (0,T)$.

Let us fix a positive integer $k \geq 2$. Since $\mathbf{u}'_{s,(-\infty,T)}, \dots, \mathbf{u}^{(k)}_{s,(-\infty,T)}$ are the strong derivatives of $\mathbf{u}$ on $(-\infty,T)$, it follows that $\mathbf{u} \in H^k(0,T;D(A))$ and the strong derivatives $\mathbf{u}'_{s,(0,T)}, \dots, \mathbf{u}^{(k)}_{s,(0,T)}$ on $(0,T)$ safisfy

$$\mathbf{u}'_{s,(0,T)} = \mathbf{u}'_{s,(-\infty,T)}\big|_{(0,T)}, \dots,\ \mathbf{u}^{(k)}_{s,(0,T)} = \mathbf{u}^{(k)}_{s,(-\infty,T)}\big|_{(0,T)}.$$

Let a sequence $(\mathbf{u}_m) \subset D^k(0,T;D(A))$ be such that

$$\begin{gathered}\mathbf{u}_m \to \mathbf{u} \quad \text{in } L^2(0,T;D(A)), \mathbf{u}'_m \to \mathbf{u}'_{s,(0,T)} \quad \text{in } L^2(0,T;D(A)),\\ \mathbf{u}''_m \to \mathbf{u}''_{s,(0,T)} \quad \text{in } L^2(0,T;D(A)), \qquad (3.4)\\ \dots \qquad (3.5)\\ \mathbf{u}^{(k)}_m \to \mathbf{u}^{(k)}_{s,(0,T)} \quad \text{in } L^2(0,T;D(A)).\end{gathered}$$

From Theorem 2.4 ((a)$\Longrightarrow$(c)) and (2.4) it follows that there exists a function $\widetilde{\mathbf{u}} : [0,T] \to D(A)$ such that

$$\begin{gathered}\mathbf{u}(t) = \widetilde{\mathbf{u}}(t), \quad \text{a.e. } t\in[0,T],\\ \widetilde{\mathbf{u}}(t) = \widetilde{\mathbf{u}}(0) + \int_0^t \widetilde{\mathbf{g}}(s)ds, \quad t\in[0,T],\end{gathered} \tag{3.6}$$

with $\widetilde{\mathbf{g}} \in L^2(0,T;D(A))$ and

$$\int_0^T \widetilde{\mathbf{u}}(t)\varphi'(t)dt = -\int_0^T \widetilde{\mathbf{g}}(t)\varphi(t)dt$$

for any $\varphi \in C_c^\infty((0,T),\mathbb{R})$. From (3.6) it follows that the function $\widetilde{\mathbf{u}}$ is classically differentiable a.e. on $[0,T]$ and $\widetilde{\mathbf{u}}'(t) = \widetilde{\mathbf{g}}(t)$ a.e. on $[0,T]$. Of course, $\widetilde{\mathbf{g}}(t) = \mathbf{u}'_{Sob}(t)$

a.e. on $[0,T]$. From Corollary 2.5 it follows that $\mathbf{u}'_{Sob}(t)=\mathbf{u}'_{s,(0,T)}(t)$ a.e. on $[0,T]$. Thus,

$$\mathbf{u}'_{s,(0,T)}(t)=\widetilde{\mathbf{u}}'(t), \quad \text{a.e. } t\in(0,T)$$

It is easy to see that $\mathbf{u}''_{s,(0,T)}$ is the strong derivative of the first order of the function $\mathbf{u}'_{s,(0,T)}$, i.e.

$$(\mathbf{u}'_{s,(0,T)})'_{s,(0,T)}=\mathbf{u}''_{s,(0,T)} \quad \text{a.e. on } (0,T).$$

Therefore, (3.2), (3.4) can be written down as

$$\mathbf{u}'_m\to\widetilde{\mathbf{u}}' \quad \text{in } L^2(0,T;D(A)), \tag{3.7}$$

$$(\mathbf{u}'_m)'=\mathbf{u}''_m\to\mathbf{u}''_{s,(0,T)}=(\mathbf{u}'_{s,(0,T)})'_{s,(0,T)}=(\widetilde{\mathbf{u}}')'_{s,(0,T)} \quad \text{in } L^2(0,T;D(A)). \tag{3.8}$$

Using (3.7) and (3.8), in the same way as above, we prove that there exists a function $\widetilde{\widetilde{\mathbf{u}}}:[0,T]\to D(A)$ such that

$$\begin{gathered}\widetilde{\mathbf{u}}'(t)=\widetilde{\widetilde{\mathbf{u}}}(t), \quad \text{a.e. } t\in[0,T],\\ \widetilde{\widetilde{\mathbf{u}}}(t)=\widetilde{\widetilde{\mathbf{u}}}(0)+\int_0^t\widetilde{\widetilde{\mathbf{g}}}(s)ds, \quad t\in[0,T].\end{gathered}\tag{3.9}$$

with $\widetilde{\widetilde{\mathbf{g}}}\in L^2(-\infty,T;D(A))$, as well as

$$(\widetilde{\mathbf{u}}')'_{s,(0,T)}=\widetilde{\widetilde{\mathbf{u}}}' \quad \text{a.e. on } [0,T].$$

Now, let us observe (see (3.9)) that, for $t\in[0,T]$,

$$\widetilde{\mathbf{u}}(t)=\widetilde{\mathbf{u}}(0)+\int_0^t\widetilde{\mathbf{u}}'(t_1)dt_1=\widetilde{\mathbf{u}}(0)+\int_0^t\widetilde{\widetilde{\mathbf{u}}}(t_1)dt_1. \tag{3.10}$$

Continuity of $\widetilde{\widetilde{\mathbf{u}}}$ implies that $\widetilde{\mathbf{u}}$ is continuously differentiable everywhere on $[0,T]$ and $\widetilde{\mathbf{u}}'(t)=\widetilde{\widetilde{\mathbf{u}}}(t)$ for all $t\in[0,T]$.

In the same way as above, we check that $\widetilde{\mathbf{u}}'$ is continuously differentiable everywhere on $[0,T]$. So, $\widetilde{\mathbf{u}}$ is twice continuously differentiable on $[0,T]$.

Repeating the above reasoning we assert that $\widetilde{\mathbf{u}}$ is $(k-1)$-times continuously differentiable. Since $k\geq 2$ was arbitrary, $\widetilde{\mathbf{u}}$ is infinitely continuously differentiable.

Now, (3.3) can be written down as

$$\int_\Omega A\widetilde{\mathbf{u}}(t)(x)Av(x)dx=\int_\Omega(\mathbf{f}(t)-\widetilde{\widetilde{\mathbf{u}}}(t))(x)v(x)dx \tag{3.11}$$

for a.e. $t\in(0,T)$. Since $\widetilde{\mathbf{u}},\mathbf{f},\widetilde{\widetilde{\mathbf{u}}}$ are continuous on $[0,T)$, it follows that the mappings

$$[0,T)\ni t\mapsto\int_\Omega A\widetilde{\mathbf{u}}(t)(x)Av(x)dx\in\mathbb{R}, \tag{3.12}$$

$$[0,T)\ni t\mapsto\int_\Omega(\mathbf{f}(t)-\widetilde{\widetilde{\mathbf{u}}}(t))(x)v(x)dx\in\mathbb{R} \tag{3.13}$$

are continuous, too. Indeed, the continuity of (3.12) follows from the estimate

$$\begin{aligned}&\Big|\int_\Omega A\widetilde{\mathbf{u}}(t_1)(x)Av(x)dx-\int_\Omega A\widetilde{\mathbf{u}}(t_2)(x)Av(x)dx\Big|\\ &\leq\int_\Omega|A\widetilde{\mathbf{u}}(t_1)(x)-A\widetilde{\mathbf{u}}(t_2)(x)|\,|Av(x)|dx\\ &\leq\|A\widetilde{\mathbf{u}}(t_1)-A\widetilde{\mathbf{u}}(t_2)\|_{L^2(\Omega,\mathbb{R})}\|Av\|_{L^2(\Omega,\mathbb{R})}\\ &\leq\|\widetilde{\mathbf{u}}(t_1)-\widetilde{\mathbf{u}}(t_2)\|_{D(A)}\|Av\|_{L^2(\Omega,\mathbb{R})}\end{aligned}$$

for any t_1, $t_2 \in [0,T)$. The continuity of (3.13) follows from the estimation (see [13])

$$\begin{aligned}
&\Big|\int_\Omega (\mathbf{f}(t_1) - \widetilde{\widetilde{\mathbf{u}}}(t_1))(x)v(x)dx - \int_\Omega (\mathbf{f}(t_2) - \widetilde{\widetilde{\mathbf{u}}}(t_2))(x)v(x)dx\Big| \\
&\le \int_\Omega (|\mathbf{f}(t_1)(x) - \mathbf{f}(t_2)(x)| + |\widetilde{\widetilde{\mathbf{u}}}(t_1)(x) - \widetilde{\widetilde{\mathbf{u}}}(t_2)(x)|)|v(x)|dx \\
&\le \Big(\|\mathbf{f}(t_1) - \mathbf{f}(t_2)\|_{L^2(\Omega,\mathbb{R})} + \|\widetilde{\widetilde{\mathbf{u}}}(t_1) - \widetilde{\widetilde{\mathbf{u}}}(t_2)\|_{L^2(\Omega,\mathbb{R})}\Big)\|v\|_{L^2(\Omega,\mathbb{R})} \\
&\le \Big(\|\mathbf{f}(t_1) - \mathbf{f}(t_2)\|_{D(A)} + \|\widetilde{\widetilde{\mathbf{u}}}(t_1) - \widetilde{\widetilde{\mathbf{u}}}(t_2)\|_{D(A)}\Big)\|v\|_{L^2(\Omega,\mathbb{R})}
\end{aligned}$$

for any t_1, $t_2 \in [0,T)$. So, equality (3.11) holds everywhere on $[0,T)$.

From Theorem 4.5 it follows that, for $t \in [0,T)$, $\widetilde{\mathbf{u}}(t) \in D(A^2)$ and

$$\widetilde{\widetilde{\mathbf{u}}}(t)(x) + A^2\widetilde{\mathbf{u}}(t)(x) = \mathbf{f}(t)(x), \quad \text{a.e. } x \in \Omega\,.$$

Consequently (see (3.10)),

$$\widetilde{\mathbf{u}}'(t)(x) + A^2\widetilde{\mathbf{u}}(t)(x) = \mathbf{f}(t)(x), \quad \text{a.e. } x \in \Omega \tag{3.14}$$

for $t \in [0,T)$.

Initial conditions. Now, we shall show that $\widetilde{\mathbf{u}}$ satisfies the zero initial conditions. Indeed, consider restrictions $\mathbf{u}'_{s,(-\infty,T)}\big|_{(a,T)}$, $\mathbf{u}''_{s,(-\infty,T)}\big|_{(a,T)}$ for some $-\infty < a < 0$. Clearly,

$$\mathbf{u}'_{s,(-\infty,T)}\big|_{(a,T)} = \mathbf{u}'_{s,(a,T)}, \quad \mathbf{u}''_{s,(-\infty,T)}\big|_{(a,T)} = \mathbf{u}''_{s,(a,T)}.$$

Moreover, in the same way as in the case of the function $\widetilde{\mathbf{u}}$ we obtain existence of a function $\widehat{\mathbf{u}} : [a,T] \to D(A)$ such that

$$\mathbf{u}(t) = \widehat{\mathbf{u}}(t), \quad \text{a.e. } t \in [a,T],$$
$$\widehat{\mathbf{u}}(t) = \widehat{\mathbf{u}}(a) + \int_a^t \widehat{\mathbf{g}}(s)ds, \quad t \in [a,T]$$

with $\widehat{\mathbf{g}} \in L^2(a,T;D(A))$. Since $\mathbf{u}(t) = 0$ for $t \in [a,0)$, it follows that $\widehat{\mathbf{g}} = 0$ on $[a,0)$ and, consequently, $\widehat{\mathbf{u}}(a) = 0$. So,

$$\widehat{\mathbf{u}}(t) = \int_a^t \widehat{\mathbf{g}}(s)ds = \int_a^0 \widehat{\mathbf{g}}(s)ds + \int_0^t \widehat{\mathbf{g}}(s)ds = \int_0^t \widehat{\mathbf{g}}(s)ds, \quad t \in [0,T].$$

Therefore, since $\widetilde{\mathbf{u}}(t) = \widehat{\mathbf{u}}(t)$ for $t \in [0,T]$, it follows that

$$\widetilde{\mathbf{u}}(0) + \int_0^t \widetilde{\mathbf{g}}(s)ds = \int_0^t \widehat{\mathbf{g}}(s)ds, \quad t \in [0,T].$$

From this condition it follows that $\widetilde{\mathbf{g}} = \widehat{\mathbf{g}}$ a.e. on $[0,T]$ and

$$\widetilde{\mathbf{u}}(0) = 0.$$

Repeating the above reasoning with the function $\widetilde{\mathbf{u}}$ replaced by $\widetilde{\mathbf{u}}'$,…,$\widetilde{\mathbf{u}}^{(k-2)}$ respectively, we check that

$$\widetilde{\mathbf{u}}'(0) = 0, \dots, \widetilde{\mathbf{u}}^{(k-2)}(0) = 0.$$

Since $k \ge 2$ was arbitrary, $\widetilde{\mathbf{u}}^{(k)}(0) = 0$ for $k = 0, 1, \dots$.

"Uniqueness". The assertion follows from the fact that if a function $\widetilde{\mathbf{u}} : [0,T] \to D(A)$ satisfies (3.2) and the conditions given in Theorem 3.2, then the function

$$\mathbf{u} : (-\infty,T) \ni t \mapsto \begin{cases} 0, & t < 0, \\ \widetilde{\mathbf{u}}(t), & t \in [0,T) \end{cases}$$

is continuously differentiable on $(-\infty,T)$ and $\mathbf{u}, \mathbf{u}' \in L^2(-\infty,T;D(A))$. So,

$$\mathbf{u} \in D^1(-\infty,T;D(A)) \subset H^1(-\infty,T;D(A)).$$

It means that $\mathbf{u}$ has the first strong derivative $\mathbf{u}'_{s,(-\infty,T)}$ which is equal to $\mathbf{u}'$.

From the fact that, for any $t \in [0,T)$, $\mathbf{u}(t) \in D(A^2)$ and

$$\mathbf{u}'(t)(x) + A^2\mathbf{u}(t)(x) = \mathbf{f}(t)(x),\ x \in \Omega \text{ a.e.}$$

it follows (see Theorem 4.5) that, for any $t \in [0,T)$,

$$\int_\Omega A\mathbf{u}(t)(x)Av(x)dx = \int_\Omega (\mathbf{f}(t) - \mathbf{u}'(t))(x)v(x)dx \tag{3.15}$$

for any $v \in D(A)$. Clearly, the above equality is also satisfied for any $t < 0$ and $v \in D(A)$. So, using our notations we can write

$$a(t,\mathbf{u}(t),v) + \big(\mathbf{u}'_s(t),v\big)_H = \big(\mathbf{f}(t),v\big)_H \tag{3.16}$$

for all $t \in (-\infty,T)$ and any $v \in D(A)$. Uniqueness of the solution stated in Theorem 3.1 (with $k = 1$, $V = D(A)$, $H = L^2$) completes the proof of the theorem. □

Theorem 3.3. *Let the assumptions of Theorem 3.2 be satisfied. If, additionally,*

$$\mathbf{f}(t) = 0, \quad t < \varepsilon$$

for some $\varepsilon > 0$, then the solution described in Theorem 3.2 satisfies

$$\widetilde{\mathbf{u}}(t) = 0, \quad 0 \le t < \varepsilon.$$

Proof. Let $\widetilde{\mathbf{u}} : [0,T] \to D(A)$ be the solution of (3.2), given in Theorem 3.2. Then the function $\mathbf{u}$ described in the "uniqueness" part of the proof of Theorem 3.2 is the solution to (3.16), corresponding to $\mathbf{f}$. Since the zero function defined on $(-\infty,\varepsilon)$ is the solution to (3.15), it satisfies also (3.16). From the uniqueness of the solution to (3.16) on $(-\infty,\varepsilon)$ it follows that $\widetilde{\mathbf{u}}\big|_{[0,\varepsilon)} \equiv 0$. □

Applying Theorems 3.2 and 3.3 in the case of $A = w((-\Delta)_\omega)$ (see Proposition 4.4 in subsection 4.1) we obtain the following result.

Corollary 3.4. *Assume that $\mathbf{f} : (-\infty,T) \to D(w((-\Delta)_\omega))$ is continuous, has all strong derivatives and*

$$\mathbf{f}, \mathbf{f}'_{s,(-\infty,T)}, \mathbf{f}''_{s,(-\infty,T)}, \dots \in L^2(-\infty,T;D(w((-\Delta)_\omega))),$$
$$\mathbf{f}(t) = 0, t < 0.$$

Then there exists a unique function $\widetilde{\mathbf{u}} : [0,T] \to D(w((-\Delta)_\omega))$ infinitely differentiable such that

$$\widetilde{\mathbf{u}}^{(k)}(0) = 0, \quad k = 0,1,\dots,$$
$$\widetilde{\mathbf{u}}(t) \in D(w^2((-\Delta)_\omega)), t \in (0,T),$$

and, for $t \in [0,T)$,

$$\widetilde{\mathbf{u}}'(t)(x) + w^2((-\Delta)_\omega)\widetilde{\mathbf{u}}(t)(x) = \mathbf{f}(t)(x), \quad \text{a.e. } x \in \Omega$$

If, additionally, $\mathbf{f}(t) = 0$ *for* $t < \varepsilon$*, with some* $\varepsilon > 0$*, then the solution* $\widetilde{\mathbf{u}}$ *satisfies*
$$\widetilde{\mathbf{u}}(t) = 0, \quad 0 \leq t < \varepsilon.$$

3.3. **Application to classical fractional parabolic problems.** Let us recall that if

(a) $N \geq 1$, $\Omega \subset \mathbb{R}^N$ is an open bounded set with the boundary of class $C^{1,1}$, or

(b) $\Omega \subset \mathbb{R}^2$ is an open bounded convex polygon,

then
$$(-\Delta)_\omega = -\Delta$$
(it follows from the results contained in [10] and [11]) and, consequently,
$$D((-\Delta)_\omega) = D(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega).$$

Lemma 3.5. *If* $\gamma > 0$*, one of the conditions (a) or (b) is satisfied, and* f *belongs to* $C_c^\infty((0,T) \times \Omega, \mathbb{R})$*, then the function* $\mathbf{f} : (-\infty, T) \to D((-\Delta)^\gamma)$*,*
$$\mathbf{f}(t) = \begin{cases} 0, & t \leq 0, \\ f(t, \cdot), & t \in (0,T) \end{cases}$$
has compact support contained in $(0,T)$*, belongs to* $D^k(-\infty, T; D((-\Delta)^\gamma)$ *for any* $k \in \mathbb{N}$*, and, consequently,* $\mathbf{f}$ *has all strong derivatives* $\mathbf{f}^j_{s,(-\infty,T)}$ *on* $(-\infty, T)$ *that are equal everywhere on* $(-\infty, T)$ *to classical ones* $\mathbf{f}^j$*.*

Proof. First, let us notice that for any $t \in (0,T)$, $f(t, \cdot) \in C_c^\infty \subset D((-\Delta)^\gamma)$. So, $\mathbf{f}$ is well defined. It is also clear that $\operatorname{supp} \mathbf{f} \subset (0,T)$.

Now, we shall show that $\mathbf{f} \in D^1(-\infty, T; D((-\Delta)^\gamma)$. So, let us fix a point $t \in (0,T)$ and let l be a positive integer such that $\gamma \leq l$. We know that
$$\|u\|_{\tilde{}\gamma} \leq \text{const} \cdot \|u\|_{\tilde{}l} = \text{const} \cdot \|[(-\Delta)^l u\|_{L^2}.$$
Thus, to show that
$$\|\frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h} - \mathbf{f}'(t)\|^2_{\tilde{}\gamma} \underset{h \to 0}{\to} 0$$
for some $\mathbf{f}'(t) \in D((-\Delta)^\gamma)$ it is sufficient to prove that
$$\|\frac{f(t+h, \cdot) - f(t, \cdot)}{h} - \frac{\partial f}{\partial t}(t, \cdot)\|^2_{\tilde{}l} \underset{h \to 0}{\to} 0.$$
(of course, $\frac{\partial f}{\partial t}(t, \cdot) \in C_c^\infty \subset D((-\Delta)^\gamma)$). Indeed (below, (h_m) is a sequence converging to 0),
$$\begin{aligned}
&\|\frac{f(t+h_m, \cdot) - f(t, \cdot)}{h_m} - \frac{\partial f}{\partial t}(t, \cdot)\|^2_{\tilde{}l} \\
&= \int_\Omega \Big| \frac{(-\Delta)^l f(t+h_m, x) - (-\Delta)^l f(t,x)}{h_m} - (-\Delta)^l \frac{\partial f}{\partial t}(t,x) \Big|^2 dx \\
&= \int_\Omega \Big| \frac{(-\Delta)^l f(t+h_m, x) - (-\Delta)^l f(t,x)}{h_m} - \frac{\partial}{\partial t}(-\Delta)^l f(t,x) \Big|^2 dx
\end{aligned}$$
Since $f \in C_c^\infty((0,T) \times \Omega, \mathbb{R})$, the above sequence of integrands converges pointwise to zero function. Moreover, using the fact that the function $(-\Delta)^l f$ is Lipschitzian on $(0,T) \times \Omega$ (with a constant L) we obtain
$$\Big| \frac{(-\Delta)^l f(t+h_m, x) - (-\Delta)^l f(t,x)}{h_m} - \frac{\partial}{\partial t}(-\Delta)^l f(t,x) \Big| \leq L + C$$

where C is such that $|\frac{\partial f}{\partial t}(-\Delta)^l(t,x)| \le C$ for $(t,x) \in (0,T)\times\Omega$. So, from the Lebesgue dominated convergence theorem it follows that

$$\int_\Omega \Big| \frac{(-\Delta)^l f(t_0+h_m,x) - (-\Delta)^l f(t_0,x)}{h_m} - \frac{\partial}{\partial t}(-\Delta)^l f(t_0,x)\Big|^2 dx \underset{m\to\infty}{\to} 0.$$

Thus, there exists the derivative $\mathbf{f}'(t) \in D([(-\Delta)_\omega]^\gamma)$ at any point $t\in(0,T)$ and $\mathbf{f}'(t)(x) = \frac{\partial f}{\partial t}(t,x)$ for all $x\in\Omega$.

Let us also observe that if $t_j \to t > 0$ then $\mathbf{f}'(t_j) \to \mathbf{f}'(t)$ in $D((-\Delta)^\gamma)$. Indeed,

$$\begin{aligned}\|\mathbf{f}'(t_j) - \mathbf{f}'(t)\|_{\tilde{}\gamma} &\le \text{const}\cdot \|\mathbf{f}'(t_j) - \mathbf{f}'(t)\|_{\tilde{}l}\\ &= \text{const}\cdot \int_\Omega |(-\Delta)^l \frac{\partial f}{\partial t}(t_j,x) - (-\Delta)^l\frac{\partial f}{\partial t}(t,x)|^2 dx \underset{j\to\infty}{\to} 0.\end{aligned}$$

The above convergence follows from the above mentioned Lebesgue theorem. Thus, we have just proved that $\mathbf{f}$ is continuously differentiable on $(0,T)$.

The relations $\operatorname{supp}\mathbf{f}\subset(0,T)$, $\operatorname{supp}\mathbf{f}'\subset(0,T)$, and the continuity of $\mathbf{f}$, $\mathbf{f}'$ imply that $\mathbf{f} \in D^1(-\infty,T;D((-\Delta)^\gamma))$. Consequently, $\mathbf{f}$ has one (first) strong derivative $\mathbf{f}'_{s,(-\infty,T)}$ on $(-\infty,T)$ which is equal to the classical derivative $\mathbf{f}'$.

Now, since $\frac{\partial f}{\partial t}\in C_c^\infty((0,T)\times\Omega,\mathbb{R})$, we can repeat our reasoning for the function $\mathbf{f}' : (-\infty,T)\to D((-\Delta)^\gamma)$,

$$\mathbf{f}'(t) = \begin{cases} 0, & t\le 0,\\ \frac{\partial f}{\partial t}(t,\cdot), & t\in(0,T),\end{cases}$$

to state that $\mathbf{f}'\in D^1(-\infty,T;D((-\Delta)^\gamma))$. So, $\mathbf{f}\in D^2(-\infty,T;D((-\Delta)^\gamma))$ and the classical derivatives $\mathbf{f}'$, $\mathbf{f}'' : (-\infty,T)\to D((-\Delta)^\gamma)$,

$$\mathbf{f}''(t) = \begin{cases} 0, & t\le 0,\\ \frac{\partial^2 f}{\partial t^2}(t,\cdot), & t\in(0,T)\end{cases}$$

are the strong ones. Repeating our reasoning infinitely many times we complete the proof. $\square$

It is known (see [1]) that if $N = 1,2,3$ and $\Omega\subset\mathbb{R}^N$ is an open set which has the cone property, then $H^2(\Omega)\subset C_B(\Omega)$ continuously, where $C_B(\Omega)$ is the space of continuous and bounded functions on Ω, with norm

$$\|u\|_{C_B(\Omega)} = \sup_{x\in\Omega}|u(x)|$$

of uniform convergence on Ω.

So, we assume that one of the conditions

- (A1) $N = 1,2,3$, $\Omega\subset\mathbb{R}^N$ is an open bounded set with the boundary of class $C^{1,1}$ and with the cone property
- (A2) $\Omega\subset\mathbb{R}^2$ is an open bounded convex polygon

is satisfied. Clearly, in the case (A2), Ω has the cone property.

Theorem 3.6. *If $\beta_k \ge 1$, one of the conditions* (A1), (A2) *is satisfied and $f\in C_c^\infty((0,T)\times\Omega,\mathbb{R})$, then there exist $\varepsilon>0$ and a function $\widetilde{u} : [0,T]\times\Omega\to\mathbb{R}$ satisfying*

$$\frac{\partial\widetilde{u}}{\partial t}(t,x) + \sum_{i,j=0}^{k}\alpha_i\alpha_j(-\Delta)^{\beta_i+\beta_j}\widetilde{u}(t,x) = f(t,x), \quad \text{a.e. } x\in\Omega \tag{3.17}$$

for $t \in (0,T)$ and such that

$$\widetilde{u}(t,x) = 0, \quad t \in [0,\varepsilon),\ x \in \Omega, \tag{3.18}$$

$$\widetilde{u}(t,\cdot) \in H_0^1(\Omega) \cap H^2(\Omega) \subset C_B(\Omega), \quad t \in (0,T], \tag{3.19}$$

$$(-\Delta)\widetilde{u}(t,\cdot) \in H_0^1(\Omega) \cap H^2(\Omega) \subset C_B(\Omega), \quad t \in (0,T). \tag{3.20}$$

Proof. Let $\widetilde{\mathbf{u}} : [0,T] \to D(w(-\Delta)) = D((-\Delta)^{\beta_k})$ be the solution to (3.2) given by Corollary 3.4, corresponding to function $\mathbf{f}$ given in Lemma 3.5. Define the function

$$\widetilde{u} : [0,T] \times \Omega \ni (t,x) \mapsto \widetilde{\mathbf{u}}(t)(x) \in \mathbb{R}.$$

For a $t \in [0,T]$, we have

$$\lim_{h\to 0} \| \frac{\widetilde{\mathbf{u}}(t+h) - \widetilde{\mathbf{u}}(t)}{h} - \widetilde{\mathbf{u}}'(t)\|_{D([(-\Delta)_\omega]^{\beta_k})} = 0.$$

Since the convergence in $D([(-\Delta)_\omega]^{\beta_k})$ implies the uniform convergence on Ω (because under our assumptions

$$D([(-\Delta)_\omega]^{\beta_k}) \subset D([(-\Delta)_\omega]^1) = D((-\Delta)^1) = H_0^1(\Omega) \cap H^2(\Omega) \subset C_B(\Omega) \tag{3.21}$$

and these embeddings are continuous. Recall that the norm $\|\cdot\|_{\sim 1}$ and the norm induced from $H^2(\Omega)$ are equivalent in $D((-\Delta)^1)$ (see [8, Theorem VII.1.3.2 - Banach's theorem on compatible norms]).

Therefore, for $t \in [0,T]$,

$$\lim_{h\to 0} \frac{\widetilde{u}(t+h,x) - \widetilde{u}(t,x)}{h} = \widetilde{\mathbf{u}}'(t)(x) \quad \text{uniformly on } \Omega.$$

In particular, the partial derivative $\frac{\partial \widetilde{u}}{\partial t}(t,x)$ exists for all $t \in [0,T]$, $x \in \Omega$ and

$$\frac{\partial \widetilde{u}}{\partial t}(t,x) = \widetilde{\mathbf{u}}'(t)(x)$$

for all $t \in [0,T]$, $x \in \Omega$. Thus, (3.17) holds for $t \in (0,T)$.

Condition (3.18) follows from Theorem 3.3. Condition (3.19) follows from the fact that $\widetilde{\mathbf{u}}(t) \in D((-\Delta)^{\beta_k}) \subset H_0^1(\Omega) \cap H^2(\Omega) \subset C_B(\Omega)$ for $t \in [0,T]$. Condition (3.20) follows from the relation $\widetilde{\mathbf{u}}(t) \in D((-\Delta)^{2\beta_k}) \subset D((-\Delta)^2)$ for $t \in (0,T)$ (see (3.1)). $\square$

4. Appendix

4.1. **Functions of self-adjoint operators.** This subsection contains the results from the theory of self-adjoint operators in real Hilbert space. Results presented in this section can be found, in a more general setting, in [13]. We give them here for the convenience of the reader. In [2, 17], these results are derived in the case of complex Hilbert space but their proofs can be moved without any or with small changes to the case of real Hilbert space (one can also consult the book [12]).

Let H be a real Hilbert space with a scalar product $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{R}$ and $E : \mathcal{B} \to \Pi(H)$ - a spectral measure where $\mathcal{B}$ is the σ-algebra of Borel subsets of $\mathbb{R}$, $\Pi(H)$ - the set of all projections of H on closed linear subspaces. Let $b : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function, defined E - a.e. By $\int_{-\infty}^{\infty} b(\lambda)E(d\lambda)$ we denote the operator

$$\int_{-\infty}^{\infty} b(\lambda)E(d\lambda) : D \subset H \to H$$

given by
$$\Big(\int_{-\infty}^{\infty} b(\lambda)E(d\lambda)\Big)x = \int_{-\infty}^{\infty} b(\lambda)E(d\lambda)x,$$
where D is the set of points $x \in H$ such that
$$\int_{-\infty}^{\infty} |b(\lambda)|^2 \|E(d\lambda)x\|^2 < \infty\,. \tag{4.1}$$
The above integral is taken with respect to the nonnegative measure $\mathcal{B} \ni P \mapsto \|E(P)x\|^2 \in \mathbb{R}_0^+$; the integral $\int_{-\infty}^{\infty} b(\lambda)E(d\lambda)x$ is taken with respect to the vector measure $\mathcal{B} \ni P \mapsto E(P)x \in H$. One proves that the operator $\int_{-\infty}^{\infty} b(\lambda)E(d\lambda)$ is self-adjoint.

Remark 4.1. To integrate a Borel measurable function $b : B \to \mathbb{R}$ where B is a Borel set containing the support of the measure E, it is sufficient to extend b on $\mathbb{R}$ to a whichever Borel measurable function (putting, for example, $b(\lambda) = 0$ for $\lambda \notin B$).

If $b : \mathbb{R} \to \mathbb{R}$ is Borel measurable and $\sigma \in \mathcal{B}$, then by the integral
$$\int_{\sigma} b(\lambda)E(d\lambda)$$
we mean the integral
$$\int_{-\infty}^{\infty} \chi_{\sigma}(\lambda)b(\lambda)E(d\lambda)$$
where χ_{σ} is the characteristic function of the set σ. The integral $\int_{\sigma} b(\lambda)E(d\lambda)$ can be also defined with the aid of the restriction of E to the set σ.

The next theorem plays a fundamental role in the spectral theory of self-adjoint operators (below, $\sigma(A)$ denotes the spectrum of an operator $A : D(A) \subset H \to H$).

Theorem 4.2. *If $A : D(A) \subset H \to H$ is self-adjoint and the resolvent set $\rho(A)$ is non-empty, then there exists a unique spectral measure E with the closed support $\Lambda = \sigma(A)$, such that*
$$A = \int_{-\infty}^{\infty} \lambda E(d\lambda) = \int_{\sigma(A)} \lambda E(d\lambda).$$

The basic notion in the Stone-von Neumann operator calculus is a function of a self-adjoint operator. Namely, if $A : D(A) \subset H \to H$ is self-adjoint and E is the spectral measure determined according to the above theorem, then, for any Borel measurable function $b : \mathbb{R} \to \mathbb{R}$, one defines the operator $b(A)$ by
$$b(A) = \int_{-\infty}^{\infty} b(\lambda)E(d\lambda) = \int_{\sigma(A)} b(\lambda)E(d\lambda).$$

Proposition 4.3. *If E is the spectral measure for a self-adjoint operator $A : D(A) \subset H \to H$ with non-empty resolvent set, then*
$$\alpha_k A^k + \dots + \alpha_1 A + \alpha_0 I = \int_{-\infty}^{\infty} (\alpha_k \lambda^k + \dots + \alpha_1 \lambda^1 + \alpha_0)E(d\lambda)$$
and, for any Borel measurable function $b : \mathbb{R} \to \mathbb{R}$,
$$(b(A))^n = b^n(A) \tag{4.2}$$
with any fixed positive integer $n \geq 2$.

Now, let $\beta > 0$ and $\sigma(A) \subset [0, \infty)$. According to the Remark 4.1 by A^β we mean the operator

$$A^\beta = \int_{-\infty}^{\infty} b(\lambda) E(d\lambda)$$

where

$$b : \mathbb{R} \ni \lambda \to \begin{cases} \lambda^\beta, & \lambda \geq 0, \\ 0, & \lambda < 0. \end{cases}$$

We have the following generalization of Proposition 4.3.

Proposition 4.4. *If E is the spectral measure for a self-adjoint operator $A : D(A) \subset H \to H$ with $\sigma(A) \subset [0, \infty)$, then*

$$\alpha_k A^{\beta_k} + \dots + \alpha_1 A^{\beta_1} + \alpha_0 A^{\beta_0} = \int_{-\infty}^{\infty} w(\lambda) E(d\lambda).$$

where

$$w : \mathbb{R} \ni \lambda \to \begin{cases} \alpha_k \lambda^{\beta_k} + \dots + \alpha_1 \lambda^{\beta_1} + \alpha_0 \lambda^{\beta_0}, & \lambda \geq 0, \\ 0, & \lambda < 0, \end{cases} \tag{4.3}$$

and $0 \leq \beta_0 < \beta_1 < \dots < \beta_k$. Moreover,

$$A^{\beta_2} \circ A^{\beta_1} = A^{\beta_2 + \beta_1} \tag{4.4}$$

for β_2, $\beta_1 > 0$.

If the numbers β_0, β_1,...,β_k are positive integers (including zero), then one can omit the assumption $\sigma(A) \subset [0, \infty)$ and consider the function

$$w(\lambda) = \alpha_k \lambda^{\beta_k} + \dots + \alpha_1 \lambda^{\beta_1} + \alpha_0 \lambda^{\beta_0},\ \lambda \in \mathbb{R}$$

(assuming that the resolvent set $\rho(A)$ is nonempty).

4.2. **Equivalence of weak and strong solutions.** The results presented in this section with additional remarks and comments can be found in [13]. Let E be the spectral measure for a self-adjoint operator $A : D(A) \subset H \to H$ with non-empty resolvent set and $b : \mathbb{R} \to \mathbb{R}$ - a Borel measurable function, defined E - a.e. Fact that the operator $b(A)$ is self-adjoint means that its domain satisfies the equality

$$\begin{aligned} D(b(A)) = \big\{ & u \in H : \text{there exists } z \in H \text{such that} \\ & \int_\Omega u(t) b(A) v(t) dt = \int_\Omega z(t) v(t) dt \text{ for any } v \in D(b(A)) \big\} \end{aligned} \tag{4.5}$$

and

$$b(A)u = z, \quad u \in D(b(A)). \tag{4.6}$$

From Proposition 4.3 it follows that

$$b(A)(b(A)u) = b^2(A)u. \tag{4.7}$$

In particular, $u \in D(b^2(A))$ if and only if $u \in D(b(A))$ and $b(A)u \in D(b(A))$. Using this fact and (4.5), (4.6), we obtain the following result.

Theorem 4.5. *If $g \in L^2$, then $u \in D(b^2(A))$ and*

$$b^2(A)u = g \tag{4.8}$$

if and only if $u \in D(b(A))$ and

$$\int_\Omega b(A)u(t) b(A)v(t) dt = \int_\Omega g(t) v(t) dt \tag{4.9}$$

for any $v \in D(b(A))$.

Consequently, if $A : D(A) \subset H \to H$ is self-adjoint with $\sigma(A) \subset [0, \infty)$, w is given by (4.3), then we have the following corollary, where clearly, $w^2(A) = \sum_{i,j=0}^{k} \alpha_i \alpha_j A^{\beta_i + \beta_j}$.

Corollary 4.6. *If* $g \in L^2$*, then* $u \in D(w^2(A))$ *and*

$$w^2(A)u = g \tag{4.10}$$

if and only if $u \in D(w(A))$ *and*

$$\int_{\Omega} w(A)u(t)w(A)v(t)dt = \int_{\Omega} g(t)v(t)dt \tag{4.11}$$

for any $v \in D(w(A))$.

Remark 4.7. The above theorem states that u is the strong solution to problem (4.10) if and only if it is the weak one in the sense of (4.11).

4.3. **Weak Dirichlet-Laplace operator and its fractional power.** Details about the results presented in this section can be found in [13]. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, $H_0^1 = H_0^1(\Omega, \mathbb{R})$ - the closure of $C_c^\infty = C_c^\infty(\Omega, \mathbb{R})$ in $H^1 = H^1(\Omega, \mathbb{R})$, where $H^1(\Omega, \mathbb{R}) = W^{1,2}(\Omega, \mathbb{R})$ is the classical Sobolev space, $C_c^\infty(\Omega, \mathbb{R})$ is the space of test functions. We say ([3]) that $u : \Omega \to \mathbb{R}$ has a weak (minus) Dirichlet-Laplacian if $u \in H_0^1$ and there exists a function $g \in L^2 = L^2(\Omega, \mathbb{R})$ such that

$$\int_{\Omega} \nabla u(x) \nabla v(x)dx = \int_{\Omega} g(x)v(x)dx$$

for any $v \in H_0^1$. The function g is called the weak Dirichlet-Laplacian (or shortly, Laplacian) of u and denoted by $(-\Delta)_\omega u$. In [3], g is named Dirichlet-Laplacian and denoted by $(-\Delta)u$. The operator

$$(-\Delta)_\omega : D((-\Delta)_\omega) \subset L^2 \to L^2$$

where

$$D((-\Delta)_\omega) = \{u : \Omega \to \mathbb{R};\ u \text{ has the weak Dirichlet-Laplacian}\}$$

is bijective and self-adjoint. We call it the weak Dirichlet-Laplace operator.

Remark 4.8. In fact, $(-\Delta)_\omega$ is the Fridrich's extension of the classical Dirichlet-Laplace operator $(-\Delta)_{class} : C_c^\infty \subset L^2 \to L^2$. Moreover,

$$(-\Delta)_{\text{class}} \subset (-\Delta) \subset (-\Delta)_\omega$$

where

$$-\Delta : H_0^1 \cap H^2 \subset L^2 \to L^2$$

is the Dirichlet-Laplace operator defined by generalized partial derivatives of second order of Sobolev type.

The spectrum $\sigma((-\Delta)_\omega)$ of $(-\Delta)_\omega$ contains only the eigenvalues. More precisely, $\sigma((-\Delta)_\omega)$ is a sequence $0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_j \to \infty$ (each λ_j is repeated k_j times where k_j is the multiplicity of λ_j). Moreover, to each eigenvalue λ_j corresponds an eigenfunction e_j and the system $\{e_j\}$ is the Hilbertian basis in L^2 (see [3, Theorem 8.3.2]; in [3, Proposition 8.5.3], the spectrum and eigenfunctions of $(-\Delta)_\omega$ are given in the case of $\Omega = (0,\pi)^N$ (cube in $\mathbb{R}^N$)). Thus, for any $u \in L^2$, there exist real numbers a_j, $j \in \mathbb{N}$, such that

$$u(x) = \sum a_j e_j(x)$$

in L^2 and $\|u\|_{L^2}^2 = \sum |a_j|^2$. Now, let us fix a number $\beta > 0$ and consider the operator

$$[(-\Delta)_\omega]^\beta : D([(-\Delta)_\omega]^\beta) \subset L^2 \to L^2$$

given by

$$([(-\Delta)_\omega]^\beta u)(x) = \sum \lambda_j^\beta a_j e_j(x)$$

where

$$D([(-\Delta)_\omega]^\beta) = \{u = u(x) = \sum a_j e_j(x) \in L^2 : \sum ((\lambda_j)^\beta)^2 a_j^2 < \infty\},$$

and the convergences of the functional series are meant in L^2. In fact, $[(-\Delta)_\omega]^\beta$ is the power of the operator $(-\Delta)_\omega$ in the sense of Stone-von Neumann operator calculus. So, $[(-\Delta)_\omega]^\beta$ is self-adjoint, the spectrum $\sigma([(-\Delta)_\omega]^\beta)$ consists of eigenvalues λ_j^β, $j \in \mathbb{N}$, and eigenspaces corresponding to λ_j^β-s are the same as eigenspaces for $(-\Delta)_\omega$, corresponding to λ_j-s. (These properties follow from the general results concerning the power of the self-adjoint operator in the sense of Stone-von Neumann operator calculus.)

$D([(-\Delta)_\omega]^\beta)$ with the scalar product

$$(u,v)_\beta = (u,v)_{L^2} + \big([(-\Delta)_\omega]^\beta u, [(-\Delta)_\omega]^\beta v\big)_{L^2}$$

is a Hilbert space; the corresponding norm

$$\|u\|_\beta = (\|u\|_{L^2}^2 + \|[(-\Delta)_\omega]^\beta u\|_{L^2}^2)^{1/2}.$$

Moreover, the scalar product

$$(u,v)_{\sim\beta} = \big([(-\Delta)_\omega]^\beta u, [(-\Delta)_\omega]^\beta v\big)_{L^2}$$

determines the equivalent norm

$$\|u\|_{\sim\beta} = \|[(-\Delta)_\omega]^\beta u\|_{L^2}.$$

It is easy to see that if $0 < \beta_1 < \beta_2$, then

$$D([(-\Delta)_\omega]^{\beta_2}) \subset D([(-\Delta)_\omega]^{\beta_1}). \tag{4.12}$$

continuously and $C_c^\infty \subset D([(-\Delta)_\omega]^\beta)$ for any $\beta > 0$.

The continuity of the embedding (4.12) follows from the estimations

$$\begin{aligned}
\sum_{j=1}^\infty ((\lambda_j)^{\beta_1})^2 a_j^2 &= ((\lambda_1)^{\beta_1})^2 \sum_{j=1}^\infty \frac{((\lambda_j)^{\beta_1})^2}{((\lambda_1)^{\beta_1})^2} a_j^2 \\
&\le ((\lambda_1)^{\beta_1})^2 \sum_{j=1}^\infty \frac{((\lambda_j)^{\beta_2})^2}{((\lambda_1)^{\beta_2})^2} a_j^2 \\
&= \frac{((\lambda_1)^{\beta_1})^2}{((\lambda_1)^{\beta_2})^2} \sum_{j=1}^\infty ((\lambda_j)^{\beta_2})^2 a_j^2.
\end{aligned}$$

Acknowledgement. I want to thank professors Bogdan Przeradzki and Robert Stegliński for their valuable guidance on the continuity of the embedding (4.12).

References

[1] R. A. Adams; *Sobolev Spaces*, Academic Press, New York, 1975.
[2] A. Alexiewicz; *Functional Analysis*, PWN, Warsaw, 1969 (in Polish).
[3] H. Attouch, G. Buttazzo, G. Michaille; *Variational Analysis in Sobolev and BV Spaces. Applications to PDEs and Optimization*, SIAM-MPS, Philadelphia, 2006.
[4] K. Bogdan, K. Burdzy, Z. Chen; Censored stable processes, *Probab. Theory Related Fields*, **127** (2003), No. 1, pp. 89–152.
[5] M. Bonforte, Y. Sire, J. L. Vazquez; Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains, *Discrete and Continuous Dynamical Systems*, **35** (2015), No. 12, pp. 5725-5767.
[6] H. Brezis; *Operateurs Maximaux Monotonnes et Semigroups de Contractions dans les Spaces de Hilbert*, Math. Studies, 5, North-Holland, 1973.
[7] H. Brezis; *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.
[8] M. Cotlar, M. Cignoli; *An Introduction to Functional Analysis*, North-Holland Publishing Company, Amsterdam, 1974.
[9] A. Friedman; *Partial Differential Equations of Parabolic type*, Prentice-Hall, Inc., Englewood Cliffs, New Jork, 1964.
[10] P. Grisvard; *Elliptic Problems in Nonsmooth Domains*, Pitmann, London, 1985.
[11] P. Grisvard; *Singularities in Boundary Value Problems*, Masson, Springer-Verlag, Paris, 1992.
[12] B. Helffer; *Spectral Theory and its applications*, Cambridge, United Kingdom, 2013.
[13] D. Idczak; A bipolynomial fractional Dirichlet-Laplace problem, *Electronic Journal of Differential Equations*, **2019** (2019), No. 59, pp. 1-17.
[14] D. Idczak; Sensitivity of a nonlinear ordinary BVP with fractional Dirichlet-Laplace operator, *Electronic Journal of Differential Equations*, **2021** (2021), No. 64, pp. 1-19.
[15] M. Kwaśnicki; Ten equivalent definitions of the fractional Laplace operator, *Fract. Calc. Appl. Anal.*, **20** (2017), No. 1, pp. 7–51.
[16] J.-L. Lions, E. Magenes; *Problemes aux Limites Non Homogenes et Applications*, Dunod, Paris, 1968.
[17] W. Mlak; *An Introduction to the Hilbert Space Theory*, PWN, Warsaw, 1970 (in Polish).

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